McGill University
Department of Mathematics and Statistics
Statistics Part A Comprehensive Exam
Methodology Paper

Date: Friday, August 13, 2021
Time: 9 am to 5 pm EST

Instructions

(i) Answer only two questions out of Section L. If you answer more than two questions, then only the FIRST TWO questions will be marked.

(ii) Answer only two questions out of Section G. If you answer more than two questions, then only the FIRST TWO questions will be marked.

Special Online Instructions

(i) Your solutions and this signed cover page must be scanned and returned by email to Jason Stillman and Pengfei Guan by 5:10 PM EST on August 13, 2021.

(ii) This is a closed book exam. No aides or consultations of any sort may be used.

Please sign below that you have complied with these rules.

Name: ________________________________________________________________

Signature: ______________________________________________________________

<table>
<thead>
<tr>
<th>Question</th>
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<th>Question</th>
<th>Marks</th>
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<tbody>
<tr>
<td>L1</td>
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<td>G1</td>
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<td>G3</td>
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</table>

This exam comprises 19 pages, including the cover page.
Question L1. Consider the multiple linear regression model defined, for all \( i \in \{1, \ldots, n\} \), by

\[
Y_i = x_i^\top \beta + \varepsilon_i, \quad \varepsilon_i \sim \mathcal{N}(0, \sigma^2),
\]

where \( x_i \) is a \( p \)-dimensional vector of covariates (including 1 for the intercept), and the parameters \( \beta \) and \( \sigma^2 \) are unknown. Here, \( p = k + 1 \) and \( k \) is the number of covariates. The matrix representation of the above model is

\[
Y = X\beta + \varepsilon, \quad \varepsilon \sim \mathcal{N}_n(0, \sigma^2 I).
\]

In answering all of the following questions, use least-squares estimators of the parameters.

(a) Assuming \( \beta_1 = \cdots = \beta_k = 0 \), prove that the ratio \( MS_R/MS_{Res} \) has a Fisher–Snedecor or \( F \) distribution (degrees of freedom to be specified), where \( MS_R \) and \( MS_{Res} \) are respectively the mean squared regression and residuals. Explain how to use this ratio for testing the null hypothesis \( H_0: \beta_1 = \cdots = \beta_k = 0 \) vs. \( H_1: \text{at least one } \beta_j \neq 0 \) at a predetermined significance level \( \alpha \).

5 MARKS

(b) Derive a \( 100 \times (1 - \alpha)\% \) confidence region for \( \beta \). Show all steps of your derivation.

3 MARKS

(c) Given a fixed vector-value \( x_0 \) of the covariates, let \( \mu(x_0) \) and \( Y_0 \) be respectively the corresponding mean and future value of the response variable under the above linear model. Derive \( 100 \times (1 - \alpha)\% \) confidence and prediction intervals for both \( \mu(x_0) \) and \( Y_0 \), and explain their differences (if any).

5 MARKS
(d) Show that an equivalent way of testing $\mathcal{H}_0 : \beta_1 = \cdots = \beta_k = 0$ vs. $\mathcal{H}_1 : \text{"at least one } \beta_j \neq 0\text{"}$ at significance level $\alpha$, is to use the statistic

$$F_0 = \frac{R^2(n - p)}{k(1 - R^2)},$$

where $R^2$ is the coefficient of determination. Perform the test at significance level $\alpha = 0.05$ assuming $n = 25$, $k = 2$, and $R^2 = 0.9$.

3 MARKS

(e) Derive an appropriate statistic for testing $\mathcal{H}_0 : \beta_1 = \cdots = \beta_k = \beta$ vs. $\mathcal{H}_1 : \text{"at least two of } \beta_j \text{ are different,"}$ at a predetermined significance level $\alpha$. Recall that $\beta$ is unknown.

