# Thermodynamic properties of coupled map lattices

Jérôme Losson and Michael C. Mackey

**Abstract**

This chapter presents an overview of the literature which deals with applications of models framed as coupled map lattices (CML's), and some recent results on the spectral properties of the transfer operators induced by various deterministic and stochastic CML's. These operators (one of which is the well-known Perron-Frobenius operator) govern the temporal evolution of ensemble statistics. As such, they lie at the heart of any thermodynamic description of CML's, and they provide some interesting insight into the origins of nontrivial collective behavior in these models.

## 1 Introduction

This chapter describes the statistical properties of networks of chaotic, interacting elements, whose evolution in time is discrete. Such systems can be profitably modeled by networks of coupled iterative maps, usually referred to as coupled map lattices (CML's for short). The description of CML's has been the subject of intense scrutiny in the past decade, and most (though by no means all) investigations have been primarily numerical rather than analytical. Investigators have often been concerned with the statistical properties of CML's, because a deterministic description of the motion of all the individual elements of the lattice is either out of reach or uninteresting, unless the behavior can somehow be described with a few degrees of freedom. However there is still no consistent framework, analogous to equilibrium statistical mechanics, within which one can describe the probabilistic properties of CML's possessing a large but finite number of elements. The results presented in this chapter illustrate some recent attempts to partially fill this theoretical void.

### 1.1 Coupled map lattices: Initial presentation

Models framed as coupled discrete time maps are not a novelty. Caianiello [10] proposed his "neuronic equations", which are coupled iterative maps, as generalizations of the McCulloch and Pitts neural networks more than three decades ago. Similarly, the work of Denman [22], trying to characterize the dynamics of interacting pressure and electromagnetic waves in plasmas, made use of coupled discrete maps, and related models were used in the early theory of transmission lines [82]. However, the modern body of work dealing with coupled map lattices can be traced back to the beginning of the eighties (*cf.* work by Kaneko [39], Waller and Kapral [98, 99] and Deissler [20]) as phenomenological models to study the behavior of large collections of coupled chaotic elements (we will return to more precise descriptions of these and more recent investigations of CML dynamics).

### 1.1.1 Deterministic CML's

In their most general form, deterministic coupled map lattices are mappings $\Phi : \mathbb{R}^N \longmapsto \mathbb{R}^N$ governing the evolution of a state vector $\mathbf{x}_t = (x_t^1, \cdots, x_t^N)$,

$$\mathbf{x}_{t+1} = \Phi(\mathbf{x}_t) \quad t = 0, 1, \cdots. \tag{1}$$

More specifically, the evolution of a component $x_t^{(i)}$ of the state vector $\mathbf{x}_t$ is governed by the difference equation

$$x_{t+1}^{(i)} = \Phi_{\text{local}}^{(i)}(x_t^{(i)}) + \Phi_{\text{neighbours}}^{(i)}(\cdots, x_t^{i-1}, x_t^{i+1}, \cdots)$$

where $\Phi_{\text{local}}^{(i)}$ models the local dynamics at site $i$, and $\Phi_{\text{neighbours}}^{(i)}$ denotes the mechanisms acting on $i$ from a specified neighbourhood. If those mechanisms are the same for all sites on the lattice, and if they are locally modeled by the map $S : \mathbb{R} \longmapsto \mathbb{R}$, and in the neighbourhood by the map $T : \mathbb{R} \longmapsto \mathbb{R}$ one can write

$$x_{t+1}^{(i)} = S(x_t^{(i)}) + \sum_{\substack{\text{some} \\ \text{neighbourhood}}} T(x_t^j).$$

In many situations of interest, it is possible to further simplify the formulation of the models by letting $T \equiv S$, and using a linear coupling scheme between the elements[1]. In these circumstances, we have

$$x_{t+1}^{(i)} = (1 - \varepsilon) S(x_t^{(i)}) + \frac{\varepsilon}{p} \sum_{\substack{p \\ \text{neighbours}}} S(x_t^j) \tag{2}$$

where $\varepsilon \in [0, 1]$ is the coupling term. Again, $i$ denotes a discrete spatial index (of arbitrary finite dimension), and $t$ denotes discrete time.

