McGill University

Faculty of Science

Department of Mathematics and Statistics

Part A Examination

Statistics: Methodology Paper

Date: 10th May 2019 Time: 1pm-5pm

Instructions

- Answer only **two** questions from Section L. If you answer more than two questions, then only the **FIRST TWO questions will be marked.**
- Answer only **two** questions from Section G. If you answer more than two questions, then only the **FIRST TWO questions will be marked.**

Questions	Marks
L1	
L2	
L3	
G1	
G2	
G3	

This exam comprises the cover page and sixteen pages of questions.

Notation: In Section L, the following notation will be used: for $i = 1, \ldots, n$, y_i is the observed response; Y_i is the random variable version of the response; $\mathbf{y}$ and $\mathbf{Y}$ are the $n \times 1$ vector versions of the responses; $\mathbf{x}_i$ is the row vector of predictor values, $\mathbf{X}$ is the matrix of predictor values; $\widehat{y}_i$, $\widehat{Y}_i$, $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{Y}}$ are the fitted or predicted response values or vectors arising from a given model; β is the vector of regression coefficients; $\widehat{\beta}$ is the vector of estimates or estimators. Furthermore, $\mathbf{0}_n$ is the n-dimensional vector of zeros, and $\mathbf{I}_n$ is the n-dimensional identity matrix.

L1. a. Write down the conditional mean for the simple linear regression model for data $y_1, \ldots, y_n$ to be modelled as a function of continuous predictor data $x_{11}, \ldots, x_{n1}$. 4 MARKS

b. Derive the form of the regression parameter estimates, denoted in vector form $\widehat{\beta}$, for the simple linear regression model estimated using the least squares criterion. You may use matrix and vector notation. 4 MARKS

c. The residual sum of squares corresponding to the least squares fit, SS_{Res} is defined by

$$\text{SS}_{\text{Res}} = \sum_{i=1}^{n} (y_i - \widehat{y}_i)^2.$$

Prove that

$$\text{SS}_{\text{Res}} = \mathbf{y}^\top (\mathbf{I}_n - \mathbf{H})\mathbf{y}$$

for some matrix $\mathbf{H}$; give the form of $\mathbf{H}$. 4 MARKS

d. Consider the following `R` output resulting from the analysis of a data set using simple linear regression (some output has been deleted and replaced by `XXXXX`). The data are stored in vectors `x1` and `y`.

```
> print(anova(fitQ1),digits=7)
Analysis of Variance Table

Response: y
          Df   Sum Sq  Mean Sq  F value     Pr(>F)
x1         1 4109.924 4109.924  XXXXXX 4.5121e-06 ***
Residuals 22  XXXXXXX  112.917
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
> (SST<-sum((y-mean(y))^2))
[1] 6594.099
> coef(summary(fitQ1))
             Estimate Std. Error  t value     Pr(>|t|)
(Intercept) 14.191827  2.2059296 6.433490 1.793164e-06
x1           3.693717  0.6122468 6.033053 4.512117e-06

> qf(0.95,1,22) #0.95 quantile of the Fisher(1,22) distribution
[1] 4.30095
```

Question continues on the next page.

From the output identify the following quantities. If the quantity cannot be identified, state that as your answer.

(a) the value of the F statistic omitted from line 6; 2 MARKS

(b) the value of R^2 for this model. 2 MARKS

State clearly which lines of the output you use when giving answers: marks will be deducted if this is not done.

e. The following code was utilized to carry up a follow up analysis:

```
19 > range(x1)
20 [1] -9.270657  7.921899
21 > x2<-as.numeric(x1 < 0)
22 > table(x2)
23 x2
24  0  1
25  9 15
26 > summary(lm(y~x1*x2))
27
28 Call:
29 lm(formula = y ~ x1 * x2)
30
31 Coefficients:
32             Estimate Std. Error t value Pr(>|t|)
33 (Intercept)  24.3402     2.4220  10.050 2.91e-09 ***
34 x1           -0.6396     0.6708  -0.953    0.352
35 x2            0.1825     3.0465   0.060    0.953
36 x1:x2         7.8109     0.8448   9.246 1.16e-08 ***
37 ---
38 Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
39
40 Residual standard error: 4.767 on 20 degrees of freedom
41 Multiple R-squared:  0.9311,    Adjusted R-squared:  0.9207
42 F-statistic: 90.07 on 3 and 20 DF,  p-value: 8.659e-12
```

Explain the model used, and what the results indicate. Sketch the resulting fitted model. 4 MARKS

L2. The following data set represents the results of a study into the economic impact of a change in regulatory policy on manufacturing firms of different sizes in four geographical regions. The study data arise from 270 firms, with data collected over two successive years. At the end of the first year, regulatory requirements were eased in order to stimulate economic growth, as measured by annual turnover of the firms studied.