4 MARKS
Question L2. The data in this question concern the sale price and other price-related characteristics for 24 houses sold in Erie, Pennsylvania. The variables are:

\[ y: \text{Sale price of the house/1000.} \]

\[ x_1: \text{Taxes (local, school, county)/1000.} \]
\[ x_2: \text{Number of baths.} \]
\[ x_3: \text{Lot size (sq ft \times 1000).} \]
\[ x_4: \text{Living space (sq ft \times 1000).} \]
\[ x_5: \text{Number of garage stalls.} \]
\[ x_6: \text{Number of rooms.} \]
\[ x_7: \text{Number of bedrooms.} \]
\[ x_8: \text{Age of the house (years).} \]
\[ x_9: \text{Number of fireplaces.} \]

The objective is to study the potential relationship between the regressors \( x_1 \) to \( x_9 \) and the sale price of house, \( y \), through a multiple linear regression model.

A portion of the data is presented below:

```r
> data:
<table>
<thead>
<tr>
<th>Obs</th>
<th>y</th>
<th>x1</th>
<th>x2</th>
<th>x3</th>
<th>x4</th>
<th>x5</th>
<th>x6</th>
<th>x7</th>
<th>x8</th>
<th>x9</th>
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<tbody>
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<td>1</td>
<td>29.5</td>
<td>5.0208</td>
<td>1.0</td>
<td>3.5310</td>
<td>1.500</td>
<td>2.0</td>
<td>7</td>
<td>4</td>
<td>62</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>27.9</td>
<td>4.5429</td>
<td>1.0</td>
<td>2.2750</td>
<td>1.175</td>
<td>1.0</td>
<td>6</td>
<td>3</td>
<td>40</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>25.9</td>
<td>4.5573</td>
<td>1.0</td>
<td>4.0500</td>
<td>1.232</td>
<td>1.0</td>
<td>6</td>
<td>3</td>
<td>54</td>
<td>0</td>
</tr>
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<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>22</td>
<td>36.9</td>
<td>8.1400</td>
<td>1.0</td>
<td>8.0000</td>
<td>1.504</td>
<td>2.0</td>
<td>7</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>23</td>
<td>45.8</td>
<td>9.1416</td>
<td>1.5</td>
<td>7.3262</td>
<td>1.831</td>
<td>1.5</td>
<td>8</td>
<td>4</td>
<td>31</td>
<td>0</td>
</tr>
<tr>
<td>24</td>
<td>25.9</td>
<td>4.9176</td>
<td>1.0</td>
<td>3.4720</td>
<td>0.998</td>
<td>1.0</td>
<td>7</td>
<td>4</td>
<td>42</td>
<td>0</td>
</tr>
</tbody>
</table>
```

In the following analysis, use the significance level \( \alpha = .05 \).
(a) The following model was fitted to the data:

\[
\text{MODEL 1: } \ln(y_i) = \alpha_0 + \alpha_1 x_{i1} + \alpha_2 x_{i2} + \alpha_3 x_{i3} + \alpha_4 x_{i4} + \alpha_5 x_{i5} \\
+ \alpha_6 x_{i6} + \alpha_7 x_{i7} + \alpha_8 x_{i8} + \alpha_9 x_{i9} + \varepsilon_i
\]

for all \( i \in \{1, \ldots, n\} \), with the usual assumptions on the error terms \( \varepsilon_1, \ldots, \varepsilon_n \).

(i) What is (are) the possible statistical motivation(s) for the choice to model \( \ln(y) \) instead of \( y \) as a function of the regressors \( x_1, \ldots, x_9 \)? Name a formal statistical method that may have been used to come up with this decision.

4 MARKS

(ii) The R output for MODEL1 is provided as fit1 on p. 6. Comment on the overall significance of the model, and also test for a statistically significant association of each of the covariates with the response variable \( \ln(y) \) in the presence of all other covariates in the model.

5 MARKS

(iii) Figure 1 on p. 7 displays two residuals plots for fit1. Comment on these plots.

