McGill University
Department of Mathematics and Statistics
Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS
Paper BETA
August 26, 2022
1:00 PM - 5:00 PM

GENERAL INSTRUCTIONS:

(i) This paper consists of the following 6 modules:

- [ALG] Algebra
- [AN] Analysis
- [GT] Geometry & Topology
- [NA] Numerical Analysis
- [PDE] Partial Differential Equations
- [PROB] Probability
- [COPT] Continuous Optimization

each of which comprises 4 questions. You should answer 7 questions from 3 modules, with at least 2 from each module. All questions are worth 10 points. PLEASE WRITE THE NUMBERS FOR THE QUESTIONS YOU ATTEMPTED NEXT TO THE MODULE TITLES ABOVE.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 6 pages of questions.
Let $R$ be a commutative ring and let $P_1, P_2$ be projective $R$-modules. Prove that $P_1 \oplus P_2$ and $P_1 \otimes_R P_2$ are projective $R$-modules as well.

(1) Prove that for $n \geq 5$, $A_n$ is the only non-trivial normal subgroup of $S_n$ (you may assume that $A_n$ is simple for $n \geq 5$).

(2) For $n \geq 5$, prove that $A_n$ is generated by the set of all three cycles $(ijk)$.

Let $R$ be a commutative ring with unit $1 \in R$. We define the Jacobson radical $J(R)$ to be the intersection of all maximal ideals of $R$. Let $x \in R$. Prove that $x \in J(R)$ if and only if $1 - xy$ is a unit for all $y \in R$.

(1) Determine the number of subfields in the splitting field of the polynomial $X^7 - 1$ over the field of rational numbers.

(2) Determine the number of subfields in the finite field $\mathbb{F}_{125}$ with 125 elements.

Please justify your answers for both (1) and (2).
Analysis

[AN 1] Let \( f : \mathbb{R} \to \mathbb{R} \) be a real-valued function with
\[
\limsup_{x \to x_0} f(x) \leq f(x_0)
\]
for all \( x_0 \in \mathbb{R} \). Such a function is called \textit{upper semi-continuous}. Is \( f \) Lebesgue measurable? Please carefully justify your answer.

[AN 2] Construct a sequence of functions \( f_n : \mathbb{R} \to [0, \infty) ; n = 1, 2, 3, ..., \) with \( f_n \in L^+ \) for all \( n \geq 1 \) such that
\[
\int_{\mathbb{R}} \liminf_{n \to \infty} f_n \, dm < \liminf_{n \to \infty} \int_{\mathbb{R}} f_n \, dm.
\]
Please carefully justify your answer.

[AN 3] Let \( 1 \leq p < \infty \). Given \( g \in L^p(\mathbb{R}^n, dx) \) and \( f \in L^1(\mathbb{R}^n, dx) \) where \( dx \) is Lebesgue measure, one defines the convolution by the formula
\[
f \ast g(x) := \int_{\mathbb{R}^n} f(x - t) g(t) \, dt.
\]
Show that \( f \ast g \in L^p(\mathbb{R}^n, dx) \) and
\[
\| f \ast g \|_p \leq \| f \|_1 \| g \|_p.
\]
Please carefully justify your answer.

[AN 4] Let \((X, \mathcal{M}, \mu)\) be a measure space and \( \{ f_n \}_{n=1}^\infty \) be a sequence of measurable functions on \( X \) and \( S := \{ x \in X ; \lim_{n \to \infty} f_n(x) \text{ exists} \} \). Show that \( S \in \mathcal{M} \). Please carefully justify your answer.
Geometry and Topology

[GT 1] Let $X = D^2 \times S^1$ be a solid torus. Let $p$ be a point in the interior of $X$.

(1) Prove that $X - \{p\}$ has the same homotopy type as the wedge $S^2 \vee S^1$ of a 2-sphere and a 1-sphere.

(2) Compute all homology groups of $X$.

[GT 2] Let $Y = S^1 \vee (S^1 \times S^1)$ be the wedge of a circle and a torus along a point.

(1) Draw the universal cover of $Y$.

(2) Draw three non-isomorphic covering spaces $\tilde{Y}_i \to Y$ that are regular and have $\text{Aut}(Y_i \to Y) \cong \mathbb{Z}$.

(3) Prove that $\pi_1 Y$ contains a free group of rank 2.

[GT 3] Given complex numbers $a, b, c \in \mathbb{C}$ and a positive integer $n$, define a subset of $\mathbb{CP}^2$ by the formula $W_{a,b,c} = \{ [x : y : z] \in \mathbb{CP}^2 \mid ax^n + by^n + cz^n = 0 \}$

(a) Prove that the set $W_{a,b,c}$ is well-defined.

(b) For which values of $(a, b, c)$ and $n$ is $W_{a,b,c}$ an embedded submanifold?

(c) What is the dimension of such a submanifold?

[GT 4] Suppose that $M$ and $N$ are oriented, connected, smooth manifolds, and that $F, G : M \to N$ are diffeomorphisms.

(a) Prove that if $M$ and $N$ are compact, then $F$ and $G$ are either both orientation-preserving or both orientation-reversing.

