Phil 411 *Topics in the History & Philosophy of Mathematics* Autumn (Sept.–Dec.) 2023

Mondays, Wednesdays: 16.05–17.25, FERR 408

Course Page: *myCourses*

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Summary. This course is part of an ongoing series designed to act as an introduction to some of the important philosophical questions (metaphysical, semantical, epistemological) about mathematics approached through its history. A look at some of the important developments in mathematics from ancient to modern times shows that there are not fixed and stable answers to the questions raised, and that mathematics is by no means a cut-and-dried discipline. Central problems are certainly solved, but often by making dramatic changes in the conceptual frameworks.

Notable developments are:

1. The instrumental use of 'imaginary numbers' (like $\sqrt{-1}$) in wider and wider contexts, leading to the introduction of the *complex numbers*, the subsequent proof of the Fundamental Theorem of Algebra, and then the development of complex analysis.
2. The move to see Euclid's Parallel *Postulate* as a potential theorem of Euclidean geometry which has to be proved, the failure of all attempts to do this, the subsequent discovery of surfaces in ordinary geometry which 'violate' this Postulate when 'straight line' is interpreted in a new way, and the subsequent mathematical and logical importance of such 'reinterpretation'.
3. The growth of what we now know as *real analysis*, via Cartesian analytic geometry, the differential and integral calculus, with its detour through infinitesimals, and then the modern characterisation of the limit notions and the real numbers, finally 'clarifying' ancient puzzles (e.g., Zeno) and ancient methods (Euclid, Archimedes) surrounding the limit notion.
4. The instructive use of logic and 'the axiomatic method'.
5. The importance of the 'Axiom of Choice', where it comes from (more generally, proof analysis), and what it does.

In all these developments there is an abundance of new concepts and new subject matter introduced. Moreover, in the period (roughly) 1900–1940 clarification of many of these things involved in a significant way central notions that we now consider to be *logical*, the most important of which are *precise formulation in restricted languages*, *deduction*, *satisfiability* and *deductive consistency*, all of which are involved in the 'drive for rigour' of the 19th c., culminating in Gödel's Incompleteness Theorems and the work of Tarski, Skolem, Church, Turing and others. (See PHIL 310!)

Particular Topic. In this iteration of the course, we will concentrate on the *theoretical treatment of infinity*, especially the transformation brought about in the nineteenth-century by the work of the German mathematician Georg Cantor (in the period 1872–1899). Roughly the same time saw the development of what became known as the axiomatic method, due to another German mathematician, David Hilbert from 1894 on. We will spend much less time on this, but it is important in understanding what is going on.

Central concerns with the infinite go back to the pre-Socratics (e.g., Zeno), Aristotle, Euclid and Archimedes, and also plays a part in the work of some important Islamic philosophers, and in Scholastic and Early Modern philosophy.

One important question, which will occupy us throughout the course, is this: does the infinite exist, and in what sense? What about the physical universe, moments (periods?) of time, the past, collections of abstract things, such as the natural numbers, or a line segment? Aristotle thought not, that the infinite was manifest only in the sense of the *potential* infinite, and not in the form of what came to be known in contradistinction as an *actual* infinite. The potential infinite is illustrated by a phenomenon one can see exemplified in the natural numbers: for any number n one chooses, there will be a number ($n + 1$ will do) which is greater. Note that the statement here is what we would call in logic a $\forall$-$\exists$ statement. (Famously Euclid showed that this phenomenon is exhibited in the prime numbers as well.) This apparently falls short of saying that there is *something* which is infinite (an abstract collection, say), which would with some manipulation take the form of an $\exists$-$\forall$ assertion. Note that in all this we have to distinguish between *unending* (or 'going on for ever'), or even being 'too large to comprehend', and being *actually infinite*, so (to take a simple example) between eventually becoming n years old for any n (even one larger than the number of grains of sand on the beach, or the number of electrons in the Milky Way), and being *infinitely* old.

In the oldest considerations of infinity, a second concern emerges, the concern with *limits*, structurally related to the concern with the distinction between the potential and actual infinite. In certain cases, we can see that an infinite sequence of ordinary numbers (say the fractions $1/2, 3/4, 7/8, \ldots$) have a limit, a number to which they get closer and closer, but are separate from, so here the number 1. In some cases, like this one, these limits are ordinary numbers, but in other cases (e.g., the ratio between a unit radius and the circumference of the unit circle) it is by no means clear that there are straightforward numerical limits. Archimedes, arguably the greatest ancient mathematician, wrestled with problems of this kind, and the network of questions here will be part of our concern from the start. And if there are limit numbers (even ordinary finite numbers, like 1) do they inherit or reflect the properties of the infinite sequences which approximate them?

