McGill University Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS Paper ALPHA

9 May 2017 1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) There are 12 problems. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12. Clearly indicate your choice of three questions in each group.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

(iii) Leave enough margins on all sides of the page to permit easy scanning and copying. Each question must be answered on consecutive pages without interruption, and only on the pages on the right side.

This exam comprises this cover and 3 pages of questions.

Single variable real analysis

Solve any three out of the four questions 1, 2, 3, and 4.

Problem 1. Determine whether the following sequences of functions converge uniformly or pointwise (or neither) in the regions indicated. Determine the pointwise limits (where they exist). Are the limiting functions continuous?

a)

$$f_n(x) = \begin{cases} \frac{\sin nx}{nx}, & x \neq 0\\ 1, & x = 0. \end{cases}$$

 $\begin{array}{l} \text{for } x\in [-\pi,\pi].\\ \text{b)} \ f_n(x)=e^{-nx}/n \text{ for } x\in [0,\infty). \end{array}$

Problem 2. Let (X, d) be a general metric space, $a \in X, t > 0$. Show that

- i) $\{x \in X : d(x, a) < t\}$ is open,
- ii) $\{x \in X : d(x, a) \le t\}$ is closed.

Problem 3. Does every sequence in the following sets have a subsequence converging to a limit point in the set? Explain your answers.

- a) The (middle-thirds) Cantor set.
- b) The set of all $x \in \mathbf{R}$ such that $1/2 \le \sin x \le 3/4$?

Problem 4. Find the radius of convergence of the following power series:

- a) $\sum_{n} (2^n/n!) z^n.$ b) $\sum_{n} (4^n/n^2) z^n.$ c) $\sum_{n} (n^3/3^n) z^n.$

Linear Algebra

Solve any three out of the four questions 5, 6, 7, and 8.

Problem 5.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Define the norm ||T|| of T to be $\max\{||Tx|| : x \in \mathbb{R}^n, ||x|| = 1\}$, where the norm of a vector is the usual Euclidean norm $(\sum x_i^2)^{1/2}$.

a) Let $A = T^* \circ T$, where T^* is the adjoint of T with respect to the standard inner product of \mathbb{R}^n , and let λ be the maximal eigenvalue of A. Prove that

 $||T||^2 = \lambda.$

b) Recall that T is orthogonal if $\langle Tx, Ty \rangle = \langle x, y \rangle$ for any two vectors $x, y \in \mathbb{R}^n$. Show that if T is orthogonal then ||T|| = 1. Suppose, conversely, that T is invertible and $||T|| = 1 = ||T^{-1}||$; prove that T is orthogonal.

Problem 6.

- a) Let V be a finite dimensional space, dim(V) = n. Let W_1, W_2, W_3 be three subspaces of V such that $dim(W_1) + dim(W_2) + dim(W_3) \ge 2n + 1$. Prove that $W_1 \cap W_2 \cap W_3 \ne \{0\}$. Show that this bound is optimal by providing for every n an example where $dim(W_1) + dim(W_2) + dim(W_3) = 2n$ and the intersection is $\{0\}$.
- b) Let n be a positive integer. Let $v \in \mathbb{F}^n$, where \mathbb{F} is a field. Prove that if n is big enough there is an $n \times n$ matrix A such that $A \neq 0$, Av = 0, tr(A) = 0, and the sum of the elements in every row of A is zero as well.

Problem 7. Let $A = (a_{ij})$ be an $m \times n$ matrix. Prove that that rank(A) = k if and only if there exists a $k \times k$ sub-determinant of A that is non-zero, and every $(k+1) \times (k+1)$ sub-determinant is zero.

Reminder: A $k \times k$ sub-determinant is defined as follows. We choose k columns $j_1 < j_2 < \ldots < j_k$ among the n columns, and k rows $i_1 < \ldots < i_k$ among m rows. The determinant of the matrix $(a_{i_sj_t})_{s,t=1,\ldots,k}$ is then a $k \times k$ sub-determinant.

Problem 8. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator. Suppose that $T^k = I$ for some integer $k \ge 1$, where I denotes the identity operator. Prove that T is diagonalizable.

Vector calculus, ODE, and complex analysis

Solve any three out of the four questions 9, 10, 11, and 12.

Problem 9. Find a general solution of the equation

$$y'''' - 5y'' + 4y = 80e^{3x}.$$

Problem 10. Find at least three nonzero terms in the power series expansion (about x = 0) of solutions to the equations xy'' + (2 - x)y' - y = 0.

Problem 11. Is it possible to solve the system

$$\begin{cases} xy^2 + xzu + yv^2 = 3, \\ u^3yz + 2xv - u^2v^2 = 2. \end{cases}$$

 $\begin{cases} xy + xzu + yv = 5, \\ u^3yz + 2xv - u^2v^2 = 2. \end{cases}$ for (u, v) as functions of (x, y, z) near (x, y, z) = (1, 1, 1) and (u, v) = (1, 1)? If so, compute the Jacobian $\partial(u, v) / \partial(x, y, z).$

Problem 12. Compute $\int \int_S |xyz| dS$, where S is a part of $z = x^2 + y^2$ bounded by the plane z = 1.