Phil 411 *Topics in the History & Philosophy of Mathematics* Winter 2023

Tuesdays, Thursdays: 11:35 to 13:00, Educ 338.

Course Page: *myCourses*

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Summary. This course is part of an ongoing series designed to act as an introduction to some of the important philosophical questions (metaphysical, semantical, epistemological) about mathematics approached through its history. A look at some of the important developments in mathematics from ancient to modern times shows that there are not fixed and stable answers to the questions raised, and that mathematics is by no means a cut-and-dried discipline. Central problems are certainly solved, but often by making dramatic changes in the conceptual frameworks.

Notable developments are:

1. The instrumental use of 'imaginary numbers' (like $\sqrt{-1}$) in wider and wider contexts, leading to the introduction of the *complex numbers*, the subsequent proof of the Fundamental Theorem of Algebra, and then the development of complex analysis.

2. The move to see Euclid's Parallel *Postulate* as a potential theorem of Euclidean geometry which has to be proved, the failure of all attempts to do this, the subsequent discovery of surfaces in ordinary geometry which 'violate' this Postulate when 'straight line' is interpreted in a new way, and the subsequent mathematical and logical importance of such 'reinterpretation'.

3. The growth of what we now know as *real analysis*, via Cartesian analytic geometry, the differential and integral calculus, with its detour through infinitesimals, and then the modern characterisation of the limit notions and the real numbers, finally 'clarifying' ancient puzzles (e.g., Zeno) and ancient methods (Euclid, Archimedes) surrounding the limit notion.

In all these developments there is an abundance of new concepts and new subject matter introduced. Moreover, in the period (roughly) 1900–1940 clarification of many of these things involved in a significant way central notions that we now consider to be *logical*, the most important of which are *precise formulation in restricted languages*, *deduction*, *satisfiability* and *deductive consistency*, all of which are involved in the 'drive for rigour' of the 19th c., culminating in Gödel's Incompleteness Theorems and the work of Tarski, Skolem, Church, Turing and others. (See PHIL 310!)

Particular Topic. In this iteration of the course, we will concentrate on one central aspect of the *theoretical treatment of infinity* in the later 19th century. This is the transformation brought about by the work of the German mathematician Georg Cantor (in the period 1872–1899). Roughly the same time saw the development of what became known as the axiomatic method, due to another German mathematician, David Hilbert from 1894 on. We will spend much less time on this, but it is important in understanding what is going on.

Despite the 'modernness' of these developments, it's important to realise that central concerns with the infinite go back to the pre-Socratics (e.g., Zeno), Aristotle, Euclid and Archimedes, and also plays a part in the the work of some important Islamic philosophers and the Scholastics.

One important question was this: does the infinite exist, and in what sense? What about the physical universe, the past, collections of abstract things, such as the natural numbers, or even a line segment? Aristotle thought not, that the infinite was manifest only in the sense of the potential infinite, and not in the form of what came to be known in contradistinction as an *actual* infinite. The potential infinite is illustrated by a phenomenon one can see exemplified in the natural numbers: for any number n one chooses, there will be a number ($n + 1$ will do) which is greater. Note that the statement here is what we would call in logic a $\forall$-$\exists$ statement. (Famously Euclid showed that this phenomenon is exhibited in the prime numbers as well.) This apparently falls short of saying that there is *something* which is infinite (an abstract collection, say), which would with some manipulation take the form of an $\exists$-$\forall$ assertion. But even if the existence of this thing *is* granted, it is surely a further step to saying that there is an infinite *number* which 'counts' this collection. And if there is such a number, is it 'like' the ordinary numbers? And in what ways is it 'like' them? What does it mean to say that such a collection can be 'counted', or its size gauged?

Objections to numerical assessment of the infinite were often presented in the form of contradictions or paradoxes, in effect paradoxes of infinite size. We will begin by examining some of these, especially in the work of Galileo. Cantor's treatment of infinite size directly addresses arguments such as Galileo's, and shows that by adopting the criterion of equal size based on bijective correspondence, the odd numbers, the even numbers, the natural numbers, the rational numbers and the algebraic numbers are all shown to be of the *same* size (denumerable), whereas the real numbers are bigger than all these collections. Our subsequent consideration of Cantor is devoted mostly to his famous *Grundlagen*, published in 1883, which lays out the groundwork of the theory of both transfinite ordinal and cardinal *numbers*, and sets out a relationship between them, so erecting a *numerical* framework within which one can present the earlier results on the size of collections. Cantor's crucial conceptual distinction between between *cardinal* and *ordinal* numbers reflects an important distinction between the size of the collection and the way it's arranged. Cantor's work itself depends essentially on the concept of a *set*, and Cantor's later framework was designed to deal with some new paradoxes of infinite number (of a different kind from Galileo's), and which were shown by Russell and others to be serious difficulties for the notion of set itself, giving rise to new kinds of paradoxes of infinity, despite Cantor's efforts (visible in the late correspondence of Cantor with Dedekind and Hilbert). But Cantor's insights were finally transformed into modern axiomatic set theory by Zermelo and von Neumann in the first decades of the 20th century, which, at some cost to simplicity, dissolves these new paradoxes of infinity. This will bring us to a brief consideration of the axiomatic method, and to a third kind of paradox of infinity, namely the *Skolem Paradox*, a 'paradox' (but not a contradiction) which we have to live with.

