Instructions

- Answer only **two** questions from Section P. If you answer more than two questions, then only the **FIRST TWO questions will be marked.**
- Answer only **four** questions from Section S. If you answer more than four questions, then only the **FIRST FOUR questions will be marked.**
- A distribution sheet, a sheet of useful summation results, and tables of the standard Normal distribution, are provided.
- The usual abbreviations are used: pmf indicates the probability mass function; pdf indicates the probability density function; cdf indicates the cumulative distribution function; mgf indicates the moment generating function; pgf indicates the probability generating function.
- ***You may use any result that is known to you, but you must state the name of the result (law/theorem/lemma/formula/inequality) that you are using, and show the work of verifying the condition(s) for that result to apply.***
- ***For the problems with multiple parts, you are allowed to assume the conclusion from the previous part in order to solve the next part, whether or not you have completed the previous part.***

SPECIAL ONLINE INSTRUCTIONS:

(i) The exam must be scanned and returned by email to Jason and Rustum Choksi by 6:10 PM on August 4, 2020.

(ii) This is a closed book exam - NO AIDES OR CONSULTATIONS of any sort may be used. PLEASE SIGN BELOW that you have complied with these rules.

Name:

Signature:

This exam comprises the cover page, nine pages of questions, and two pages of tables.

P1. (a) Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, assume that $\{A_n : n \geq 1\}$ is a sequence of **independent** events such that, if $\alpha_n := \min\{\mathbb{P}(A_n), 1 - \mathbb{P}(A_n)\}$, then $\sum_{n\geq 1} \alpha_n = \infty$. Prove that all the singletons of $(\Omega, \mathcal{F}, \mathbb{P})$ are $\mathbb{P}-$null sets, i.e., for every $\omega \in \Omega$ such that $\{\omega\} \in \mathcal{F}$, $\mathbb{P}(\{\omega\}) = 0$.

10 MARKS

(b) Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{X_n : n \geq 1\}$ be a sequence of **independent** random variables and F_n be the distribution function of X_n for every $n \geq 1$. Prove that $\lim_{n\to\infty} X_n = 0$ almost surely if and only if

$$\forall \epsilon > 0, \sum_{n\geq 1} [1 - F_n(\epsilon) + F_n(-\epsilon)] < \infty.$$

P2. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{X_n : n \geq 1\}$ be a sequence of **independent** and **identically distributed** random variables such that $\mathbb{E}[|X_1|] < \infty$. For every $n \geq 1$, set $S_n := \sum_{j=1}^{n} X_j$.

(a) Prove that $\left\{\frac{S_n}{n} : n \geq 1\right\}$ is uniformly integrable.

10 MARKS

(b) Prove that $\frac{S_n}{n} \to \mathbb{E}[X_1]$ (as $n \to \infty$) almost surely as well as in $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

10 MARKS

P3. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{Y_n : n \geq 1\}$ be a sequence of **independent** and **identically distributed** random variables with the common distribution

$$\mathbb{P}(Y_n = 1) = \mathbb{P}(Y_n = -1) = \frac{1}{2}.$$

Let $\{a_n : n \geq 1\}$ be a sequence of positive numbers such that $a_n > a_{n+1}$ for every $n \geq 1$ and $\sum_{n=1}^{\infty} a_n^2 < \infty$. Set $X_0 \equiv 1$, and

$$X_n := \frac{\exp\left(\sum_{j=1}^{n} a_j Y_j\right)}{\prod_{j=1}^{n} \cosh(a_j)} \text{ for every } n \geq 1.$$

(a) Prove that $\{X_n : n \geq 0\}$ is a martingale (with respect to the natural filtration).

10 MARKS

(b) Prove that there exists $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \to X$ (as $n \to \infty$) almost surely as well as in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

10 MARKS

S1. (a) If X is an random variable (rv) that is positive with probability 1, and $\mathbb{E}(X) = \mu < \infty$, directly prove that if $Y = \log X$, then

$$\mathbb{E}(Y) \leq \log \mu.$$

6 MARKS

(b) If f_0 and f_1 are probability mass functions (pmfs) for a discrete random variable taking integer values, then the Kullback-Leibler divergence between f_0 and f_1, $KL(f_0, f_1)$ is defined by

$$KL(f_0, f_1) = \sum_{x=-\infty}^{\infty} \log\left(\frac{f_0(x)}{f_1(x)}\right) f_0(x)$$

and is finite provided that $f_1(x) > 0$ at every x for which $f_0(x) > 0$.