In our description of CML's, we view the sites of the lattice as being located on the nodes of a regular body centered cubic lattice, and in this chapter periodic boundary conditions are always enforced. There are investigations of coupled map lattices in which the underlying lattice is not as simple as the body-centered-cubic example chosen here, and possesses intrinsically "complex" (sometimes called hierarchical) structure. In these cases, it was demonstrated [18, 19] that the bifurcation structure of the CML can depend on the topology of the lattice, but we will not dwell on this point, since most of the analytical tools discussed in this chapter do not depend on the properties of the underlying lattice topology.

We consider cases where the phase space $\mathbb{X}$ of $\Phi$ is a restriction of $\mathbb{R}^N$ to the $N$-dimensional hypercube: $\mathbb{X} = [0, 1] \times \cdots \times [0, 1]$. In two spatial dimensions, the evolution of each site of a deterministic coupled map lattice with linear interelement coupling is given by

$$x_{t+1}^{(kl)} = \Phi^{(kl)}(\mathbf{x}_t) = (1 - \varepsilon) S(x_t^{(kl)}) + \frac{\varepsilon}{p} \sum_{\substack{p \text{ nearest} \\ \text{neighbours}}} S(x_t^{(ij)}), \quad \varepsilon \in (0, 1), \tag{3}$$

[1] The coupling scheme of equation (2) is called *linear* because $x_{t+1}^{(i)}$ is linearly proportional to $S(x_t^{(i)})$. Some authors [46] would call such architectures nonlinear but we will adhere to this convention.

where $S : [0,1] \longmapsto [0,1]$ describes the local dynamics. When $p = 4$, the coupling in (3) mimics a discrete version of the diffusion operator, and when the $p$-neighbourhood encompasses the entire lattice, the coupling is known as mean-field.

To allow for the possibility that stochastic perturbations influence the evolution of the CML, we now introduce a class of stochastic CML's which will be investigated numerically and analytically in Section 5.

### 1.1.2 Stochastic CML's

It is of interest to understand and clarify the influence of noisy perturbations on the evolution of these CML's. The perturbations considered here are random vectors of $N$ numbers (for an $N$ element CML), whose components are independent of one another, each being distributed according to a one dimensional probability density $g$. The density $\mathbf{g}$ of the vector random variable $\boldsymbol{\xi} = (\xi^{(1)}, \cdots, \xi^{(N)})$ will therefore be constructed as the product of independent (identical) components:

$$\mathbf{g}(\boldsymbol{\xi}) = \prod_{i=1}^{N} g^{(i)}(\xi^{(i)}) = \prod_{i=1}^{N} g(\xi^{(i)}). \tag{4}$$

There are various ways in which a stochastic perturbation can influence the evolution of a coupled map lattice: the perturbation can be additive or multiplicative, and it can be applied constantly or randomly. The influence of the noise on the dynamics depends on which of these is considered.

When the stochastic perturbations are applied at each iteration step, they can be either added to, or multiply, the original transformation $\Phi$. In the former case, the evolution of a lattice site is given by a relation of the form

$$x_{t+1}^{(kl)} = \Phi^{(kl)}(\mathbf{x}_t) + \xi_t^{(kl)} \equiv \Phi_{\text{add}}^{(kl)}(\mathbf{x}_t) \tag{5}$$

and $\boldsymbol{\xi}$ is then referred to as *additive* noise. In the latter, we have

$$x_{t+1}^{(kl)} = \Phi^{(kl)}(\mathbf{x}_t) \times \xi_t^{(kl)} \equiv \Phi_{\text{mul}}^{(kl)}(\mathbf{x}_t) \tag{6}$$

and $\boldsymbol{\xi}$ is then referred to as *multiplicative* or *parametric noise*. In general, the effects of additive and multiplicative noise on CML's can be different, since they model different perturbing mechanisms. The density (4) of the perturbations present in (5) and (6) is always defined so that the phase space of the perturbed transformations remains the $N$ dimensional hypercube $\mathbb{X}$ defined above. In other words, $\Phi_{\text{add}} : \mathbb{X} \longmapsto \mathbb{X}$ and $\Phi_{\text{mul}} : \mathbb{X} \longmapsto \mathbb{X}$.

