McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS
Paper ALPHA

7 May, 2019
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) There are 12 questions. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.
Linear Algebra

Solve any three out of the four questions 1, 2, 3, 4.

**Question 1.**
Let $A$ be a real $n \times n$ matrix acting on $\mathbb{R}^n$ in the usual way. We think about $\mathbb{R}^n$ as column vectors and equip $\mathbb{R}^n$ with the usual Euclidean norm $||v||$. The norm of $A$ is defined as

$$
\sup_{v \neq 0} \frac{||Av||}{||v||}.
$$

Prove that the norm of $A$ is the square root of the maximal eigenvalue of the matrix $A^t \cdot A$ where $A^t$ is the transpose of $A$.

**Question 2.**
Let $V_1, V_2$ be vector subspaces of $\mathbb{R}^n$. Show that

$$
\dim(V_1 \cap V_2) \geq \dim V_1 + \dim V_2 - n.
$$

**Question 3.**
Let $A$ be $n \times n$ complex matrix satisfying $A^k = \text{Id}_n$ for some positive integer $k$ (where $\text{Id}_n$ is the $n \times n$ identity matrix). Show that the trace of $A$ satisfies

$$
|\text{tr}(A)| \leq n.
$$

Here $|x + iy| = \sqrt{x^2 + y^2}$ is the absolute value for complex numbers.

**Question 4.**
Let $V$ be a finite-dimensional vector space over $\mathbb{R}$, equipped with an inner product $\langle - , - \rangle$. Let $T$ be a self-adjoint operator on $V$, i.e. for any vectors $u, v \in V$ we have $\langle Tu, v \rangle = \langle u, Tv \rangle$.

Prove or disprove the following statements:

(a) For any basis $B = (e_1, \ldots, e_n)$ of $V$, the matrix $T_B$ of $T$ in the basis $V$ is symmetric;

(b) If $v_1, v_2$ be two eigenvectors of $T$ corresponding to different eigenvalues $\lambda_1 \neq \lambda_2$, then $v_1$ and $v_2$ are orthogonal.
Single variable real analysis

Solve any three out of the four questions 5, 6, 7, 8.

Question 5.
Let \( f_n(x) = nx/(1 + n^2 x^2) \). Determine the pointwise limit of \( f_n \) for \( x \in [0, \infty) \). Does that sequence of functions converge uniformly?

Question 6.
(a) State Hölder’s inequality for sequences.
(b) Let \( a_n \geq 0 \). Suppose \( \sum_n a_n \) converges. Show that
\[
\sum_n \frac{a_n^{1/3}}{n}
\]
also converges.

Question 7.
Consider the power series
\[
f(x) = x - \frac{3}{4}x^4 + \frac{3^2}{4 \cdot 7}x^7 - \frac{3^3}{4 \cdot 7 \cdot 10}x^{10} \pm \ldots
\]
Find the radius of convergence for \( f(x) \).

Question 8.
Suppose that \( a_n \) is a bounded sequence of real numbers and \( b \) is a real number such that any subsequence of \( a_n \) which converges at all has limit \( b \). Prove that \( a_n \to b \) as \( n \to \infty \).
Solve any three out of the four questions 9, 10, 11, 12.

**Question 9.**
Find a general solution $y(t)$ of the differential equation
$$y'' - 2y' - 3y = 3t^2 - 5.$$  

**Question 10.**
Find a general solution $y(x)$ of the differential equation
$$y''' - 9y'' + 27y' - 27y = 0.$$  

**Question 11.**
Find the volume of the region lying inside the ellipsoid $x^2 + y^2/4 + z^2/9 = 9$ and outside the ellipsoid $x^2 + y^2/4 + z^2/9 = 3$.

**Question 12.**
Compute the double integral
$$\int_0^{\pi/2} dy \int_y^{\pi/2} \frac{\sin x}{x} \, dx.$$