4 MARKS
R Code and Output for Question L2 (MODEL 1)

```
######## MODEL 1:
fit1<-lm(log(y)~x1+x2+x3+x4+x5+x6+x7+x8+x9, data=data)

summary(fit1)
Call:
  lm(formula = log(y)~x1+x2+x3+x4+x5+x6+x7+x8+x9, data = data)
Residuals:
   Min       1Q   Median       3Q      Max
-0.096061 -0.052793  0.000923  0.042088  0.115485

Coefficients:
                     Estimate   Std. Error
(Intercept)  2.999741  0.162444
          x1   0.051869  0.028295
          x2   0.210366  0.118145
          x3   0.004836  0.013472
          x4   0.096770  0.119770
          x5   0.070985  0.037735
          x6  -0.015694  0.065346
          x7  -0.054705  0.093286
          x8  -0.001061  0.001833
          x9   0.045225  0.053229
---
Residual standard error: 0.08102
Multiple R-squared: 0.8655, Adjusted R-squared: 0.779
F-statistic: ----

anova(fit1)
Analysis of Variance Table
Response: log(y)
            Df Sum Sq Mean Sq
x1           1  0.51828  0.51828
x2           1  0.02431  0.02431
x3           1  0.00472  0.00472
x4           1  0.00004  0.00004
x5           1  0.00974  0.00974
x6           1  0.01855  0.01855
x7           1  0.01043  0.01043
x8           1  0.00051  0.00051
x9           1  0.00474  0.00474
Residuals  14  0.09189  0.00656
---
```
Figure 1: Residual plots for MODEL1 defined on p. 5.

(b) The second model fitted to this data set is

MODEL 2: \( \ln(y_i) = \gamma_0 + \gamma_1 x_{i1} + \varepsilon_i \)

for all \( i \in \{1, \ldots, n\} \), and with the usual assumptions on the error terms \( \varepsilon_1, \ldots, \varepsilon_n \).

The R output for this model is \texttt{fit2} on p. 8. Figure 2 on p. 9 displays two residuals plots for \texttt{fit2}.
Using the outputs fit1 and fit2, test the following hypothesis in MODEL1 defined on p. 5:

\[ H_0 : \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = 0. \]

Comment on your findings.  

4 MARKS

(c) Of the two models, which one do you suggest for these data? Explain. 

3 MARKS

R Code and Output for Question L2 (MODEL 2)

```
######## MODEL 2:
fit2<-lm(log(y)~x1, data=data)
summary(fit2)

Call:
  lm(formula = log(y) ~ x1, data = data)

Residuals:
  Min   1Q   Median   3Q   Max
-0.134563 -0.063478 -0.003664 0.039192 0.177646

Coefficients:
             Estimate Std. Error
(Intercept)  2.92218   0.07520
x1           0.09489   0.01141

Residual standard error: 0.08658
Multiple R-squared: 0.7586, Adjusted R-squared: 0.7476
F-statistic: -----,

anova(fit2)

Analysis of Variance Table

Response: log(y)
             Df  Sum Sq Mean Sq
x1              1 0.51828  0.51828
Residuals      22 0.16493  0.00750
```
Figure 2: Residual plots for MODEL 2 defined on p. 7.
Question L3. Consider two independent samples

Sample 1: \((x_{i1}, x_{i2}, Y_i), \quad i \in \{1, \ldots, n_1\}\),
Sample 2: \((x_{i1}, x_{i2}, Y_i), \quad i \in \{n_1 + 1, \ldots, n_1 + n_2\}\).

The following two models can be fitted to these two samples:

\begin{align*}
Y_i &= \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i, \quad i \in \{1, \ldots, n_1\}, \\
Y_i &= \alpha_0 + \alpha_1 x_{i1} + \alpha_2 x_{i2} + \varepsilon_i, \quad i \in \{n_1 + 1, \ldots, n_1 + n_2\},
\end{align*}

where \(\varepsilon_1, \ldots, \varepsilon_{n_1+n_2}\) are iid \(\mathcal{N}(0, \sigma^2)\).

(a) Show how these two separate models can be written as a single model.

