PURE AND APPLIED MATHEMATICS

Paper BETA

August 28, 2020
10:00 a.m. - 6:00 p.m.

SPECIAL ONLINE INSTRUCTIONS:

(i) Your solutions and this signed cover page must be scanned and returned by email to Jason and Rustum Choksi by 6:10 PM on August 7, 2020.

(ii) This is a closed book exam - NO AIDES OR CONSULTATIONS of any sort may be used. PLEASE SIGN BELOW that you have complied with these rules.

Name:

Signature:

GENERAL INSTRUCTIONS:

(i) This paper consists of the following 6 modules:

- [ALG] Algebra
- [AN] Analysis
- [GT] Geometry & Topology
- [NA] Numerical Analysis
- [PDE] Partial Differential Equations
- [PROB] Probability;

each of which comprises 4 questions. You should answer 7 questions from 3 modules, with at least 2 from each module. All questions are worth 10 points. PLEASE WRITE THE NUMBERS FOR THE QUESTIONS YOU ATTEMPTED NEXT TO THE MODULE TITLES ABOVE.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 8 pages of questions.
Algebra Module

[ALG 1]
Let \( H \) be a non-commutative division algebra over \( \mathbb{R} \) which is finite-dimensional over \( \mathbb{R} \).
(a) Show that the center of \( H \) is isomorphic either to \( \mathbb{R} \) or to \( \mathbb{C} \).
(b) Assume that the center of \( H \) is equal to \( \mathbb{R} \). Show that any element of \( H - \mathbb{R} \) satisfies a degree two polynomial with coefficients in \( \mathbb{R} \).
(c) Use this to conclude that \( H \) is necessarily four-dimensional over \( \mathbb{R} \).

[ALG 2]
Let \( G \) be the group of upper-triangular unipotent \( 3 \times 3 \) matrices with entries in the field with \( p \) elements.
(a) Show that \( G \) has a center \( Z \) of order \( p \) and that \( G/Z \) is isomorphic to \( (\mathbb{Z}/p\mathbb{Z})^2 \).
(b) Show that \( G \) has \( p^2 \) irreducible complex representations of dimension 1, and \( p - 1 \) irreducible representations of dimension \( p \). Conclude that an irreducible representation of \( G \) of dimension \( p \) is completely determined (up to isomorphism) by its restriction to \( Z \).

[ALG 3]
Let \( p \) be a prime number. Let \( \mathbb{F}_p \) denote the finite field with \( p \) elements. Let \( f(X) \) denote the polynomial \( X^{p^4} - X \). Prove that all irreducible factors of \( f \) are distinct. How many irreducible factors of each degree does \( f \) have?

[ALG 4]
Consider the below commutative diagram

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
\cdots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \cdots \\
\downarrow & \downarrow & \downarrow \\
\cdots & \longrightarrow & B_{n+1} & \longrightarrow & B_n & \longrightarrow & B_{n-1} & \longrightarrow & \cdots \\
\downarrow & \downarrow & \downarrow \\
\cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0,
\end{array}
\]

where all columns are exact. Prove that if two of the three rows are exact, then the third row is exact too.
Analysis Module

[AN 1] Suppose \( f : \mathbb{R} \to \mathbb{R} \) is monotone increasing. Is the function \( f \) \((\mathcal{B}_\mathbb{R}, \mathcal{B}_\mathbb{R})\)-measurable, where \( \mathcal{B}_\mathbb{R} \) denotes the Borel \( \sigma \)-algebra on \( \mathbb{R} \)? Please carefully justify your answer.

[AN 2] Let \( f \in L^1(\mathbb{R}) \) with respect to Lebesgue measure. Compute

\[
\lim_{n \to \infty} \int_{[0,\infty)} e^{-nx^3} f(x) \, dx
\]

and carefully justify your answer.

[AN 3] Consider the Hardy-Littlewood maximal function

\[
Mf(x) := \sup_{r>0} \frac{1}{B(x,r)} \int_{B(x,r)} |f(y)| \, dy,
\]

where \( B(x,r) = \{ y \in \mathbb{R}^n ; |y-x| < r \} \). Is the operator \( M \) bounded on \( L^1(\mathbb{R}^n, dy) \)? Please carefully justify your answer.