The variables recorded in the data set are as follows:

- x_1: size: a continuous predictor recording the total number of employees averaged over the two years of the study;
- x_2: region, factor predictor with four levels – denoted `A`, `B`, `C` and `D` – for four regions, respectively;
- x_3: turnover, a continuous predictor recording the total annual turnover (in thousands of dollars) in the first year of the study;
- Y: response, change in turnover (in thousands of dollars) from year 1 to year 2.

The following sequence of models was fitted to these data in `R`.

```
> fit.1<-lm(y ~ x1)
> fit.2<-lm(y ~ x2)
> fit.3<-lm(y ~ x3)
> fit.4<-lm(y ~ x1+x2)
> fit.5<-lm(y ~ x1+x3)
> fit.6<-lm(y ~ x2+x3)
> fit.7<-lm(y ~ x1+x2+x3)
> fit.8<-lm(y ~ x1+x2+x3+x1:x2+x2:x3)
> fit.9<-lm(y ~ x1+x2+x3+x1:x2+x2:x3+x1:x3)
> fit.full<-lm(y ~ x1*x2*x3)
```

Further information was recovered from the fits:

```
anova(fit.full,fit.9,fit.8,fit.7,fit.6,fit.5,fit.4,fit.3,fit.2,fit.1)
Analysis of Variance Table

Model  1: y ~ x1 * x2 * x3
Model  2: y ~ x1 + x2 + x3 + x1:x2 + x2:x3 + x1:x3
Model  3: y ~ x1 + x2 + x3 + x1:x2 + x2:x3
Model  4: y ~ x1 + x2 + x3
Model  5: y ~ x2 + x3
Model  6: y ~ x1 + x3
Model  7: y ~ x1 + x2
Model  8: y ~ x3
Model  9: y ~ x2
Model 10: y ~ x1
```

Question continues on the next page.

```
Res.Df       RSS Df   Sum of Sq       F   Pr(>F)
1      254 137053.13
2      257 140203.08 -3   -3149.948 1.94593 0.122657
3      258 141034.06 -1    -830.978 1.54005 0.215756
4      264 147095.86 -6   -6061.803 1.87239 0.085966 .
5      265 147134.32 -1     -38.457 0.07127 0.789709
6      267 147792.91 -2    -658.596 0.61029 0.543990
7      265 148474.19  2    -681.282
8      268 147848.09 -3     626.106
9      266 375949.08  2 -228100.994
10     268 149112.71 -2  226836.369
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Model selection quantities were also compute d:

```
Model Res. df      RSS      AIC      BIC    Rsq Adj.Rsq
 fit.full     254 137053.1 2482.246 2543.420 0.6394  0.6181
    fit.9     257 140203.1 2482.382 2532.760 0.6311  0.6139
    fit.8     258 141034.1 2481.977 2528.757 0.6289  0.6131
    fit.7     264 147095.9 2481.340 2506.529 0.6130  0.6057
    fit.6     265 147134.3 2479.410 2501.001 0.6129  0.6070
    fit.5     267 147792.9 2476.616 2491.010 0.6112  0.6082
    fit.4     265 148474.2 2481.858 2503.448 0.6094  0.6035
    fit.3     268 147848.1 2474.717 2485.512 0.6110  0.6096
    fit.2     266 375949.1 2730.699 2748.691 0.0109 -0.0003
    fit.1     268 149112.7 2477.016 2487.812 0.6077  0.6062
```

In the output, `Res. df` is the residual degrees of freedom in the model, and the table also includes the residual sum of squares (`RSS`, which is also termed the sum of squares of the residuals, SS_{Res}) AIC, BIC, R^2 and R^2_{Adj} quantities.

a. Compute the F-statistic to compare the two models

$$X_1 + X_2 \qquad \text{and} \qquad X_1 + X_2 + X_3.$$

4 MARKS

b. Compute the F-statistics to compare the two models

$$X_1 \qquad \text{and} \qquad X_1 + X_2.$$

first assuming that X_3 is potentially an important predictor, and then assuming that it is **known** that X_3 is **not** an influential predictor. 4 MARKS

c. On the basis of the analyses above, identify the most appropriate model to represent the variation in response, and write down precisely (in terms of β parameters) the form of the conditional expectation, $\mathbb{E}_{Y_i|X_i}[Y_i|\mathbf{x}_i]$, for the selected model. 2 MARKS

Question continues on the next page.

d. One of the models was investigated further:

