PHIL 310 *Intermediate Logic* Winter 2024

Tuesdays, Thursdays: 13.05–14.35. Room: LEA 15

Course Page: *MyCourses*

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Brief Overview. In a first course on logic, the emphasis is on presenting and working *within* a standard logical system for first-order logic, usually (as in Phil 210) some version of the propositional (truth-functional) and first-order languages together with a Natural Deduction derivation system. In this course, however, we concentrate much more on proving theorems *about* this (or equivalent) standard logical systems, and not *in* it. Because of this, the course is very different in character from an introductory logic course: above all, it is more *mathematical* in nature, since we use mathematics to study the language, the interpretations and the derivation system. Because of this, *mathematical aptitude* (although not necessarily any special mathematical *knowledge*) is very important. This is meant to be a necessary **WARNING**, since many students (by no means all, of course!) are misled by having found introductory logic easy, as many do. On the other hand, if you found introductory logic difficult, then the chances are that you will find this difficult, too.

Prerequisites. Introduction to Deductive Logic (Phil 210), Comp 230, or equivalents. Not open to students who have taken Math 318.

Textbook We will use a book specially written for this course, namely

Intermediate Logic, an Open Introduction, by Richard Zach and Michael Hallett.

This book was originally written by Richard Zach of the University of Calgary, for when he taught Phil 310 here in the winter of 2015. It is based on the material Dirk Schlimm and I have used for this course for at least the past 20 years, material which is fairly standard for a course of this kind, and which can be found (sometimes in rather different presentations) in many good books. (See below.)

The Zach/Hallett text can be found immediately on the *myCourses* site. As its title might suggest, it's part of a project, *The Open Logic Project*, which aims to provide free or inexpensive open source textbooks across a wide swathe of logic and mathematical logic, and which we very much support, as should you! Anyone not yet registered who would like to see the book can find it **here**: Textbook

NB: The Parts, Chapters, and Appendices mentioned in what follows refer to this book. The whole book has been posted, but I've put the Contents and Introduction as separate parts in the Introductory Reading section of *MyC*. I will later post the Parts we will use separately just to make the book more manageably for you on your computer screens. (If there are new versions of the text, I will let you know, as with any other supplements. And conversely, if you spot typos or come across clumsy or misleading formulations or arrangements, we'd be very grateful if you'd let us know.)

There are many good books on the material covered here. Four I would single out are: (a) Hubert Enderton: *A Mathematical Introduction to Logic;* (b) Elliot Mendelson: *Introduction to Mathematical Logic*; (c) George Boolos, John Burgess and Richard Jeffrey: *Computability and Logic* (4th/5th edition, Cambridge University Press); and (d) the second half of Moshé Machover: *Set Theory, Logic and Their Limitations* (Cambridge, 1996), a book we used for years to teach Phil 310 from. A very nice book (which, however, doesn't go as far as we will) is Dirk van Dalen: *Logic and Structure* (3rd edition, Springer-Verlag). The best book in French is S. C. Kleene, *Logique mathématique* (Paris, Armand Colin, 1971, a translation of Kleene's *Mathematical Logic*).

Summary of the Material We will begin with a brief introduction to some basic set theory (Part I) of the book, and interspersed with this a quick review of ways of proving informally (more details in Appendix B), and a rather more detailed (though still breezy) treatment of the principle of mathematical induction (Appendix C in the book). The next focus (covered in Part II of the book) will then be on *classical first-order predicate logic* in the form you are more or less familiar with from Phil 210, i.e., FOL with a Natural Deduction derivation system. This will culminate in the major *Completeness Theorem for first-order logic* (Chapter 10), a theorem first proved by Gödel, which we relied on implicitly in our first course, which says roughly that the semantics and the proof-system for FOL *match*, more precisely, that a formula is an FOL truth if, and only if, it is derivable from no premises, and (more strongly) that a formula is a logical consequence (semantical!) of a set of formulas Σ if, and only if, it is derivable from Σ in the proof-system (syntactical!). (So crucially this applies when Σ is designed as a set of axioms, or first principles, for a specific theory, such as arithmetic.) This means that looking for a proof in the system is the right way to try to discover if a formula is a logical consequence of a given set of assumptions (or is a first-order truth), for the Completeness Theorem tells us that if a proposition is a logical consequence of Σ, there *will* be a proof it, though there's no guarantee that we'll find one. (So here we work on the syntactic side.) On the other hand, if the proposition is *not* a logical consequence, it must be possible to design (again, in theory!) an interpretation (a 'model' as we say) which makes the starting points (so, the axioms) all true and the proposition itself false.