(b) Give an example showing that the conclusion of part a) can fail if $M$ and $N$ are not compact.
Numerical Analysis

[NA 1] Vandemonde Systems. Consider a set of 7 distinct points on the plane, \( \{x_0, \cdots, x_6\} \).

We write the approximation to \( u_{xy} \) at the point \( x_0 \) as a linear combination, \( u_{xy}(x_0) \approx \sum_{i=0}^{6} \alpha_i u(x_i) \).

(1) Derive the Vandermonde matrix \( V \), and right hand side vector \( b \) such that \( V \alpha = b \), where \( \alpha \) is the vector of coefficients \( \alpha_i \) defined above.

(2) How would you solve the system above? Give a short description with the various choices and considerations one must make.

[NA 2] FD Scheme. Consider the heat equation, \( u_t - \alpha u_{xx} = 0 \), on the periodic domain \( x \in [0, 2\pi] \), with \( \alpha > 0 \), and \( t \geq 0 \).

(1) Provide an explicit Finite-Difference discretization of this problem using 2 levels in time (e.g. \( n \) and \( n+1 \)) and 5 points in space (e.g. \( i-2, \cdots, i+2 \)).

(2) Derive stability (Von Neumann) for your scheme and provide a stability criterion involving \( \Delta x \), \( \Delta t \), and \( \alpha \).

[NA 3] Semi-Lagrangian Scheme. Consider the linear advection equation, \( u_t + v(x) u_x = 0 \), on the periodic domain \( x \in [0, 2\pi] \), with \( v(x) \) a known \( C^1 \) velocity field and \( t \geq 0 \).

(1) Describe briefly the semi-Lagrangian method to solve this problem on a grid with spacing \( \Delta x \), and time step \( \Delta t \).

(2) Write a specific semi-Lagrangian algorithm of your choice (of your own design) to solve this equation. Note that this requires you to make two choices: one to approximate characteristics and one for the projection step. Provide details of your actual choices.

(3) Analyse the local truncation error of your algorithm and provide an estimate.

[NA 4] Fourier Spectral Method. Consider the heat equation \( u_t = -u_{xxxx} + f(x,t) \), on the periodic domain \( x \in [0, 2\pi] \), and \( t \geq 0 \).

(1) Give a Fourier-Spectral discretization (\( \hat{u}^{n+1} = \cdots \)) for the case \( f = 0 \).

(2) Modify your scheme to include an arbitrary source term \( f(x,t) \). You may use the following notation: \( \mathcal{F}(u^n) \) as the Fourier transform of \( u^n \), and \( \mathcal{F}^{-1}(\hat{u}^n) \) as the inverse Fourier transform of \( \hat{u}^n \).

(3) Find a pair of functions \( u(x,t) \) and \( f(x,t) \) that solve this PDE (this is the so-called “method of manufactured solutions”. Comment on how you would use this pair to evaluate the global truncation error of the scheme in an implementation of this scheme.)
Partial Differential Equations

[**PDE 1**] For $M > 0$ a constant, consider $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, satisfying

$$-\Delta u + u \leq M \quad \text{in } \mathbb{R}^n.$$  

Prove that $u \leq M$. **Hint:** $\max_{x \in \mathbb{R}^n} u(x)$ may not exist, but for $u^\varepsilon(x) := u(x) - \varepsilon |x|^2$, $\max_{x \in \mathbb{R}^n} u^\varepsilon(x)$ always exists.

[**PDE 2**] Consider the heat equation on the entire line,

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}, \end{cases}$$

where $u_0 \in C(\mathbb{R})$, satisfies

$$\lim_{x \to \infty} u_0(x) = 1 \quad \text{and} \quad \lim_{x \to -\infty} u_0(x) = 0.$$  

For every $x \in \mathbb{R}$, compute $\lim_{t \to \infty} u(x, t)$. Justify your argument carefully. **Hint:** You may want to recall that

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = 1.$$

[**PDE 3**] Let $p$ be a nontrivial polynomial of $n$ variables, and let $f$ be a real analytic function defined in an open neighbourhood of 0. Show that there is an open neighbourhood of 0, in which the equation $p(\partial)u = f$ has a solution.

[**PDE 4**] Let $\Omega \subset \mathbb{R}^3$ be an open connected set, and let $\Sigma \subset \Omega$ be a smooth (nontrivial) surface. Let $u \in C^2(\Omega)$ satisfy $\Delta u = 0$ in $\Omega$ and $u = \partial_{\nu} u = 0$ on $\Sigma$, where $\partial_{\nu}$ is a normal derivative operator at $\Sigma$. Show that $u$ vanishes identically in $\Omega$. 
Probability

Instructions:
• You may use any result that is known to you, but you must state the name of the result (law/theorem/lemma/formula/inequality) that you are using, and show the work of verifying the condition(s) for that result to apply.
• For the problems with two parts (i) and (ii), you may assume the conclusion in (i) in solving (ii).