Prerequisites. Having done PHIL 210 or COMP 230 is *essential*, and having done PHIL 310 (or equivalent) and PHIL 311 is *highly recommended*; it would also be greatly beneficial to have done a course in the history of mathematics (e.g., the course sometimes offered in the McGill Mathematics Department).

Course Material/Readings. The lectures will concentrate on close reading and discussion of original texts, all of which will be made available through the *myCourses* Website, occasionally supplemented by Handouts on various things. The readings will be *essential*, and many of the lectures will consider them in some detail, and will assume that they have been read beforehand.

Marking and Assessment: The final mark is composed of three short assignments (20% in total), a short sketch paper due around the middle of the term (30%), based on set questions, and a final paper, also based on set questions, due in the exam period with deadline as set by the university's rules on 'take-home exams' (50%). (This will be officially a take-home exam, with a date for the 'exam' and date of submission set by the University, although in practice it is really a paper.)

Provisional timing is as follows:

- First Short Assignment: Week 3.
- Second Short Assignment, Week 6.
- Sketch Essay, Week 8 (so due before the break).
- Third Short Assignment, Week 12.
- Final Essay, Week 13.

Note that according to my way of counting, Week 1 of the course begins on 2nd January. All work will be *assigned* on *myCourses*, and must be *submitted* through *myCourses*. The Short Assignments will require only short answers ($\leq$ 100 words per question) to specific questions asked about the reading material assigned, and may focus on readings not discussed at length in lectures. The short/sketch paper is to be $\leq$ 500 words in length on one of the topics to be assigned. The final paper (the take-home 'exam') is expected to be $\leq$ 2500 words in length on a topic to be assigned.

More detailed instructions for each assignment will be given at the time.

Policy for Late Work: Extensions to deadlines set will be granted only in **exceptional** circumstances, usually only for medical reasons or other, similar emergencies (which of course include COVID-related difficulties), and with a medical note or other appropriate documentation wherever appropriate. Late work will be penalised at the rate of 5 percentage points per day overdue, so half a grade-scale per day.

Important: Students experiencing difficulties for any reason, particularly with assignments, and especially for reasons connected directly or indirectly with COVID and the now semi-permanent 'unusual' circumstances, ***should contact me as early as they can***. My experience suggests that delay simply makes it more difficult, and greatly increases stress for you and the workload for you *and* me. In cases of clear difficulties of this sort, deadlines will be treated with flexibility.

For specific problems concerning completion of assignments or with exams, you are encouraged to contact the office for *Student Accessibility and Achievement* (Formerly known as the *Office for Students with Disabilities*); please see `https://www.mcgill.ca/access-achieve/`. *NB*

Submission of Work. *All* work is to be submitted *electronically*, to *myCourses* as PDF documents. (WORD files are NOT acceptable: PDFs can be created very simply from any word processor files.) The titles of the files submitted are to be of the form 'Bloggs-G-350-X', where 'Bloggs' is here a placeholder for *your **surname** as it appears on the course registration*, 'G' is a placeholder for your first **given name** as it appears on the registration sheet, 'X' is a placeholder either for (as appropriate) 'Sketch', 'Final' or 'Assignment-*n*', where '*n*' will be either '1', '2', '3', again as appropriate.

Delivery of Lecture Material The course will be in person; there will be three lecture-hours per week, on Tuesdays and Thursdays, together with some Office Hours, as yet unscheduled. These will be sometimes in person, sometimes via Zoom. I will post a link to a sign-up sheet, and signing up will be mandatory.

The lectures will focus on the material in the readings, the aim being to explain the more difficult parts of this, and perhaps to elaborate, too. I will sometimes use supplementary Handouts for the same purpose. Note that

- I will also occasionally make use of the same (posted-video) format in the case of (I hope rare) enforced absence from the classroom. This will, of course, will be managed/announced on an *ad hoc* basis.
- I will invite questions to be submitted to me, and when I consider these of general importance, they will be discussed in lectures, otherwise I'll just reply privately.

McGill Policies

1. *McGill University values academic integrity. Therefore all students must understand the meaning and consequences of cheating, plagiarism and other academic offences under the Code of Student Conduct and Disciplinary Procedures. (See* `www.mcgill.ca/integrity` *for more information.)* $\underline{\mathcal{NB}}$

2. *In the event of extraordinary circumstances beyond the University's control, the content and/or evaluation scheme in this course is subject to change.*

3. *In accord with McGill University's Charter of Students' Rights, students in this course have the right, without seeking permission, to submit in English or in French any written work that is to be graded.*

4. *As instructors of this course, the Lecturer and (where appropriate) TAs endeavour to provide an inclusive learning environment. If you experience barriers to learning in this course, do not hesitate to discuss them with us or with Student Affairs or the* Office for Students with Disabilities, `https://www.mcgill.ca/osd`, *514-398-6009.*

5. *McGill University is on land which is the traditional and unceded territory of the Kanien'keha:ka (Mohawk), a place which has long served as a site of meeting and exchange amongst nations.*