Note that Part II of the textbook is long. This is in part because a variety of deduction systems are presented in the book; we, however, will concentrate only on the Natural Deduction system with which you are (in all probability) familiar.

There is another equivalent formulation of the completeness theorem which is important in the way we prove it (in our case through Henkin's work), the version which says that if a set of formulas is syntactically consistent, then there is an interpretation which satisfies this set, which incidentally gives us a version of the traditional claim 'consistency $\Rightarrow$ existence', which says that if a set of axioms is consistent, then it must describe *something*.

This leads to a related question: How can we show in general that an axiom system we give is *consistent*, i.e., does not lead to the proof of contradictions? (Logic itself doesn't, as we will show, but special axioms for numbers or sets added to the logical system might.) In some cases, this question can be answered, for example, by translation into another system which is accepted as consistent (e.g., taking Euclidean geometry and translating it into analyic, Cartesian geometry). But not for all. Take the relatively simple system for the arithmetic of natural numbers; can we prove its consistency? If it's *not* consistent, then it is clearly inadequate, so asking this question is asking something about its adequacy. But there is another adequacy question, too. Are there truths about numbers which *cannot* be derived from the natural axiom system for arithmetic? We can derive some truths, for example

$$2 + 2 = 4$$

or

$$\forall m, n(m + n = n + m)$$

and it's good that we can derive truths like these, because they're important, and close to the heart of the subject. But do we know that we can derive in principle *all* of the truths, obvious or not? We have reason to think the Twin Prime Conjecture (which says that there are infinitely many prime numbers p

such that $p + 2$ is also prime, for example, 11 and 13 or 29 and 31) is true; but is it possible that this for some reason is *not* provable in our system of ordinary arithmetic?

The questions are addressed by what we cover in Part III of the book, 'Incompleteness'. In fact, both questions were addressed by Gödel.

Gödel's First Incompleteness Theorem (proved in Chapter 18) presents a general method (and its generality is the centrally important point) for producing sentences of arithmetic which are true but not provable, and this method applies to a very wide range of theories which go well beyond arithmetic. Leading up to this theorem, we will look at recursive functions, coding and representability (Chapter 15–17), all of which are crucial in the set up and proof of the Gödel theorem. *Gödel's Second Incompleteness Theorem* addresses the first question raised above. Essentially what Gödel showed is that the consistency of arithmetic can only be shown by use of a theory which has *stronger* theoretical resources than those available to arithmetic itself. This in a sense says that we cannot really prove the consistency of arithmetic at all, at least, not if we are doing so to guarantee the 'health and safety' of formal arithmetic. Again, Gödel's Second Incompleteness Theorem has extraordinary range, and applies to a very wide swathe of ordinary, working theories, the theory of sets among them.

Closely related to the both the First and the Second Theorems is a result called *Löb's Theorem* (in Chapter 18), which we will also look at briefly. We will also examine another result which in a way presages Gödel's First Incompleteness Theorem, often called *Skolem's Theorem*, which concerns the austere and strange world of non-standard models of arithmetic. These contain the usual natural numbers, which behave exactly as we expect them to behave, but much (very much!) more, namely a vast multiplicity of what we call *non-standard numbers* with very odd structure. (See Chapter 19.) And tied up with all these things is another classic result, namely *Tarski's Theorem* on the undefinability of truth of arithmetical sentences within the language of arithmetic itself (also Chapter 18).