```
> summary(fit.1)

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 11.670602   1.538799   7.584 5.51e-13 ***
x1           0.089157   0.004376  20.374  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 23.59 on 268 degrees of freedom
Multiple R-squared:  0.6077,    Adjusted R-squared:  0.6062
F-statistic: 415.1 on 1 and 268 DF,  p-value: < 2.2e-16
```

In light of the earlier results, comment on the importance of predictor `x1` in explaining the variation in response.

4 MARKS

e. Penalization may be used to reduce the complexity of a multiple regression model, and a common form of penalization is given by the *ridge* penalty; in the model with parameters $\beta = (\beta_0, \beta_1, \ldots, \beta_k)^\top$, we typically choose β to minimize

$$S_\lambda(\beta) = \sum_{i=1}^{n} (y_i - \mathbf{x}_i\beta)^2 + \lambda \sum_{j=1}^{k} \beta_j^2$$

(that is, the intercept β_0 is **not** penalized), where $\lambda > 0$ is a parameter to be specified. In this formulation, the predictors are all continuous (that is, are not factor predictors), and are recorded on a standard scale.

(a) Derive, for fixed λ, the form of the estimate $\widehat{\beta}_\lambda$ that mimizes $S_\lambda(\beta)$.

2 MARKS

(b) In a standard least squares analysis (that is, with $\lambda = 0$ in the ridge penalty), the complexity of the fitted model is equal to the number of parameters $p = \text{Trace}(\mathbf{H})$ where $\mathbf{H}$ is the hat matrix.

Derive the hat matrix, $\mathbf{H}_\lambda$, for the ridge penalized least squares solution, and show that

$$\text{Trace}(\mathbf{H}) \geq \text{Trace}(\mathbf{H}_\lambda)$$

Recall that the singular value decomposition of a rectangular $(n \times p)$ matrix $\mathbf{X}$ is

$$\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$$

where $\mathbf{U}$ is an orthonormal $n \times p$ matrix ($\mathbf{U}^\top\mathbf{U} = \mathbf{I}_p$), $\mathbf{V}$ is an orthonormal $p \times p$ matrix ($\mathbf{V}^\top\mathbf{V} = \mathbf{I}_p$) and $\mathbf{D}$ is a diagonal matrix with non-negative entries.