The developments which followed the introduction of CML's have established the usefulness of these models to investigate the dynamics of a wide variety of systems in various areas of research ranging from population dynamics to solid state physics. Our own research was motivated in part by this activity, and we therefore give a fairly extensive though by no means exhaustive review of the literature before proceeding to a description of CML thermodynamics.

# 2 Overview of models framed as CML's

Two collections of papers on the subject, both edited by K. Kaneko [38, 24] are available, so the present section focuses primarily on some of the more recent published works on CML dynamics.

## 2.1 Biological applications

There are many biological systems which can be thought of as collections of interacting elements with intrinsic nontrivial dynamics. When this is case, and if the local dynamics can reasonably be modeled by discrete time maps, it is feasible to introduce models framed as CML's.

This approach has been fruitful in population dynamics, in which the discrete time occurs naturally if generations do not overlap (insect populations constitute one possible example). The investigations of Solé *et al.* [87, 88] have led these authors to conclude that CML's provided the simplest models for discrete ecological models. Franke and Yakubu [26] have recently proposed a CML to investigate the inter-species competition of large bird populations. These CML's are crude models for the evolution of species competing for shared resources, which are obtained by straightforward (albeit not very realistic) multidimensional generalizations of proposed one-dimensional maps [70, 96]. They open the way for more realistic population competition models which could be framed as CML's in which the underlying lattice is not regular, perhaps taking into account some of the spatial features observed in the field. Ikegami and Kaneko [45] have also proposed a model for host-parasitoid networks, and their study of the corresponding CML have led them to introduce a generalization of the idea of homeostasis. The proposed alternative, "homeochaos", describes an asymptotic state reached by networks of evolving and mutating host-parasite populations in which chaotic fluctuations in the numbers of hosts and parasites are observed at equilibrium.

Beyond population dynamics, the mathematical description of neural behavior has also benefited from discrete time, discrete space models. The foundations of the modeling of cortical function were laid in two seminal papers by Wilson and Cowan [102, 103]. However, the original models presented by these authors are computationally costly, and are not easily amenable to analytic investigations. As a result, there have been attempts to reduce the original networks of integro-differential equations to simpler spatially extended models. Reduction to CML's are presently being considered by some of the same authors [68]. In its methodology, this work [*ibid*] is typical of investigations in which the CML is proposed as a discrete-time version of previously considered continuous-time systems. For example, Molgedey *et al.* [71] made use of coupled map lattices to examine the effects of noise on spatiotemporal chaotic behavior in a neural network which was originally proposed (in its continuous-time version) by Sompolinsky *et al.* [89]. Following a similar path, Nozawa [73] has presented a CML model, obtained by using the Euler approximation to the original Hopfield equations.

One of the outstanding problems motivating this neural oriented research is the identification of organizing principles to explain the synchronization of large populations of neurons possessing individually complex dynamics. Such synchronizations are thought to take place in pathological situations (*e.g.* epileptic seizures) as well as in the normal brain. For example, Andersen and Andersson [1], and later Steriade and Deschênes [90] have hypothesized that such a synchronized activity of the reticular thalamic nucleus (RTN) acted as a pacemaker for the so-called "spindle oscillations" observed during various sleep stages. Models of the RTN framed as networks of coupled differential delay equations have been proposed by Destexhe [23], and these can be reduced, by a straightforward singular perturbation procedure [34], to CML's. Models of the RTN framed as coupled ODE's have also been considered recently, and provide a motivation for the theoretical description of globally coupled arrays of oscillators [27]. An interesting review of the mathematical description of cortical behavior in terms of coupled nonlinear units is given in [104]. A less recent, but somewhat broader view of the contemporary efforts to mathematically describe the behavior of neural networks using the conceptual tools of nonlinear dynamics is presented in [84]. For the sake of completeness, we also refer the reader to the review by Herz [30], which describes some of the earlier neural modeling attempts which made use of CML's, as well as some of the models based on delay differential equations.