[PROB 1] Given a probability space \((\Omega, \mathcal{F}, P)\), let \(\{X_n : n \geq 1\}\) be a sequence of independent and identically distributed exponential random variables with rate 1, i.e., \(X_n\)'s have the common probability density function 
\[ f(x) = e^{-x}I_{(0,\infty)}(x) \] for \(x \in \mathbb{R}\). Prove that, if 
\[ M := \limsup_{n \to \infty} \frac{X_n - \ln n}{\ln \ln n} \] then \(M = 1\) a.s. 
(Hint: Consider \(P(X_n - \ln n > \alpha \cdot \ln (\ln n))\) for \(\alpha > 1\) and \(\alpha \leq 1\) separately.)

[PROB 2] Let \(\{X_n : n \geq 1\}\) be a sequence of \(\mathbb{R}\)-valued random variables on a probability space \((\Omega, \mathcal{F}, P)\). Assume that \(\{X_n : n \geq 1\}\) is a Cauchy sequence in probability, i.e., \(\forall \epsilon > 0,\) 
\[ \lim_{n \to \infty} \sup_{m \geq n} P(|X_m - X_n| > \epsilon) = 0. \] Prove that there exists a \(\mathbb{R}\)-valued random variable \(X\) on \((\Omega, \mathcal{F}, P)\) such that \(X_n \to X\) as \(n \to \infty\) in probability.

[PROB 3] Let \(X\) and \(Y\) be two independent and identically distributed \(\mathbb{R}\)-valued random variables on a probability space \((\Omega, \mathcal{F}, P)\). Further assume that \(E[X] = 0\) and \(E[X^2] = 1\). Prove that, if \(X + Y\) and \(X - Y\) are independent, then \(X\) is a standard Gaussian random variable, i.e., the distribution of \(X\) is \(N(0, 1)\).

[PROB 4] Given a probability space \((\Omega, \mathcal{F}, P)\), for every \(n \geq 1\), let \(\{B_{n,j} : j = 1, 2, \ldots, 2^n\}\) be a family of independent and identically distributed Gaussian random variables with the common distribution \(N(0, 2^{-n})\).
(i) (5 points) Prove that 
\[ \lim_{n \to \infty} \max_{1 \leq m \leq 2^n} \left| \sum_{j=1}^{m} B_{n,j}^2 - m2^{-n} \right| = 0 \text{ a.s.} \]

(ii) (5 points) Prove that 
\[ \limsup_{n \to \infty} \sum_{j=1}^{2^n} |B_{n,j}| = \infty \text{ a.s.} \]
Continuous Optimization

[COPT 1] (Constrained minimization) Consider a $C^1$-smooth function $f : \mathbb{R}^n \to \mathbb{R}$, a point $x_0 \in \mathbb{R}^n$, and a linear subspace $L \subset \mathbb{R}^n$.

1. (6 points) Show that any point $x^*$ solving the optimization problem

$$\min_{x \in x_0 + L} f(x)$$

must satisfy the inclusion

$$-\nabla f(x^*) \in L^\perp = \{ y \in \mathbb{R}^n : y^Tv = 0 \quad \text{for all} \quad v \in L \}.$$  

2. (4 points) Suppose now that $f$ is, in addition, convex. Show that the conditions

$$\begin{cases} 
-\nabla f(x^*) \in L^\perp \\
x^* - x_0 \in L
\end{cases}$$

are necessary and sufficient for $x^*$ to be a global solution of (1).

[COPT 2] (Unconstrained minimization) Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \ f(x, y) = 2y^2 - x(x-1)^2.$$  

Find all critical points and determine which of these are local and global minimizers, respectively.

[COPT 3] (Constraint qualifications and KKT) Consider the nonlinear program

$$\min_{x \in \Omega} -x_2 \quad \text{s.t.} \quad \Omega = \{ x \in \mathbb{R}^2 : x_2 \geq 0, x_2 \leq x_1^2 \}.$$  

1. (3 points) For $x^* = (0,0)^T$, write down $T_\Omega(x^*)$, the tangent cone at $x^*$ and $\mathcal{F}(x^*)$, the cone of linearized feasible directions at $x^*$.

2. (3 points) Is LICQ (linear independence constraint qualification) satisfied at $x^*$? Is MFCQ (Mangasarian-Fromovitz constraint qualification) satisfied?

3. (4 points) Verify that the KKT conditions are satisfied at $x^*$.

[COPT 4] (Duality) Consider the problem

$$\mathcal{P} \quad \min_{x \in \mathbb{R}^n} x^TQ_0x$$

subject to $x^TQ_jx = b_j$ for $j = 1, \ldots, m$,

where $Q_j$ (for $j = 0, 1, \ldots, m$) are $n \times n$ symmetric matrices and $b \in \mathbb{R}^m$ is a vector ($b_j$ is the $j$th coordinate of $b$). Prove that the Lagrangian dual program of $\mathcal{P}$ is the semi-definite program

$$\mathcal{D} \quad \max_{y \in \mathbb{R}^m} y^Tb$$

subject to $Q_0 - \sum_{j=1}^m y_j Q_j \succeq 0$.

Here $y_j$ is the $j$th coordinate of $y$. 