The considerations adduced above bring out another important distinction. Even if the existence of something actually infinite is granted, it is surely a further step to saying that there are infinite *numbers* which 'count' or 'measure' this collection. What does it mean to say that such a collection can be 'counted', or its size gauged? And are these the same? If there is such a number, is it 'like' the ordinary numbers? And in what ways is it 'like' them, so how are they related?

As we will see from the beginning, objections to the existence of actual infinity were often presented in the form of *contradictions* or *paradoxes* (as we will see, this reflects a structural problem with arguing for the impossibility of the actual infinite), and closely related objections were raised against the idea of the numerical assessment of the infinite, in effect paradoxes of infinite size. These continue to be important in various guises, and we will examine some of the historically most important of these. The modern theory of infinity treated mathematically is due to the German mathematician Georg Cantor, who maintained that the actual infinite in many manifestations was deeply embedded in, and necessary for, modern mathematics. Cantor adopts the criterion of *sameness of size* based on bijective correspondence, a criterion which was often at the base of the 'paradoxes' mentioned above, and as opposed to a criterion of size based on the notion of co-extensive, or more- or less-extensive. This construal of size directly addresses many of the old arguments, and shows that when we adopt we can show that natural numbers, the odd numbers, the even numbers, the rational numbers, and the algebraic numbers are all of the *same* size (*denumerable*), whereas the real numbers are of different size, so bigger than all these collections (*non-denumerable*). (Cantor gave a famous proof of this, using the so-called 'diagonal argument', which we'll look at closely.) What we see here is something genuinely new in the study of infinity, the manifestation of radically different classes of infinite size.

This is the first of Cantor's major achievements. The second is the pursuance of the idea of infinite number. Our subsequent consideration of Cantor is devoted mostly to his famous *Grundlagen*, published in 1883, which lays out the groundwork of the theory of both transfinite ordinal and cardinal *numbers*, and posits a relationship between them, so erecting a *numerical* framework within which one can present the earlier results on the size of collections. Cantor isolated a crucial conceptual distinction between cardinal and ordinal numbers, the latter, but not the former, being in effect generalised counting numbers. The distinction is between the size of a collection and the way it's arranged in an ordering, a distinction which, in the case of finite collections, is hidden. Cantor's work itself depends essentially on the concept of a *set*, and the idea of using the infinite ordinal numbers to construct an account of infinite numerical size depends crucially on the *well-ordering hypothesis*, the assumption that every set can be arranged in a well-ordering ('can be counted', an association which we'll explore). Cantor's later justification of this framework (visible in the late correspondence with Dedekind and Hilbert) was designed to justify this, whilst at the same time dealing with some new paradoxes of infinite number (different from, but related to, the earlier paradoxes). These were shown by Russell and others to be serious difficulties for the notion of set itself, not just infinite number. Alongside this, various philosophers and mathematicians (Bolzano, Dedekind, Frege) tried to prove the existence of infinite sets, all unsuccessfully. All these efforts were replaced by the modern axiomatic set theory developed by Zermelo and von Neumann in the first decades of the 20th century, which, at some cost to simplicity, dissolves these new paradoxes, and incorporates most of Cantor's intentions, including the dependence on the well-ordering hypothesis. We will spend some time on this system, which will bring us to a brief consideration of the axiomatic method, and finally to a third kind of 'paradox' of infinity, namely the *Skolem Paradox*, a 'paradox' (but not a contradiction), and one that we have to live with.

Prerequisites. Having done PHIL 210 or COMP 230 is *essential*, and having done PHIL 310 (or equivalent) and PHIL 311 is *highly recommended*; it would also be greatly beneficial to have done a course in the history of mathematics (e.g., the course sometimes offered in the McGill Mathematics Department).

Course Material/Readings. The lectures will concentrate on close reading and discussion of original texts, all of which will be made available through the *myCourses* Website, supplemented by detailed Handouts on some of the subjects covered. The readings will be *essential*; many of the lectures will consider them in some detail, and will assume that they have been read beforehand.