In sum, we will see that these results (some of the most important theorems of twentieth-century logic) tells us a good deal about the power and also the limitations of first-order logic. (But what's the alternative to first-order logic? This is touched on in Chapter 13.)

From these central results, technical in nature, wider *philosophical* consequences start to flow, certainly important consequences for the philosophy of mathematics, for instance concerning the role of proof and provability, and the connections between proof and truth, and for Hilbert's programme, some of which we touched on above, but also consequences for philosophy more generally, for instance for the computational theory of mind, for the theory of truth (and thus philosophy of language), and consequences concerning the nature of mathematical and scientific theories.

The material is very much cumulative and difficult, and the results emerge slowly. Hence, it's important both to keep up, and to be patient. Believe us, it's worth it! And remember, what we're doing is really proving things in an *informal* way about *formal* proof systems, even though it helps to be familiar with doing formal proofs (following rules of proof) and with the semantics of first-order languages.

Marking and Assessment: The final mark is composed of the results of 5 assignments worth 15% each, and a final exam worth 25%.

The timing of the Assignments will be roughly as follows:

- Assignment 1: Week 4 (15%).
- Assignment 2: Week 6 (15%).

- Assignment 3: Week 8 (15%).
- Assignment 4: Week 10 (15%).
- Assignment 5: Week 12/13 (15%).
- Final (Take-Home Final, regulated by Final Exam Schedule), 25%.

Remember we have a Study-Break Week, beginning Monday, 4th March.

The Final Exam will be largely informal in nature, concentrating more on the conceptual side of what we've looked at, rather than the technical side.

Note that according to my way of counting, Week 1 of the course begins on Monday, 8th January, 2024, though there will be a lecture before this on 4th January.

All work will be *assigned* on *myCourses*, and must be *submitted* through *myCourses*.

The basic means for *Submission of Work* will be through PDF documents uploaded to *myCourses/Assignments*. (*Word* or *Pages* files are NOT themselves acceptable, although they can of course be used to create PDF files.) We encourage you very strongly to use the LaTeX typesetting system to produce the PDF. This sounds intimidating, but is really not. (See below.) Failing this, *Word* and *Pages* probably have enough symbols available for the preparation of adequate PDF documents; if necessary, symbols can be written in by hand or by using some acceptable key, such as the conventions we have become used to through the Carnap system. In the last resort, we will accept PDF documents produced from handwritten pages photographed with a smartphone camera or the like. But if you choose this route, you must produce PDF versions of your photographs, AND moreover you should weave all these pages into ONE PDF document. Failure to do this will be treated as failure to submit the assignment.

The file titles for Assignments should be of the form 'Bloggs-G-310-*n*', where 'Bloggs' is here a placeholder for *your* ***surname*** *only*, 'G' a placeholder for your first (given) name as it appears on the registration sheet, and '*n*' will be either '1', '2', '3', '4', '5', or 'Final' as required. In the case where an Assignment is split, you will use '*n*-1' or '*n*-2', as appropriate.

The textbook has an abundance of problems in it; some of them might well figure in formal assignments, *but you should in any case do them unbidden and as a matter of course as you come across them*.

Learning LaTeX As should be clear from the above, we take this opportunity to encourage students to use the type-setting system LaTeX to produce PDFs for submission. LaTeX is a must for anyone studying mathematics, physics, statistics, economics, computer science, or philosophy of science, and we also think for people in the arts as well, especially philosophy. Not only can you set beautifully anything formal, but it gives lovely, fully controllable layouts for ordinary text, like this one, including[1], which appears as if it's been professionally typeset, and also excellent, clean Slide presentations (through LaTeX Beamer), which have become very *à la mode*.