[AN 4] Let \( X \) and \( Y \) be Banach spaces. If \( T : X \to Y \) is a linear map such that \( f \circ T \in X^* \) for every \( f \in Y^* \), prove that \( T \) is bounded. Please carefully justify your argument.
Geometry and Topology Module

[GT 1]
Let $T$ be a torus, and let $K$ be a klein bottle.

(1) Find presentations for $\pi_1 T$ and $\pi_1 K$.

(2) Show that $K$ and $T$ are not homeomorphic by finding a finite group that is a quotient of $\pi_1 T$ but not a quotient of $\pi_1 K$.

(3) Show that there are three isomorphism types of degree 2 connected covering maps $\hat{K} \to K$.

(4) Show that one of these has domain homeomorphic to $T$ but the other two have domain homeomorphic to $K$.

[GT 2]
Let $L_1$ and $L_2$ be disjoint parallel lines in $\mathbb{R}^3$. Let $M = \mathbb{R}^3 - (L_1 \cup L_2)$. Let $q \in M$.
Let $C_1$ and $C_2$ be disjoint parallel lines in $\mathbb{R}^3$. Let $N = \mathbb{R}^3 - (C_1 \cup C_2)$.

(1) Show that $\pi_1 M \cong \pi_1 N$.

(2) Show that $M$ and $N$ do not have the same homotopy type.

(3) Show that $M - \{q\}$ has the same homotopy type as $N$.

[GT 3]
Let $S^2$ denote the unit sphere entered at the origin in $\mathbb{R}^3$ and consider the map $F : S^2 \to \mathbb{R}^4$ defined by $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Show that $f$ passes to the quotient of $S^2$ under the antipodal map, and that it induces an embedding of the real projective plane $\mathbb{P}_2(\mathbb{R})$ into $\mathbb{R}^4$.

[GT 4]
Suppose $(M, \omega)$ is a $2n$-dimensional compact symplectic manifold.

a) Show that $\omega^n$ (the $n$-fold wedge product of $\omega$ with itself) is not exact.

b) Show that $H^p_{dR}(M) \neq 0$ for all $0 \leq p \leq n$.

c) Show that $S^2$ is the only sphere that admits a symplectic structure.
Numerical Analysis Module

[NA 1] Transport equation in 1D
Consider the linear transport equation, \( u_t = 2u_x \), with initial condition \( u(x,0) = u_0(x) = 1 + \cos 2x \). Consider the periodic domain \([0, 2\pi)\), and final time \( T = 1 \).

1. What is the exact solution \( u(x,T) \)?
2. Derive the Lax-Wendrof scheme.
   Recall that this scheme is forward in time/centered in space and is derived by cancelling the second order terms in the Taylor expansion.
3. Derive the stability criterion for the scheme.
4. Derive the modified equation for this scheme and show the scheme is dispersive.

[NA 2] Fourier-Spectral Solution

1D Case: Consider the following equation,
\[
(\beta(x)u_x)_x = g(x)
\]
where, \( u : [0, 2\pi) \to \mathbb{R}, x \in [0, 2\pi), \beta : [0, 2\pi) \to \mathbb{R}, g : [0, 2\pi) \to \mathbb{R} \), and \( u \) is periodic.

1. Assume \( \beta(x) = 2 + \sin(x) \), Provide two non-trivial functions \( g(x) \), and \( u(x) \) which solve the problem above.
2. Write the solution \( u(x) \) making use of the operator \( \mathcal{F} \), the Fourier transform operator (and \( \mathcal{F}^{-1} \) the inverse Fourier operator) - e.g. \( \hat{u} = \mathcal{F}u \).
3. What convergence order (as the number of point \( N \) increases) do you expect?
4. If \( \beta(x) \in C^k(S^1) \) and \( \beta(x) \notin C^{k+1}(S^1) \), what convergence order do you expect? Why?

2D Case: Consider the following equation on the flat 2-torus \( T^2 \),
\[
\nabla \cdot (\beta \nabla u) = f,
\]
where, \( u : [0, 2\pi]^2 \to \mathbb{R}, f : [0, 2\pi]^2 \to \mathbb{R}, \beta : [0, 2\pi]^2 \to \mathbb{R}, \beta > 0 \), assume \( f(x) \), sufficiently smooth and \( \beta \) is not constant.

1. Setup the fixed point iteration scheme for this problem.
2. Write a short pseudo-code to solve this problem (e.g. making use of the operator \( \mathcal{F} \), the Fourier transform operator and \( \mathcal{F}^{-1} \) the inverse Fourier operator as done above).