4 MARKS

(b) Based on the model in (a) and using the general linear hypothesis, show how to test the equality of “\(\beta_1\) and \(\alpha_1\)” and “\(\beta_2\) and \(\alpha_2\)”. Provide all the details.

4 MARKS

(c) Based on the model in (a) and using the general linear hypothesis, show how to test that both “\(\beta_1\) and \(\alpha_1\) are equal to \(c_1\)” and that “\(\beta_2\) and \(\alpha_2\) are equal to \(c_2\)” for some known \(c_1\) and \(c_2\).

4 MARKS

(d) Based on the model in (a) and using the general linear hypothesis, show how to test the equality of the two models.

4 MARKS

(e) Given \(x_0 = (x_0^1, x_0^2)^\top\), and using the least squares estimates of all the parameters based on the above data, explain how one can predict its corresponding value \(Y_0\). Is it even possible, or you would need more information?

4 MARKS
Section G (Generalized Linear Models)
Answer only two questions out of G1–G3

Question G1.

(a) Explain the meaning of the term overdispersion in non-negative count data. If the data is modeled using a Poisson GLM and overdispersion is present, what can be said about the estimated standard errors of the parameters and the p-values of the Wald tests of the hypotheses $\beta_1 = 0, \ldots, \beta_p = 0$?

2 MARKS

(b) Detail how the Negative Binomial distribution derives from the Poisson distribution and calculate its mean and variance. Formulate the Negative Binomial model with the log link, list its parameters, and explain how it accounts for overdispersion. You can use, without proof, the fact that the mean and variance of a Gamma distribution with shape $\alpha$ and scale $\beta$ are, respectively, $\alpha/\beta$ and $\alpha/\beta^2$.

5 MARKS

Consider the following study on the occurrence of herpes encephalitis in children. The data are from Bavaria and Lower Saxony, Germany, between 1980 and 1993; they consist of 26 observations of count, the number of cases with herpes encephalitis, year from 1980 to 1993 (labeled 1–14), and country (Bavaria = 1, Lower Saxony = 2).

(c) Models mod1 (output lines 1–20) and mod2 (output lines 22–43) were fitted to the data. Describe in detail a test that can be used to decide whether the Poisson model mod1 is an adequate simplification of mod2, and explain how to compute the p-value of the test. Run this test at the 5% level and interpret the result.

4 MARKS

(d) Another model mod3 (output lines 44–69) was fitted to the data, including the square of year as a predictor, year.sq. Decide, on a basis of suitable statistical tests at the 5% level, which one of the three fitted models is the most suitable for the data. Does your selected model fit the data well?

4 MARKS

(e) Sketch the fitted number of cases $\hat{\mu}$ in mod3 as a function of year for the two Länder (regions) Bavaria and Lower Saxony, and formulate the conclusions from this model for the data at hand.