At the molecular level, Cocho *et al.* [15] proposed a CML model to describe the evolution of genetic sequences. A comprehensive account of the development of this idea can be found in [16]. In this simplified formalism, each genetic sequence is made up of $m$ nucleotides, which come in four flavors. The latter is determined by which of four possible bases (guanine, cytosine, adenine, thymine) complements the phosphate and deoxyribose groups which make-up the nucleotide. The building block of a genetic sequence is then a triplet of nucleotides, called a codon (which codes for an amino acid). Cocho *et al.* established that for certain viruses, it is relevant to restrict attention to sequences containing only two types of codons, denoted type I and II. Hence a sequence of length $L = m/3$ codons is uniquely characterized by the number $i_I$ of type I codons it possesses. $i_I$ can also be thought of as a position index in a configuration space, and in this case two sequences are "close" if they differ by a small number of codons. Under specified fitness constraints (whose meaningful definition imposes the most important limits on this approach), sequences can mutate: a type I codon becomes type II, or *vice-versa*. The CML model for genetic sequence evolution describes the evolution of the number of sequences at location $i_I$ in the configuration space, and therefore, local interactions are due to mutations, whereas ecological constraints (*i.e.* coming from limited food supplies) generate long range coupling. Recently [17], the same authors have extended this approach to study the mutations of the HIV1 virus, and their predictions concerning the regularity of the chemical compositions of this virus' RNA sequences agree with statistical analyses of gene data.

The use of CML's, though interesting from the mathematical biologist's point of view, is not restricted to biological models. Contemporary developments in the theory of image processing have led to the introduction of various algorithms which are in fact coupled map lattices.

## 2.2 Image processing applications

One of the basic challenges in image processing is the so-called "shape from shading" problem [8], which surfaces both in computer graphics, where shading is used to enhance realism, and computer vision, where the study of shading is crucial for the proper interpretation of a pattern's two dimensional projection (its picture). In computer vision, a typical task is the classification of patterns into classes (*e.g.* faces *vs.* landscapes), where the input patterns possess underlying "shapes" describing their essential features (nose, eyes, *vs.* trees or clouds) which are immersed in secondary information due to the shading of the image.

Several approaches to this problem [8, 92, 97] make use of algorithms which are coupled map lattices, although there appears to be no explicit awareness in this literature of the link between the structure of the algorithms and their formulation as CML's. We illustrate this link with a frequently encountered model used to approach shape from shading, which was introduced by Brooks and Horn and is known as the B-H algorithm [7]. To derive the model, the shape of an object is thought of as a function which minimizes a given functional. After minimization of the proper errors [97], the B-H algorithm is written

$$x_{t+1}^{(ij)} = \bar{x}_t^{(ij)} + \frac{\varepsilon^2}{4\lambda}\left(E^{(ij)} - x_t^{(ij)} \cdot \mathbf{S}\right)\mathbf{S} \tag{7}$$

where $E^{(ij)}$ is the shading, $x_t^{(ij)}$ is the surface normal at site $(i, j)$ of the image, $\lambda$ and $\varepsilon$ describe the role of a smoothness constraint, and $\mathbf{S}$ is the light source vector (the light source being responsible for the presence of shade), and $\bar{x}_t^{(ij)}$ is the average of the normals in a neighbourhood of site $(i, j)$. The local coupling comes from this latter term, and as a result, the evolution of the initial image under the action of the B-H shade from shading algorithm is akin to the evolution of an initial vector under the action of a CML. There are more recent descriptions of this problem which do not make use of the variational techniques used to derive (7), and which lead to different CML's (one example is given in [97]).

The treatment of fuzzy images is not limited to the shape from shading problem. In fact, prior to this analysis, "dirty" images, possibly obtained with remote sensing equipment must be "cleaned". This procedure, known as the segmentation of an image, is an attempt to highlight edges while smoothing the noise in regions devoid of edges. A "physicist-friendly" presentation of the segmentation problem is given by Price *et al.* [78]. They introduce a coupled map lattice designed as an alternative to the costlier and more unstable segmentation algorithms obtained by the minimization of a cost function. Their work is an additional illustration of the potential benefits to the image-processing community which could follow from an increased awareness of the wealth of dynamics displayed by high-dimensional nonlinear discrete time maps: the stability properties of the algorithms, and their possible pathological treatments of real images can sometimes be determined beforehand by an in-depth investigation of the corresponding CML.