4 MARKS

L3. Consider a multiple regression model for response data $\mathbf{y}$ specified in terms of a p-dimensional parameter vector β that includes the intercept, and the $(n \times p)$ matrix $\mathbf{X}$. Suppose that fit is constructed using least squares, and that the model is correctly specified.

a. Show that the vector of fitted value random variables, $\widehat{\mathbf{Y}}$, has expectation $\mathbf{X}\beta$, and find the variance-covariance matrix of $\widehat{\mathbf{Y}}$, $\mathrm{Var}_{\mathbf{Y}|\mathbf{X}}[\widehat{\mathbf{Y}}|\mathbf{X}]$. 5 MARKS

b. Prove that

$$\mathbb{E}_{\mathbf{Y}|\mathbf{X}}[\mathbf{X}^\top(\mathbf{Y} - \widehat{\mathbf{Y}})|\mathbf{X}] = \mathbf{0}_p.$$

Does this result still hold if the model is **incorrectly** specified, that is, is the true data generating model is actually given by

$$\mathbb{E}_{\mathbf{Y}|\mathbf{X}}[\mathbf{Y}|\mathbf{X}] = \mathbf{X}_{\text{TRUE}}\beta_{\text{TRUE}}$$

– justify your answer. 5 MARKS

c. Suppose that a vector of predictions, $\widehat{\mathbf{Y}}^{\text{new}}$ at a new set of predictor values contained in the $m \times p$ matrix $\mathbf{X}^{\text{new}}$ is to be computed using the fit to the original data.

Write down the expectation and variance-covariance matrix of $\widehat{\mathbf{Y}}^{\text{new}}$ conditional on $\mathbf{X}$ and $\mathbf{X}^{\text{new}}$. 4 MARKS

d. Derive the mathematical forms of the 95% *confidence* and *prediction* intervals for this model at new predictor value x_1^{new}, and explain the differences in their construction.

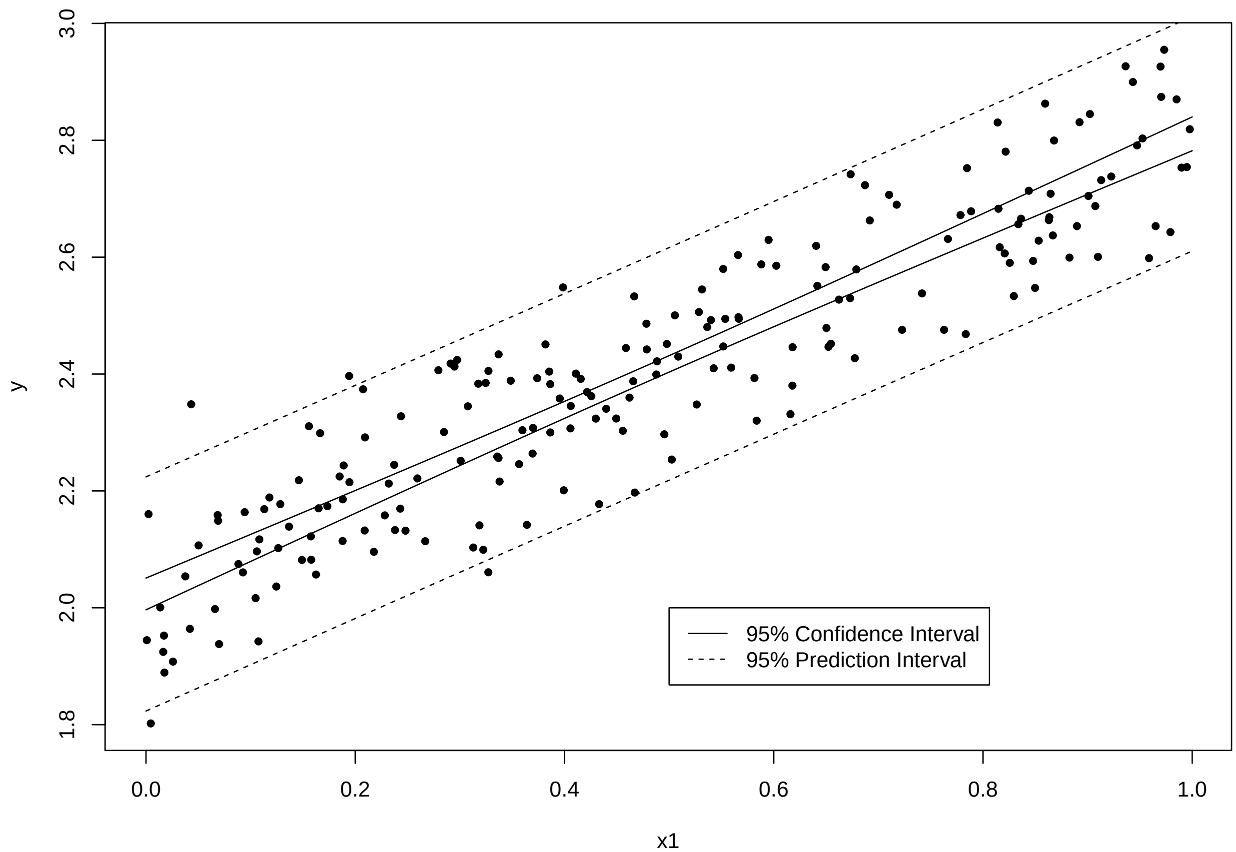
2 MARKS

e. The following code in R computes the fit of a simple linear regression model on $n = 200$ data points, and then plots the 95% confidence and prediction intervals for the predictions on a grid of 101 new predictor values equally spaced on the interval $(0, 1)$:

```
fitQ2<-lm(y ~ x1)
xv<-seq(0,1,by=0.01)
newx1<-data.frame(x1=xv)
y.conf<-predict(fitQ2,newdata=newx1,interval='confidence')
y.pred<-predict(fitQ2,newdata=newx1,interval='prediction')
plot(x1,y,pch=19,cex=0.7)
for(j in 2:3){lines(xv,y.conf[,j])}
for(j in 2:3){lines(xv,y.pred[,j],lty=2)}
```

Question continues on the next page.

The following figure is produced:



At new predictor value $x_1^{\text{new}} = 0.50$, the 95% confidence interval derived from the fit is $(2.4033 : 2.4314)$. If the 0.975 quantile of the Student-t distribution with 198 degrees of freedom is 1.972, and the estimate of σ from the fit is $\widehat{\sigma} = 0.1006$, calculate the 95% prediction interval at $x_1^{\text{new}} = 0.50$. 4 MARKS

G1. Consider a three-way contingency table with factors X, Y and Z, and a log-linear model $1 + X + Y + Z + X : Z + Y : Z$.

(a) Show that the MLE of the model parameters are determined by solving the equations

$$\mu_{i+k} = y_{i+k}, \quad i = 1, \ldots, I, k = 1, \ldots, K,$$
$$\mu_{+jk} = y_{+jk}, \quad j = 1, \ldots, J, k = 1, \ldots, K$$

4 MARKS

(b) Identify the dependence pattern described by the model and show that $\hat{\mu}_{ijk} = y_{i+k}y_{+jk}/y_{++k}$, $i = 1, \ldots, I$, $j = 1, \ldots, J$, $k = 1, \ldots, K$ are the maximum likelihood estimates. 3 MARKS

(c) Show that the Pearson and likelihood ratio statistics for testing this model's fit have the form $T^2 = \sum_{k=1}^{K} T_k^2$, where T_k^2 is a statistic for testing the independence between X and Y given $Z = k$. 4 MARKS

The following data reflect the results of a prospective study into the frequency of use of three types of cognitive therapy on four personality types, in male and female subjects. The results of various

Personality Type	Therapy	Sex	
		F	M
A	T1	3	2
	T2	3	5
	T3	2	3
B	T1	0	5
	T2	5	9
	T3	4	5
C	T1	3	7
	T2	14	14
	T3	5	8
D	T1	11	9
	T2	19	22
	T3	7	11

log-linear models fitted to the data appear overleaf.

(d) Decide which of the fitted models is best suited for the data at hand. Test at the 5% level. Discuss the independence, partial independence, or conditional independence structure of the identified model. 5 MARKS

(e) The researchers who carried out the study were primarily interested in discovering whether significantly different therapeutic strategies were adopted for males and females. Briefly describe an alternative analysis to log-linear modeling that could have been used. 4 MARKS

