SPECIAL ONLINE INSTRUCTIONS:

(i) Your solutions and this signed cover page must be scanned and returned by email to Jason and Pengfei Guan by 5:10 PM on August 10, 2021.

(ii) This is a closed book exam - NO AIDES OR CONSULTATIONS of any sort may be used. PLEASE SIGN BELOW that you have complied with these rules.

Name:

Signature:

INSTRUCTIONS:

(i) There are 12 questions. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.
Linear Algebra

Solve any three out of the four questions 1, 2, 3, 4.

Problem 1. Let \( v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3 \). Let \( \beta := \begin{pmatrix} e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix} \), an ordered basis of \( \mathbb{R}^3 \). Let \( \theta \) be a real number. Find the matrix representing rotation by either \( \theta \) or \( -\theta \) around the line spanned by \( v \).

Problem 2.
(a) Let \( (V, \langle \cdot, \cdot \rangle) \) be a finite dimensional inner-product space over the real numbers. Prove that every linear functional \( \alpha : V \to \mathbb{R} \) is of the form \( \alpha(x) = \langle x, v \rangle \) for a unique vector \( v \in V \).

(b) Let \( (V, \langle \cdot, \cdot \rangle) \) be as in (a). Suppose \( \alpha : V \to \mathbb{R} \) is a linear functional. Let \( A = \{ x \in V : \alpha(x) = c \} \). What is the minimum norm, with respect to the norm \( ||x|| := \sqrt{\langle x, x \rangle} \), among all vectors \( x \in A \)? Express your answer in terms of \( c \) and the vector \( v \) from (a).

(c) Let \( A \) be the collection of all real polynomials \( p \) of degree at most 3 satisfying \( p(1) = 1 \). Find the minimum value of
\[
\int_0^1 f(x)^2 \, dx
\]
among all \( f \in A \).

Problem 3. Let \( V, \langle \cdot, \cdot \rangle \) be a finite dimensional complex inner-product space. Let \( A : V \to V \) be a self-adjoint linear transformation, i.e. \( A = A^* \). Let \( f \) be the function
\[
f : V \setminus \{0\} \to \mathbb{C} \\
x \mapsto \frac{\langle Ax, x \rangle}{\langle x, x \rangle}.
\]
(a) Prove that \( f \) is real-valued.

(b) Prove that if \( x \) maximizes the function \( f \), then \( x \) is an eigenvector of \( A \).

Problem 4.
Let \( (V, \langle \cdot, \cdot \rangle) \) be a finite dimensional complex inner-product space. Let \( U : V \to V \) be a unitary linear transformation, i.e. \( \langle Ux, Uy \rangle = \langle x, y \rangle \) for all \( x, y \in V \).

(a) Suppose \( v \) is an eigenvector of \( U \). Prove that the orthogonal complement \( v^\perp \) is an invariant subspace of \( U \).

(b) Prove that \( U \) is diagonalizable.
Undergraduate Real Analysis

Solve any three out of the four questions 5, 6, 7, 8.

**Problem 5.** Suppose \( f : [0, 1] \to \mathbb{R} \) is differentiable and \( f(0) = 0 \). Suppose
\[
|f'(x)| \leq |f(x)|^2, \quad \forall x \in [0, 1].
\]
Prove that \( f(x) = 0, \quad \forall x \in [0, 1] \).

**Problem 6.** Suppose \( a_n > 0, \forall n = 1, 2, \ldots \), prove that,
\[
(1) \text{ If there is } A > 2 \text{ such that } \frac{a_{n+1}}{a_n} \leq \sqrt{1 - \frac{A}{n}}, \forall n = 1, 2, \ldots, \text{ then } \sum_{n=1}^{\infty} a_n \text{ converges.}
\]
\[
(2) \text{ If there is } A \leq 2 \text{ such that } \frac{a_{n+1}}{a_n} \geq \sqrt{1 - \frac{A}{n}}, \forall n = 1, 2, \ldots, \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges.}
\]

**Problem 7.** Suppose \( \{f_n\} \subset C([0, 1]) \) is a sequence of continuous functions. Suppose \( f'_n(x) \) exists for \( 0 < x \leq 1 \) and there is \( 0 < \alpha < 1 \) such that
\[
f_n(0) = 0, \quad |f'_n(x)| \leq x^{-\alpha}, \quad \forall 0 < x \leq 1, \quad \forall n = 1, 2, \ldots.
\]
Prove that \( \{f_n\} \) has a subsequence \( \{f_{n_k}\} \) which is uniformly convergent in \([0, 1]\).

**Problem 8.** Let \( f \in L^1(\mathbb{R}) \) be a real-valued function, suppose that \( f^{2021} \) is locally integrable in the sense that \( f \in L^{2021}(I) \) for every bounded subinterval \( I \subset \mathbb{R} \). Show that, if
\[
\int_I f^{2021} = 0,
\]
for every bounded subinterval \( I \subset \mathbb{R} \), then \( f = 0 \) a.e.
ODE and Advanced Calculus

Solve any three out of the four questions 9, 10, 11, 12.

**Problem 9.** Consider the initial value problem: \( \ddot{x} - 3 \dot{x} + 2x = 6t^2 - 14t + 2 \) with \( x(0) = 0 \) and \( \dot{x}(0) = 0 \). (Note that, \( \dot{x} = \frac{dx}{dt} \) and \( \ddot{x} = \frac{d^2x}{dt^2} \).)

1. Write the homogeneous solution.
2. Compute the particular solution.
3. Compute the general solution of the initial value problem.

**Problem 10.**
Find a function \( f(x) \) with \( x \in (0, \infty) \) such that \( \frac{df}{dx} = f^{-1}(x) \). (Note that \( f^{-1}(x) \) is the inverse function of \( f(x) \), that is \( f^{-1}(f(x)) = x \).)

**Problem 11.** Suppose \( f \) is a \( C^2 \) function in \( \mathbb{R}^n \) satisfying
\[
f(x) = 1, \quad \forall \|x\| = 1; \quad \|\nabla f(x)\| \leq 1, \quad \text{div}(\nabla f(x)) \geq 0, \quad \forall x \in B_1(0) = \{\|x\| < 1, \ x \in \mathbb{R}^n\}.
\]
Prove that
\[
0 < f(x) \leq 1, \forall x \in \bar{B}_1(0).
\]

**Problem 12.** Let \( \mathbf{F}(x, y) = (\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}) \) be a vector field in \( \mathbb{R}^2 \setminus \{0\} \). Evaluate the \( \int_\gamma \mathbf{F} \cdot ds \), where \( \gamma = \{(e^{t^2}\cos(t), e^{\cos(3t)}\sin(t)) \mid 0 \leq t \leq \frac{3\pi}{2}\} \) with counterclockwise orientation.