## 2.3 Phenomenological models

In spite of the obvious interest generated by CML's for their many potential applications, the main motivation for their investigation from a physicist's point of view undoubtedly lies in their use as phenomenological models for the study of more general spatially extended systems

### 2.3.1 Spatiotemporal intermittency and weak turbulence

An example of the fruitful application of CML's to study fluid dynamics is given by the work of Chaté and Manneville, concerning the transition to turbulence *via* spatiotemporal intermittency [11, 12]. In this work, the CML's are constructed to reflect what are thought to be the essential features of a fluid undergoing the transition from laminar flow to turbulent flow *via* the so-called intermittency scenario, according to which a laminar flow gradually becomes turbulent by the growth of regions in the laminar regime in which the flow is turbulent. Hence, the essential features of the Chaté-Manneville models are the partition of the local phase space into two regimes: one laminar, and the other turbulent. Their analysis of the corresponding CML's lead to the identification of universality classes describing the "contamination process" of the laminar flows by turbulent "islands" [11, 13, 12]. The usefulness of the CML approach is that these models capture much of the phenomenology while remaining amenable to extensive numerical simulations.

The destabilization of laminar flows does not always occur *via* spatiotemporal intermittency. Various convective instabilities can result in alternate destabilizing mechanisms, and some of the recent work on CML's focuses on the dynamics of these instabilities in so-called "open flow" models [21]. Convective instabilities grow as they are transported downstream, and they are localized in the sense that a laboratory observer sees them pass by from upstream to downstream as localized defects [66]. Such situations are encountered, for example, in the modeling of shear flows and boundary layers, and they provide situations in which spatial order can be coexistent with temporal chaos. Given the complexity of the full equations of motion, it has been helpful to consider reduced models framed as CML's. In [4], Biferale *et al.* describe the convective instabilities of a unidirectionally coupled CML by focusing on the tangent vector associated with a trajectory of the CML. This analysis resulted in a relatively simple description of the localization of temporal chaos around the defects of the lattice. Other descriptions of asymmetrically coupled CML's include the works of Jensen [37, 36], Aranson *et al.* [2], and Willeboordse [101]. In all these, the coupling between the elements of the CML is not isotropic, and there is a preferred spatial direction in the lattice along which information is more easily transmitted. More recently, we have used CML's with unidirectional coupling to investigate the statistical properties of some differential delay equations [61].

### 2.3.2 Reaction diffusion models

Reaction-diffusion models play an important role in the description of real spatially extended systems because the competition between these two general mechanisms is ubiquitous in nature. In one dimension, they are modeled by the generic PDE

$$\frac{du(x,t)}{dt} = D\nabla^2 u(x,t) + F(u(x,t)), \tag{8}$$

where $F$ is the reaction term. In a seminal work, Turing [95] established that this competition was at the origin of many pattern-forming instabilities. Reaction diffusion systems have been the subject of many descriptions in terms of CML's because diffusion is approximated by a nearest neighbour coupling in CML's of the form (2) (examples of this reduction are given by Puri *et al.* [81] for the one dimensional Cahn-Hilliard equation, and by some of the same authors for the Fischer equation [75]). We note that the reduction of models framed as PDE's to their CML counterparts is usually not a rigorous procedure, although there are special circumstances (for some externally forced models) in which the CML provides a close approximation to the PDE [48]. As mentioned in [58], the benefits of using CML's in the majority of investigations stem from the fact that they reproduce most of the interesting phenomenology, without requiring the prohibitively large computing resources associated with PDE simulations. In addition, it is likely that as those resources increase with technological breakthroughs, so will the complexity of the problems considered by the modeling community, so that there is some intrinsic virtue in trying to understand reduced systems, such as CML's, to help in the study of more complicated ones.

Because of their computational efficiency, CML's are well-suited for the introduction of new quantifiers of spatiotemporal dynamics, or for the multidimensional generalizations of one-dimensional concepts [93] (this was an important motivation for the early discussions [20, 98, 99]). In this spirit, Kaneko has introduced such concepts as the "comoving mutual information flow" [40], and various "pattern entropies" and "pattern distribution functions" [41], to mention a few of the frequently encountered statistical descriptors of the motion. Reaction diffusion CML's are then usually of the form (2) with $p = 2$ in one spatial dimension, or $p = 4$ in two dimensions, and used to explore in great detail the behavior of the quantifiers of spatio-temporal motion more efficiently than if PDE's were considered.