5 MARKS
R Code and Output for Question G1

1 Call:
2 glm(formula = count ~ year + as.factor(country) + as.factor(country):year,
3   family = "poisson")
4
5 Coefficients:
6 Estimate Std. Error z value Pr(>|z|)
7 (Intercept) 0.13359 0.38552 0.347 0.7290
8 year 0.19702 0.03810 5.171 2.33e-07 ***
9 as.factor(country)2 0.87373 0.54157 1.613 0.1067
10 year:as.factor(country)2 -0.13328 0.05592 -2.383 0.0172 *
11 ---
12 Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
13
14 (Dispersion parameter for poisson family taken to be 1)
15
16 Null deviance: 60.390 on 21 degrees of freedom
17 Residual deviance: 24.709 on 18 degrees of freedom
18 AIC: 104.03
19
20 'log Lik.' -48.01353 (df=4)
21
22 Call:
23 glm.nb(formula = count ~ year + as.factor(country) + as.factor(country):year,
24   init.theta = 33.65183353, link = log)
25
26 Coefficients:
27 Estimate Std. Error z value Pr(>|z|)
28 (Intercept) 0.05147 0.41579 0.124 0.901
29 year 0.20649 0.04192 4.926 8.41e-07 ***
30 as.factor(country)2 0.93872 0.57850 1.623 0.105
31 year:as.factor(country)2 -0.14065 0.06047 -2.326 0.020 *
32 ---
33 Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
34
35 (Dispersion parameter for Negative Binomial(33.6518) family taken to be 1)
36
37 Null deviance: 52.511 on 21 degrees of freedom
38 Residual deviance: 21.144 on 18 degrees of freedom
39 AIC: 105.71
40
41 Theta: 33.7
42 Std. Err.: 62.9
43
44 2 x log-likelihood:  -95.708
R Code and Output for Question G1

```r
Call:
glm(formula = count ~ year + year.sq + as.factor(country) + as.factor(country):year,
    family = "poisson")

Deviance Residuals:
     Min       1Q   Median       3Q      Max
-1.73391 -0.44777 -0.01956  0.39270  1.09159

Coefficients:
                     Estimate Std. Error z value Pr(>|z|)
(Intercept)       -2.49401   0.91891  -2.714  0.006646 **
year               0.88439   0.20266   4.364  1.28e-05 ***
year.sq           -0.03975   0.01114  -3.567  0.000361 ***
as.factor(country)2 1.61982   0.68188   2.376  0.017524 *
year:as.factor(country)2 -0.20953   0.07162  -2.926  0.003438 **

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 60.390  on 21  degrees of freedom
Residual deviance: 10.147  on 17  degrees of freedom
AIC: 91.465

Number of Fisher Scoring iterations: 4
```
Question G2.

(a) Show that the deviance for a binomial GLM with binary responses and the logit link is given by

$$D(y, \hat{\pi}) = -2 \sum_{i=1}^{n} \{(X_i \hat{\beta})\hat{\pi}_i + \ln(1 - \hat{\pi}_i)\}.$$ 

Explain the consequences of this result for model validation.

6 MARKS

(b) A study on the duration of unemployment with \(n = 982\) participants distinguished between short term unemployment (\(\leq 6\) months, coded as 0) and long term unemployment (> 6 months, coded as 1). It is of interest to determine the effect of age, ranging from 16 to 61 years, on the duration of unemployment. Consider the model whose output is on p. 15, lines 1–22.

(i) Describe which GLM was used for the data analysis and determine whether the data were entered in a grouped or ungrouped form. Decide, on a basis of suitable tests at the 5% level, whether the model is adequate for the data at hand and whether the effect of age is significant. Describe all tests used.

5 MARKS

(ii) Formulate the effect of age in terms of odds and report a 95% confidence interval for the odds ratio corresponding to a 10-year increase in age. You can use the fact that the 97.5% quantile of the standard Gaussian distribution is approximately 1.96.

4 MARKS

(c) Suppose that a binary response \(Y\) has been generated from a latent (unobservable) variable \(Z\) in the sense that \(Y = 0\) when \(Z \leq 0\) and \(Y = 1\) when \(Z > 0\). Suppose also that \(Z\) satisfies the regression model \(Z = X\beta + \epsilon\), where \(\epsilon \sim N(0, 1)\). Formulate a suitable generalized linear model that can be used to estimate \(\beta\). Give full details of the mathematical derivations.

5 MARKS
R Code and Output for Question G2

```r
Call: 
  glm(formula = cbind(short, long) ~ age, family = "binomial")

Deviance Residuals:
  Min 1Q Median 3Q Max
-2.04395 -0.81071 0.00009 0.60754 1.68543

Coefficients:
  Estimate Std. Error z value Pr(>|z|)
(Intercept) 1.46624 0.19672 7.453 9.09e-14
age -0.02686 0.00591 -4.546 5.48e-06

(Dispersion parameter for binomial family taken to be 1)

Null deviance: 54.008 on 45 degrees of freedom
Residual deviance: 33.269 on 44 degrees of freedom
AIC: 178.34

Number of Fisher Scoring iterations: 3
```
Question G3.