If you're not familiar with this system, the best place to begin (without installing LaTeX on your computers) is with an on-line site called *Overleaf*. In fact, Richard Zach has an Introductory Video **here**: Zach's Introduction to *Overleaf*. (*Overleaf* provides a very good introduction to LaTeX generally, and is not just a means to do this straightforwardly.) We will provide some templates for practice, and files for completing each of the practice files and Assignments.

Policy for Late Work: Extensions to deadlines set will be granted only in **exceptional** circumstances, usually only for medical reasons or other, similar emergencies (which of course include COVID-related

[1]… footnotes …

difficulties), and with a medical note or other appropriate documentation wherever appropriate. Late work will be penalised at the rate of 5 percentage points per day overdue, so half a grade-scale per day.

Important: Students experiencing difficulties for any reason, particularly with assignments, and especially for reasons connected directly or indirectly with COVID and the now semi-permanent unusual circumstances, ***should contact me as early as they can***. Experience suggests that delay simply makes it more difficult, and greatly increases stress for you and workload for you *and* us. In cases of clear difficulties of this sort, deadlines will be treated with flexibility.

For specific problems concerning completion of assignments or with exams, you are encourage to contact the office for *Student Accessibility and Achievement* (formerly known as the *Office for Students with Disabilities*); please see *SAA Office*. NB

Reading and Handouts The textbook, background and supplementary readings, Handouts if and when issued, will all be made available through *myCourses* (divided into self-explanatory sections), so you should keep a steady eye on this.

Initial background material should be read *as soon as possible*, certainly before the first week. The most interesting of these readings is an article 'Gödel's proof' by Ernest Nagel and James Newman, published in 1957 in *Scientific American*. This presents the very important background which culminates in Gödel's Incompleteness Theorem, and has an informal presentation of the result itself. (The article was later expanded into a small book, which you can find in the library, perhaps even online there, and the Concluding section of this book is also included here in the Introductory Reading.) Gödel's Incompleteness Theorem is the final important goal of this course, so understanding the background and having some idea of the way the result is proved cannot but be beneficial. There will also be some readings from Frege, the originator of modern logic, and also some material on the nature of logic itself. (Frege's view of logic is very instructive, although it differs considerably from the modern view.) These readings will be placed on the *myCourses* site in the 'Introductory Readings' folder.

Delivery of Lecture Material

The course will be in person; there will be three lecture-hours per week, on Tuesdays and Thursdays, together with some Office Hours, as yet unscheduled. These will take place via Zoom.

The lectures will focus on the material as presented in the textbook, the aim being to explain the more difficult parts of this, and perhaps to elaborate, too. I will sometimes use supplementary Handouts for the same purpose. Note that:

- I may make use of the video format (not live, but posted) for discussion of extra matters, which are germane, but perhaps tangential to the core subject matter.
- I will also occasionally make use of the same (posted-video) format in the case of (I hope rare) enforced absence from the classroom. This will, of course, will be managed/announced on an *ad hoc* basis.

McGill Policies

1. *McGill University values academic integrity. Therefore all students must understand the meaning and consequences of cheating, plagiarism and other academic offences under the Code of Student Conduct and Disciplinary Procedures. (See* `www.mcgill.ca/integrity` *for more information.)* NB

2. *In the event of extraordinary circumstances beyond the University's control, the content and/or evaluation scheme in this course is subject to change.*

3. *In accord with McGill University's Charter of Students' Rights, students in this course have the right, without seeking permission, to submit in English or in French any written work that is to be graded.*

4. *As instructors of this course, the Lecturer and (where appropriate) TAs endeavour to provide an inclusive learning environment. If you experience barriers to learning in this course, do not hesitate to discuss them with us or with Student Affairs or the* Student Accessibility and Achievement *(*SAA Office*, 514-398-6009).*

5. *McGill University is on land which is the traditional and unceded territory of the Kanien'keha:ka (Mohawk), a place which has long served as a site of meeting and exchange amongst nations.*