(i) Prove that $KL(f_0, f_1) \geq 0$. 6 MARKS

(ii) Compute $KL(f_0, f_1)$ if $f_0(x)$ is the $Bernoulli(\theta_0)$ pmf and $f_1(x)$ is the $Bernoulli(\theta_1)$ pmf for $0 < \theta_0, \theta_1 < 1$. 4 MARKS

(c) Let X be a rv with a moment generating function (mfg) $M(t)$ defined on interval $(-\delta, \delta)$, for some $\delta > 0$. Show that, for any real number a

$$P(X \geq a) \leq e^{-at} M(t)$$

for $0 < t < \delta$.

4 MARKS

S2. (a) Suppose that $X \sim Gamma(\alpha, \beta)$.

(i) Show that X has an Exponential Family distribution. 3 MARKS

(ii) Find $\mathbb{E}(\log X)$. 3 MARKS

(b) Suppose that $X \sim Gamma(\alpha, \beta)$, and $Y = 1/X$.

(i) Does Y have an Exponential Family distribution ? Justify your answer. 3 MARKS

(ii) Find $\mathbb{E}(Y)$ and $\mathbb{E}(\log Y)$. 5 MARKS

(c) Find the form of the Fisher information, $\mathcal{I}(\eta)$, of an m-parameter Exponential Family distribution in its canonical (or natural) parameterization based on parameter $\eta = (\eta_1, \ldots, \eta_m)$.

Recall that for the Exponential Family

$$\mathcal{I}(\eta) = -\mathbb{E}[\Psi(X;\eta)]$$

where $\Psi(x;\eta)$ is the $m \times m$ matrix with (j,k)th entry

$$\frac{\partial^2}{\partial \eta_j \partial \eta_k} \{\log f(x;\eta)\}$$

for $1 \leq j, k \leq m$. 6 MARKS

In (a) and (b), leave your answers in terms of special functions and their derivatives where necessary. In (c), leave your answer in terms of the functions used to define the Exponential Family distribution.

S3. (a) Suppose that $X_1, \ldots, X_n$ are a random sample of size n from the distribution with cumulative distribution function (cdf)

$$F_X(x) = \left(\frac{x}{1+x}\right) \qquad x > 0$$

and zero otherwise. Let $Y_1 = X_{(1)}$ and $Y_n = X_{(n)}$ be the minimum and maximum order statistics derived from $X_1, \ldots, X_n$, respectively.

(i) Find the cdfs of Y_1 and Y_n. 4 MARKS

(ii) Find the limiting distributions (if they exist) of Y_1 and Y_n as $n \longrightarrow \infty$. 6 MARKS

(iii) Find the limiting distribution of rv $V_n = Y_n/n$ as $n \longrightarrow \infty$, and hence find an approximation to the distribution of Y_n for large n. 4 MARKS

(b) Suppose that $X_1, \ldots, X_n, \ldots$ are a sequence of iid rvs from a distribution with characteristic function (cf) $\varphi_X(t)$ and expectation

$$\mathbb{E}(X) = \mu.$$

For $n = 1, 2, \ldots$, let M_n be the sample mean rv derived from $X_1, \ldots, X_n$

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Show that for arbitrary $t \in \mathbb{R}$,

$$\varphi_{M_n}(t) \longrightarrow \exp\{i\mu t\}$$

where $i = \sqrt{-1}$, and $\varphi_{M_n}(t)$ is cf of M_n. Give full details of your reasoning.

Recall that if for rv X we have $\mathbb{E}(X^r) < \infty$ for positive integer r, then by a Taylor expansion,

$$\varphi_X(t) = 1 + \sum_{j=1}^{r} \frac{(it)^j}{j!}\mathbb{E}(X^j) + O(t^{r+1})$$

as $|t| \longrightarrow 0$. 6 MARKS

S4.