[NA 3] Consider the boundary value problem \( u'' + f = 0 \) on the interval \([0, 1]\) with the boundary conditions \( u(0) = u(1) = 0 \), where \( f \) is, say, a continuous function.
1. Describe the piecewise quadratic finite element discretization of this problem.
2. Calculate the stiffness matrix corresponding to the uniform mesh with spacing \( h \).
3. Estimate the condition number of the stiffness matrix.

[NA 4] Let \( V \) and \( Q \) be reflexive Banach spaces, and \( a : V \times V \to \mathbb{R} \) and \( b : V \times Q \to \mathbb{R} \) be bounded bilinear forms, satisfying
\[
a(u,u) \geq \alpha \|u\|^2 \quad \text{for all } u \in V,
\]
and
\[
\sup_{v \in V} \frac{b(v,p)}{\|v\|} \geq \beta \|p\| \quad \text{for all } p \in Q,
\]
where $\alpha > 0$ and $\beta > 0$ are constants. Let $f \in V'$ and $g \in Q'$, and suppose that $(u, p) \in V \times Q$ satisfy
\[
\begin{align*}
  a(u, v) + b(v, p) &= \langle f, v \rangle \quad \forall v \in V, \\
  b(u, q) &= \langle g, q \rangle \quad \forall q \in Q.
\end{align*}
\]

Now let $\hat{V} \subset V$ and $\hat{Q} \subset Q$ be closed subspaces, satisfying the discrete inf-sup condition
\[
\sup_{v \in \hat{V}} \frac{b(v, p)}{\|v\|} \geq \hat{\beta} \|p\| \quad \text{for all } p \in \hat{Q}.
\]

(a) Show that the Galerkin problem
\[
\begin{align*}
  a(\hat{u}, v) + b(v, \hat{p}) &= \langle f, v \rangle \quad \forall v \in \hat{V}, \\
  b(\hat{u}, q) &= \langle g, q \rangle \quad \forall q \in \hat{Q},
\end{align*}
\]
has a unique solution $(\hat{u}, \hat{p}) \in \hat{V} \times \hat{Q}$.

(b) Establish the Céa-type estimate
\[
\|u - \hat{u}\| + \|p - \hat{p}\| \leq C \inf_{(v, q) \in \hat{V} \times \hat{Q}} \left(\|u - v\| + \|p - q\|\right).
\]

How does the constant $C$ depend on the constants $\alpha$, $\beta$, $\hat{\beta}$, and the norms of $a$ and $b$?
PDE Module

[PDE 1]

a) Use the method of characteristics to solve for \( u(x_1, x_2) \) where
\[ u_{x_1} u_{x_2} = u \quad \text{on} \quad \{(x_1, x_2) \mid x_1 > 0\}, \quad \text{with} \quad u(0, x_2) = x_2^2. \]

b) Find the entropy solution to the inviscid Burgers’ equation
\[
\begin{cases}
  u_t + u u_x = 0 \\
  u(x, 0) = g(x),
\end{cases}
\]
where
\[
g(x) = \begin{cases} 
  0 & \text{if } x \leq -1 \\
  1 + x & \text{if } -1 < x \leq 0 \\
  1 - x & \text{if } 0 < x \leq 1 \\
  0 & \text{if } x > 1.
\end{cases}
\]
That is, find a piecewise smooth distributional solution whose shocks satisfy the entropy condition.

[PDE 2]

Let \( c > 0 \) and \( f \in C_c^2(\mathbb{R}^3) \). Derive an explicit formula for the solution of
\[ \Delta u + cu = f, \]
in \( \mathbb{R}^3 \). First find the Fundamental solution \( \Phi(x) \) where
\[ \Delta \Phi + c\Phi = \delta_0 \]
in the sense of distributions.

Hint: look for radially symmetric solutions of the \( \Delta u + cu = 0 \).

[PDE 3]

Construct a fundamental matrix for the Stokes operator
\[ S : \begin{bmatrix} u \\ p \end{bmatrix} \mapsto \begin{bmatrix} -\Delta u + \nabla p \\ \nabla \cdot u \end{bmatrix}, \]
where \( u \) is a vector field, and \( p \) is a scalar field in \( \mathbb{R}^n \). Recall that a fundamental matrix for a linear differential operator \( A : D'(\mathbb{R}^n, \mathbb{R}^m) \to D'(\mathbb{R}^n, \mathbb{R}^m) \) is a matrix-valued distribution (or equivalently, a matrix of distributions) \( E \in D'(\mathbb{R}^n, \mathbb{R}^{m \times m}) \) satisfying \( AE = \delta I \), where \( I \) is the \( m \times m \) identity matrix, and \( \delta \in D'(\mathbb{R}^n) \) is the Dirac distribution.