Output for Question G1

Model	Residual DF	Deviance
`1`	23	90.184
`1 + Per`	20	40.143
`1 + The`	21	64.797
`1 + Sex`	22	86.902
`1 + Per + The`	18	14.756
`1 + Per + Sex`	19	36.860
`1 + The + Sex`	20	61.514
`1 + Per + The + The:Per`	12	12.770
`1 + Per + Sex + Sex:Per`	16	34.989
`1 + The + Sex + Sex:The`	18	61.190
`1 + Per + The + Sex`	17	11.473
`1 + Per + The + Sex + The:Per`	11	9.487
`1 + Per + The + Sex + Sex:Per`	14	9.601
`1 + Per + The + Sex + Sex:The`	15	11.149
`1 + Per + The + Sex + The:Per + Sex:Per`	8	7.616
`1 + Per + The + Sex + The:Per + Sex:The`	9	9.163
`1 + Per + The + Sex + Sex:Per + Sex:The`	12	9.277
`1 + Per + The + Sex + The:Per + Sex:Per + Sex:The`	6	7.340

G2. (a) Consider a Binomial GLM with some arbitrary link function g. Show that the Fisher information matrix is of the form $X^\top W X$, and identify the matrices X and W.

4 MARKS

(b) In the context of a Binomial GLM, explain the term overdispersion, and how to diagnose whether overdispersion is present. Describe in detail one method for accounting for overdispersion in a Binomial-type model. 5 MARKS

A method of predicting survival `STA` (1 = died, 0 = survived) after admission to an intensive care unit (ICU) is required. The available predictors are:

- `SEX` (0 = Male,1 = Female)
- `AGE` (age in years, covariate, ranging from 16 to 92 with median age 63)
- `SYS` (systolic blood pressure at admission, in mmHg, ranging from 36 to 236 with median pressure 130mmHg)
- `TYP` (type of admission, 0 = Elective, 1 = Emergency)
- `HRA` (heart rate at admission in beats per minute, covariate, ranging from 39 to 192 with median rate 96)

(c) Based on the output lines 1–11, decide which one of the GLMs fitted to the data is most appropriate. Test at the 5% level. Clearly formulate the GLM that you selected and the test you used. 4 MARKS

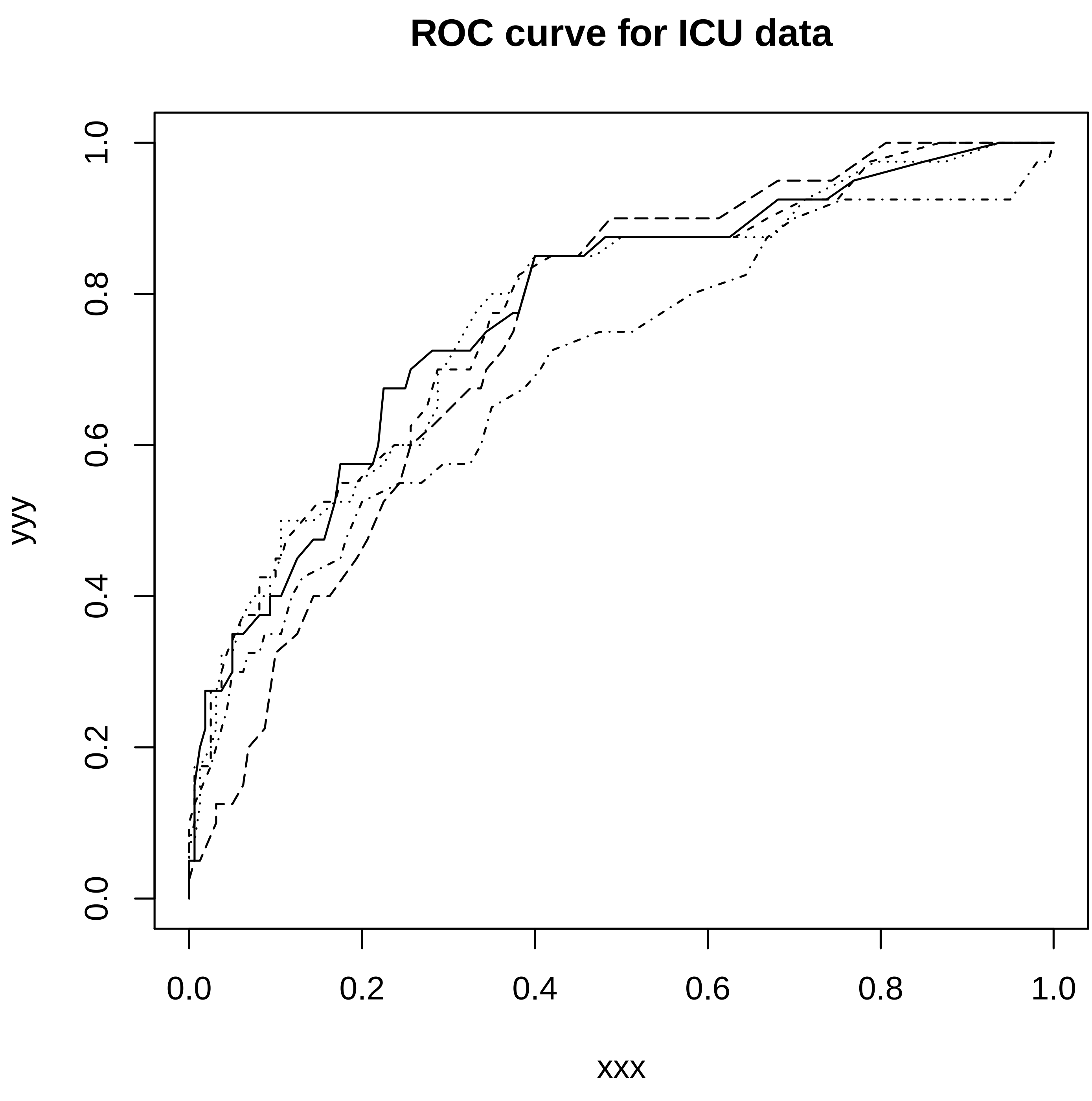
(f) A fifth model fitted is `Model 5`; the `R` output appears on page 6, lines 13–33. Sketch how the probability of death varies with age in the two admission type groups based on this analysis. Sketching the qualitative behaviour is sufficient. 4 MARKS

(g) The models `Model 1`,...,`Model 5` were compared using a plot, displayed on page 7. Explain how were the curves constructed and which model seems the most appropriate based on this plot. 3 MARKS

R Code and Output for Question 2