Marking and Assessment: The final mark is composed of three short assignments (20% in total), a short sketch paper due around the middle of the term (30%), based on set questions, and a final paper, also based on set questions, due in the exam period with deadline as set by the university's rules on 'take-home exams' (50%). (This will be officially a take-home exam, with a date for the 'exam' and date of submission set by the University, although in practice it is really a paper.)

Provisional timing is as follows:

- First Short Assignment: Week 3.

- Second Short Assignment, Week 6.
- Sketch Essay, Week 8.
- Third Short Assignment, Week 12.
- Final Essay, set Week 13, due on the date set in the Final Exam Timetable.

All work will be *assigned* on *myCourses*, and must be *submitted* through *myCourses*. The Short Assignments will require only brief answers to specific questions (usually two questions, with $\leq$ 100 words per answer) asked about the reading material assigned, and may focus on readings not discussed at length in lectures. The short/sketch paper is to be $\leq$ 500 words in length on one of the topics to be assigned. The final paper (the take-home 'exam') is expected to be $\leq$ 2500 words in length on a topic to be assigned.

More detailed instructions for each assignment will be given at the time.

Policy for Late Work: Extensions to deadlines set will be granted only in **exceptional** circumstances, usually only for medical reasons or other, similar emergencies (which of course include COVID-related difficulties), and with a medical note or other appropriate documentation wherever appropriate. Late work will be penalised at the rate of 5 percentage points per day overdue, so half a grade-scale per day. Please ask for an extension as far in advance as possible.

Important: Students experiencing difficulties for any reason, particularly with assignments, and especially for reasons connected directly or indirectly with COVID and the now semi-permanent 'unusual' circumstances, ***should contact me as early as they can***. My experience suggests that delay simply makes it more difficult, and greatly increases stress for you and the workload for both you *and* me. In cases of clear difficulties of this sort, deadlines will be treated with flexibility.

For specific problems concerning completion of assignments or with exams, you are encouraged to contact the office for *Student Accessibility and Achievement* (Formerly known as the *Office for Students with Disabilities*); please see `https://www.mcgill.ca/access-achieve/`. *NB*

Submission of Work. *All* work is to be submitted *electronically*, to *myCourses* as PDF documents. (WORD or PAGES files are NOT acceptable: PDFs can be created very simply from any word processor files.) The titles of the files submitted are to be of the form 'Bloggs-G-350-X', where 'Bloggs' is here a placeholder for *your* ***surname*** *as it appears on the course registration*, 'G' is a placeholder for your first **given name** as it appears on the registration sheet, 'X' is a placeholder either for (as appropriate) 'Sketch', 'Final' or 'Assignment-*n*', where '*n*' will be either '1', '2', '3', again as appropriate.

Delivery of Lecture Material The course will be in person; there will be three lecture-hours per week, on Mondays and Wednesdays, together with some Office Hours, as yet unscheduled. The latter will be sometimes in person, sometimes via Zoom. I will post a link to a sign-up sheet, and signing up will be mandatory.

The lectures will focus on the material in the readings, the aim being to explain the more difficult parts of this, and perhaps to elaborate, too. I will sometimes use supplementary Handouts for the same purpose.

McGill Policies

1. *McGill University values academic integrity. Therefore all students must understand the meaning and* *NB*

consequences of cheating, plagiarism and other academic offences under the Code of Student Conduct and Disciplinary Procedures. (See `www.mcgill.ca/integrity` for more information.)

2. *In the event of extraordinary circumstances beyond the University's control, the content and/or evaluation scheme in this course is subject to change.*

3. *In accord with McGill University's Charter of Students' Rights, students in this course have the right, without seeking permission, to submit in English or in French any written work that is to be graded.*

4. *As instructors of this course, the Lecturer and (where appropriate) TAs endeavour to provide an inclusive learning environment. If you experience barriers to learning in this course, do not hesitate to discuss them with us or with Student Affairs or the* Office for Students with Disabilities, *`https://www.mcgill.ca/osd`, 514-398-6009.*

5. *McGill University is on land which long served as a site of meeting and exchange amongst Indigenous peoples, including the Haudenosaunee and Anishinabeg nations. We acknowledge and thank the diverse Indigenous people whose footsteps have marked this territory on which peoples of the world now gather.*