- Let $X_i \overset{iid}{\sim} N(\theta, 1)$, $i = 1, 2, \cdots, n$. Consider the sequence

$$\delta_n = \begin{cases} \overline{X}_n, & \text{if } |\overline{X}_n| \geq 1/n^{1/4}, \\ a\overline{X}_n, & \text{if } |\overline{X}_n| < 1/n^{1/4}. \end{cases}$$

Show that $\sqrt{n}(\delta_n - \theta) \xrightarrow{\mathcal{L}} N(0, \nu(\theta))$, where $\nu(\theta) = 1$ if $\theta \neq 0$ and $\nu(\theta) = a^2$ if $\theta = 0$. Suppose $|a| < 1$. Find the information bound and compare it with $\nu(\theta)$. Is there any contradiction?

20 MARKS

S5. Let $X_i \overset{iid}{\sim} N(0, \sigma^2)$ for $i = 1, 2, \cdots, n$, where σ^2 is unknown.

(a) Find $\hat{\sigma}_{ML}$, the MLE of σ, and show that $\hat{\sigma}_{ML}$ is a consistent estimator of σ.

6 MARKS

(b) Consider $T_n = T(X_1, \cdots, X_n) = \sqrt{\pi/2}\sum_{i=1}^{n} |X_i|/n$. Show that T_n is an unbiased and consistent estimator of σ.

7 MARKS

(c) Compute asymptotic relative efficiency of T_n with respect to $\hat{\sigma}_{ML}$,

$$\lim_{n\to\infty} \frac{\mathbb{V}(\hat{\sigma}_{ML})}{\mathbb{V}(T_n)}.$$

Do you choose $\hat{\sigma}_{ML}$ or T_n to make a confidence interval for σ? Explain using the asymptotic relative efficiency. 7 MARKS

S6. *Generalized Neyman-Pearson Lemma.* Let $f_0(x), f_1(x), \cdots, f_k(x)$ be $k+1$ probability density functions. Let ϕ_0 be a test function of the form

$$\phi_0(x) = \begin{cases} 1, & \text{if} \quad f_0(x) > \sum_{j=1}^{k} a_j f_j(x) \\ \gamma(x), & \text{if} \quad f_0(x) = \sum_{j=1}^{k} a_j f_j(x) \\ 0, & \text{if} \quad f_0(x) < \sum_{j=1}^{k} a_j f_j(x) \end{cases}$$

where $a_j \geq 0$ for $j = 1, \cdots, k$. Show that ϕ_0 maximizes

$$\int \phi(x) f_0(x) dx$$

among all ϕ, $0 \leq \phi \leq 1$, such that

$$\int \phi(x) f_j(x) dx \leq \int \phi_0(x) f_j(x) dx, \qquad j = 1, 2, \cdots, k$$

20 MARKS

DISCRETE DISTRIBUTIONS

	SUPPORT $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathbb{E}_X[X]$	$\text{Var}_X[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$ Version I	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or Version II	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION** for $\alpha > 0$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathbb{E}_Y[Y] = \mu + \sigma\mathbb{E}_X[X] \qquad \text{Var}_Y[Y] = \sigma^2\text{Var}_X[X]$$

CONTINUOUS DISTRIBUTIONS

	SUPPORT $\mathbb{X}$	PARAMETERS	PDF f_X	CDF F_X	$\mathbb{E}_X[X]$	$\mathrm{Var}_X[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\frac{1}{\beta-\alpha}$	$\frac{x-\alpha}{\beta-\alpha}$	$\frac{(\alpha+\beta)}{2}$	$\frac{(\beta-\alpha)^2}{12}$	$\frac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\left(\frac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\frac{\Gamma\left(1+\frac{1}{\alpha}\right)}{\beta^{1/\alpha}}$	$\frac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R}$ $\sigma\in\mathbb{R}^+$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$\exp\left\{\mu t+\frac{\sigma^2t^2}{2}\right\}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\frac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\frac{\theta}{\theta+x}\right)^\alpha$	$\frac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\frac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	