```
Analysis of Deviance Table

Model 1: STA ~ AGE + factor(SEX) + SYS + factor(TYP) + HRA
Model 2: STA ~ AGE + factor(SEX) + SYS + factor(TYP)
Model 3: STA ~ AGE + SYS + factor(TYP)
Model 4: STA ~ AGE + SYS
  Resid. Df Resid. Dev Df Deviance
1       194     166.91
2       195     167.70 -1  -0.7888
3       196     167.82 -1  -0.1239
4       197     183.25 -1 -15.4309

Call:
glm(formula = STA ~ AGE * factor(TYP), family = binomial, data = ICU)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.1851  -0.7760  -0.4283  -0.1656   2.5305

Coefficients:
                 Estimate Std. Error z value Pr(>|z|)
(Intercept)      -8.31541    6.37531  -1.304    0.192
AGE               0.07365    0.08765   0.840    0.401
factor(TYP)1      5.30784    6.41312   0.828    0.408
AGE:factor(TYP)1 -0.04040    0.08830  -0.457    0.647

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 200.16  on 199  degrees of freedom
Residual deviance: 172.83  on 196  degrees of freedom
AIC: 180.83

Number of Fisher Scoring iterations: 7
```

Graphical Output for Question 2



`Model 1`: full line; `Model 2`: short-dashed line; `Model 3`: dotted line; `Model 4`: dash-dotted line; `Model 5`: long-dashed line.

G3. (a) Explain why modelling via the *cumulative logits*

$$\psi_{ik} = \log\left(\frac{\mathrm{P}[Y_i \le k]}{1 - \mathrm{P}[Y_i \le k]}\right), \qquad k = 1, \ldots, K-1$$

is appropriate for ordinal categorical multinomial data having K (ordered) categories. 4 MARKS

(b) If a regression model is constructed for the cumulative logits, yielding quantities $\psi_{ik}(\beta) = \psi_k(x_i; \beta)$ for $i = 1, \ldots, n$ and $k = 1, \ldots, K-1$, write down the likelihood for β based on the observed data $(y_i, x_i), i = 1, \ldots, n$, where y_i takes values in the set $\{1, \ldots, K\}$. 5 MARKS

The following study investigates 68,694 passengers in autos and light trucks involved in accidents during one year in the state of Maine. The response categories are (1) not injured, (2) injured but not transported by emergency medical services, (3) injured and transported by emergency medical services but not hospitalized, (4) injured and hospitalized but did not die, (5) injured and died. Explanatory variables collected were `gender` of the passenger (male or female), `location` (rural or urban) and whether or not the passenger wore a `seatbelt`.

(c) The output of the data analysis appears on page 9. Formulate the model that was used. Why are there six lines in the table in which the parameter estimate and its standard deviation are exactly zero? 3 MARKS

(d) Why are there four intercepts? Explain how they determine the estimated response distribution for males in urban areas wearing seatbelts. How does this estimated response distribution compare to that for males in urban areas wearing no seatbelt? Provide a sketch of the two distributions and interpret. 4 MARKS

(e) Find the cumulative odds ratio between the response and seatbelt use for those in rural locations, given gender. Based on this, explain how the effect of seatbelt varies by region and explain how to interpret the interaction estimate, -0.1244. 4 MARKS

R Code and Output for Question Q3

Parameter			DF	Estimate	Std Error
Intercept1			1	3.3074	0.0351
Intercept2			1	3.4818	0.0355
Intercept3			1	5.3494	0.0470
Intercept4			1	7.2563	0.0914
gender	female		1	−0.5463	0.0272
gender	male		0	0.0000	0.0000
location	rural		1	−0.6988	0.0424
location	urban		0	0.0000	0.0000
seatbelt	no		1	−0.7602	0.0393
seatbelt	yes		0	0.0000	0.0000
location*seatbelt	rural	no	1	−0.1244	0.0548
location*seatbelt	rural	yes	0	0.0000	0.0000
location*seatbelt	urban	no	0	0.0000	0.0000
location*seatbelt	urban	yes	0	0.0000	0.0000

Table of the Chi-squared distribution

Entries in table are $\chi^2_\alpha(\nu)$: the α tail quantile of Chi-squared(ν) distribution