(a) Formulate the log-linear model $AB + AC$, where $A$, $B$, and $C$ are factor predictors. Determine its number of parameters, calculate the fitted values using the iterative proportional fitting algorithm, and interpret the model in terms of independence patterns.

4 MARKS

(b) Write down the asymptotic distribution of the parameter estimates, including a formula for the asymptotic variance-covariance matrix. Specify what is the meaning of asymptotic, i.e., what "tends to infinity"?

4 MARKS

In a survey study, several variables were collected that are linked to the birth process: gender of the child $G$ (0: male, 1: female), whether membranes ruptured before the beginning of labor $M$ (1: yes, 0: no), whether a Cesarean section was done $C$ (1: yes, 0: no), and whether birth was induced $I$ (1: yes, 0: no). The association between the four variables was unknown and investigated.

(c) Using the $R$ output on pp. 17–18, select the best fitting model for these data. Comment on the global fit of this model and write down any other models that you would consider fitting.

5 MARKS

(d) Interpret the log-linear model $G + MC + MI + CI$. In particular, interpret all two-way interaction parameters. Does there seem to be any dependence between the baby’s gender and the variables membranes, Cesarean section, and induced birth?

5 MARKS

(e) Suppose it is of interest to investigate the effect of $G$, $M$, and $I$ on whether a Cesarean section was carried out. Is there a logistic model that would be equivalent to the log-linear model $G + MC + MI + CI$? Explain.

2 MARKS
R Code and Output for Question Q3

1 m0 <- glm(Y ~ C*G*M + C*G*I + G*M*I+C*M*I)
2 m1 <- glm(Y ~ G*M+C*I+G*C+G*I+C*M+M*I, family="poisson")
3 m2 <- glm(Y ~ C*I + G*C + G*I+C*M + M*I, family="poisson")
4 m3 <- glm(Y ~ C*I+G*C+C*M+M*I, family="poisson")
5 m4 <- glm(Y ~ G+M+C+I, family="poisson")
6 m5 <- glm(Y ~ M+I, family="poisson")
7 m6 <- glm(Y ~ I, family="poisson")
8
c(deviance(m0), deviance(m1), deviance(m2),
  deviance(m3), deviance(m4), deviance(m5), deviance(m6))


[1] 1 5 6 7 8 9 11
R Code and Output for Question Q3

```r
1 Call:
2 glm(formula = Y ~ C * I + G + C * M + M * I, family = "poisson")
3
4 Deviance Residuals:
5       Min 1Q Median 3Q Max
6 -1.2756 -0.5190 -0.2503 0.2515 1.6721
7
8 Coefficients:
9               Estimate Std. Error z value Pr(>|z|)
10 (Intercept)   5.16925   0.06498  79.547  < 2e-16 ***
11          C   -1.67722   0.13727  -12.219  < 2e-16 ***
12          I   -1.17828   0.11361  -10.371  < 2e-16 ***
13          G   -0.23850   0.07440   -3.206   0.00135 **
14          M   -0.61013   0.09363   -6.517  7.19e-11 ***
15         C:I   0.57928   0.22107   2.620   0.00878 **
16         C:M   0.55783   0.24284   2.297   0.02161 *
17         I:M   0.39891   0.20028   1.992   0.04640 *
18        ---
19 Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
20
21 (Dispersion parameter for poisson family taken to be 1)
22
23 Null deviance: 736.4601 on 15 degrees of freedom
24 Residual deviance: 9.3964 on 8 degrees of freedom
25 AIC: 106.34
26
27 Number of Fisher Scoring iterations: 4
```
## Table of the Chi-squared distribution

Entries in table are $\chi^2_\nu(\alpha)$: the $\alpha$ tail quantile of Chi-squared($\nu$) distribution $\alpha$ given in columns, $\nu$ given in rows.

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<th>$\nu$</th>
<th>0.99500</th>
<th>0.99000</th>
<th>0.97500</th>
<th>0.95000</th>
<th>0.90000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00004</td>
<td>0.00016</td>
<td>0.00098</td>
<td>0.00393</td>
<td>0.01579</td>
</tr>
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