[PDE 4]

Show that if \( p \) is an \( n \)-variable polynomial whose set of real zeroes is bounded, then every tempered distribution solution of \( p(D)u = 0 \) is an entire function satisfying the growth estimate
\[ |u(x)| \leq C(1 + |x|^N) e^{\alpha |x|}, \quad x \in \mathbb{R}^n, \]
for some constants \( C, N \) and \( \alpha \).
Probability Module

[PROB 1]
In this question, we let \((X_n,n \geq 1)\) be independent, non-negative random variables defined on a common space \((\Omega,F,P)\). For \(n \geq 1\) write \(T_n = \sigma(X_m,m \geq n) \subset F\) for the \(\sigma\)-algebra generated by \((X_m,m \geq n)\).

(i) (4 points) Let \(Y\) be another real random variable, and suppose that \(Y\) is \(T_n/B(R)-\)measurable for all \(n \geq 1\). Prove that \(Y\) is almost surely constant: there is \(c\) such that \(P(Y = c) = 1\).

(ii) (6 points) Define the random Taylor series
\[
F_z(\omega) := \sum_{n \geq 1} X_n(\omega) z^n,
\]
and let \(R = R(\omega)\) be its radius of convergence. Prove that \(R\) is almost surely constant.

[PROB 2]
In this question \((X_i,i \geq 1)\) is an arbitrary sequence of real random variables.

(i) (1 point) What does it mean for \(X_i\) to converge in distribution to a random variable \(X\) as \(i \to \infty\)?

(ii) (2 points) Show that there exist positive constants \(a_1,a_2,...\) such that \(a_nX_n\) converges in distribution to 0 as \(n \to \infty\).

(iii) (4 points) Let \(X_1,X_2,...\) be identically distributed random variables. Suppose that \(E[|X_1|^3]<\infty\). Show that for all \(\epsilon > 0\), \(nP[|X_1| > \epsilon n^{1/3}] \to 0\).

(iv) (3 points) Let \(X_1,X_2,...\) be identically distributed random variables with finite second moment. Show that \(n^{-1/2}\max_{1 \leq k \leq n} X_k\) converges in probability to 0.

[PROB 3]

i) (2 points) State the non-negative supermartingale convergence theorem.

ii) (4 points) Let \((\Omega,F,(F_n)_{n \geq 0},P)\) be a filtered probability space, and let \((X_n,n \geq 1)\) be an \(F_n\) martingale. Suppose that almost surely \(|X_{n+1} - X_n| \leq 1\) for all \(n \geq 1\). Writing
\[
\{X_n \text{ converges}\} := \{\omega \in \Omega : X_n(\omega) \text{ converges to a finite limit}\},
\]
and
\[
\{X_n \text{ oscillates}\} := \{\omega \in \Omega : \limsup_n X_n(\omega) = +\infty, \liminf_n X_n(\omega) = -\infty\},
\]
prove that
\[
P(\{X_n \text{ converges}\} \cup \{X_n \text{ oscillates}\}) = 1.
\]

Hint. Consider the modified martingale formed by stopping the first time \(X_n\) exceeds a given value.

iii) (4 points) Give an example of a martingale \((X_n)_{n \geq 1}\) such that
\[
P(\sup_n |X_n| < \infty) = 1
\]
and such that
\[
P(X_n \text{ converges}) = 0.
\]

[PROB 4]
In this question we write \(P_\lambda\) for a Poisson(\(\lambda\))-distributed random variable; in other words \(P(P_\lambda = k) = \lambda^k e^{-\lambda}/k!\) for integer \(k \geq 0\).

(i) (4 points) Derive the moment generating function of \(P_\lambda\). Prove that if \(P_\lambda\) and \(P_{\lambda'}\) are independent, then \(P_\lambda + P_{\lambda'}\) is Poisson(\(\lambda + \lambda'\))-distributed.

(ii) (6 points) Prove that \((P_\lambda - \lambda)/\lambda^{3/2}\) converges in distribution to the Normal(0,1) distribution as \(\lambda \to \infty\).