α given in columns, ν given in rows.

	Left-tail					Right-tail				
ν	0.99500	0.99000	0.97500	0.95000	0.90000	0.10000	0.05000	0.02500	0.01000	0.00500
1	0.00004	0.00016	0.00098	0.00393	0.01579	2.70554	3.84146	5.02389	6.63490	7.87944
2	0.01003	0.02010	0.05064	0.10259	0.21072	4.60517	5.99146	7.37776	9.21034	10.59663
3	0.07172	0.11483	0.21580	0.35185	0.58437	6.25139	7.81473	9.34840	11.34487	12.83816
4	0.20699	0.29711	0.48442	0.71072	1.06362	7.77944	9.48773	11.14329	13.27670	14.86026
5	0.41174	0.55430	0.83121	1.14548	1.61031	9.23636	11.07050	12.83250	15.08627	16.74960
6	0.67573	0.87209	1.23734	1.63538	2.20413	10.64464	12.59159	14.44938	16.81189	18.54758
7	0.98926	1.23904	1.68987	2.16735	2.83311	12.01704	14.06714	16.01276	18.47531	20.27774
8	1.34441	1.64650	2.17973	2.73264	3.48954	13.36157	15.50731	17.53455	20.09024	21.95495
9	1.73493	2.08790	2.70039	3.32511	4.16816	14.68366	16.91898	19.02277	21.66599	23.58935
10	2.15586	2.55821	3.24697	3.94030	4.86518	15.98718	18.30704	20.48318	23.20925	25.18818
11	2.60322	3.05348	3.81575	4.57481	5.57778	17.27501	19.67514	21.92005	24.72497	26.75685
12	3.07382	3.57057	4.40379	5.22603	6.30380	18.54935	21.02607	23.33666	26.21697	28.29952
13	3.56503	4.10692	5.00875	5.89186	7.04150	19.81193	22.36203	24.73560	27.68825	29.81947
14	4.07467	4.66043	5.62873	6.57063	7.78953	21.06414	23.68479	26.11895	29.14124	31.31935
15	4.60092	5.22935	6.26214	7.26094	8.54676	22.30713	24.99579	27.48839	30.57791	32.80132
16	5.14221	5.81221	6.90766	7.96165	9.31224	23.54183	26.29623	28.84535	31.99993	34.26719
17	5.69722	6.40776	7.56419	8.67176	10.08519	24.76904	27.58711	30.19101	33.40866	35.71847
18	6.26480	7.01491	8.23075	9.39046	10.86494	25.98942	28.86930	31.52638	34.80531	37.15645
19	6.84397	7.63273	8.90652	10.11701	11.65091	27.20357	30.14353	32.85233	36.19087	38.58226
20	7.43384	8.26040	9.59078	10.85081	12.44261	28.41198	31.41043	34.16961	37.56623	39.99685
21	8.03365	8.89720	10.28290	11.59131	13.23960	29.61509	32.67057	35.47888	38.93217	41.40106
22	8.64272	9.54249	10.98232	12.33801	14.04149	30.81328	33.92444	36.78071	40.28936	42.79565
23	9.26042	10.19572	11.68855	13.09051	14.84796	32.00690	35.17246	38.07563	41.63840	44.18128
24	9.88623	10.85636	12.40115	13.84843	15.65868	33.19624	36.41503	39.36408	42.97982	45.55851
25	10.51965	11.52398	13.11972	14.61141	16.47341	34.38159	37.65248	40.64647	44.31410	46.92789
26	11.16024	12.19815	13.84390	15.37916	17.29188	35.56317	38.88514	41.92317	45.64168	48.28988
27	11.80759	12.87850	14.57338	16.15140	18.11390	36.74122	40.11327	43.19451	46.96294	49.64492
28	12.46134	13.56471	15.30786	16.92788	18.93924	37.91592	41.33714	44.46079	48.27824	50.99338
29	13.12115	14.25645	16.04707	17.70837	19.76774	39.08747	42.55697	45.72229	49.58788	52.33562
30	13.78672	14.95346	16.79077	18.49266	20.59923	40.25602	43.77297	46.97924	50.89218	53.67196
31	14.45777	15.65546	17.53874	19.28057	21.43356	41.42174	44.98534	48.23189	52.19139	55.00270
32	15.13403	16.36222	18.29076	20.07191	22.27059	42.58475	46.19426	49.48044	53.48577	56.32811
33	15.81527	17.07351	19.04666	20.86653	23.11020	43.74518	47.39988	50.72508	54.77554	57.64845
34	16.50127	17.78915	19.80625	21.66428	23.95225	44.90316	48.60237	51.96600	56.06091	58.96393
35	17.19182	18.50893	20.56938	22.46502	24.79665	46.05879	49.80185	53.20335	57.34207	60.27477
36	17.88673	19.23268	21.33588	23.26861	25.64330	47.21217	50.99846	54.43729	58.61921	61.58118
37	18.58581	19.96023	22.10563	24.07494	26.49209	48.36341	52.19232	55.66797	59.89250	62.88334
38	19.28891	20.69144	22.87848	24.88390	27.34295	49.51258	53.38354	56.89552	61.16209	64.18141
39	19.99587	21.42616	23.65432	25.69539	28.19579	50.65977	54.57223	58.12006	62.42812	65.47557
40	20.70654	22.16426	24.43304	26.50930	29.05052	51.80506	55.75848	59.34171	63.69074	66.76596
50	27.99075	29.70668	32.35736	34.76425	37.68865	63.16712	67.50481	71.42020	76.15389	79.48998