Similar lattices have been used to simulate interfacial phenomena in reaction diffusion systems [58]. In these investigations, the CML's usually arise from the phenomenological simplification of PDE's of the form (8), and they provide the simplest models which retain the disparate length and time scales necessary for the appearance of rich interfacial dynamics. Other typical examples of this approach are given for crystal growth by Oono and Puri [74] and for chemical waves by Barkley [3]. A phenomenological description of interfacial phenomena was recently given by Kapral *et al.* [47], using a piecewise linear CML (with a branch with slope zero in the local map) which displays some of the interfacial structures associated with continuous time, continuous space models. In a similar spirit, the behavior of liquids at the boiling transition was studied by Yanagita [105] with another reaction-diffusion CML. To conclude, we refer the interested reader to the comprehensive review of the applications of CML's to capture the essential features of pattern formation in chemically reacting systems given by Kapral in [46]

### 2.3.3 Arrays of globally coupled oscillators

The introduction of all-to-all (or mean-field, or global) coupling in theoretical physics to investigate the dynamics of spatially extended systems is not novel; it has always been one of the standard techniques used to describe the magnetic properties of spin systems. As experimentalists probe ever deeper into the behavior of systems with a large number of degrees of freedom, new models of globally coupled oscillator arrays are introduced, in which the individual oscillators are either continuous or discrete in time. Some of the experimental situations in which global coupling arises naturally are related to nonlinear optics, with examples ranging from solid-state laser arrays [107], to multimode lasers [35]. In electronics, a number of experiments on Josephson junction arrays coupled in series or in parallel have indicated the presence of very rich dynamics, often related to the multiplicity of attractors, or the linear stability properties of fully synchronized states (*cf.* [72] and references therein). The majority of models proposed to describe these dynamics are framed as globally coupled sets of ordinary differential equations [29, 85, 91]. The ODE's are usually not rigorously reduced to CML's, and the introduction of the discrete-time map lattices is often motivated by the desire to improve the phenomenological insight into the evolution of the continuous-time oscillators. For example, Wiesenfeld and Hadley [100] found that CML's provided useful reduced systems to investigate the effects of low levels of noise on large globally coupled arrays which possess an even larger number of attractors. More recently, discrete maps were used to describe the dynamical properties of periodic attractors in arrays of $p-n$ diode junctions [25], and the stability regions of various solution types for the CML's agreed qualitatively with the experimental data obtained from two coupled diode junctions. We close this admittedly incomplete presentation of some contemporary discussions of global coupling in the physical sciences, by mentioning that CML's have recently been used to study theoretically the remarkable phenomenon of mutually destructive fluctuations in which the activity of the mean field is observed to have a much *smaller* variance than the individual trajectories [72]. This phenomenon is extremely interesting for researchers trying to understand the role of noise in the transmission of information in spatially extended processing systems. For example, it is well known that the behavior of individual neurons can sometimes be more erratic than that of the average behavior of a population of neurons [69].

We now turn to the presentation of some of the conceptual tools which will be used throughout the remainder of this chapter to discuss the statistical properties of models framed as CML's.

# 3 CML's and probability densities

Suppose that the dynamics of a physical system are modeled by a (deterministic or stochastic) dynamical system denoted by $\mathcal{T} : \mathbb{X} \longmapsto \mathbb{X}$ (many examples of such situations are described in Sections 2 and 2.3). Suppose further that some observable $\mathcal{O}(\mathbf{x}_n)$, which depends on the state $\mathbf{x}_n$ of $\mathcal{T}$, is being measured at time $n$ (The observable $\mathcal{O}$ is arbitrary, though it must be a bounded measurable function). The expectation value of this observable, denoted by $E(\mathcal{O}_n)$, is the mean value of $\mathcal{O}(\mathbf{x}_n)$ when the measurement is repeated a large (ideally infinite) number of times. Mathematically it is given by

$$E(\mathcal{O}_n) = \int_{\mathbb{X}} \mathbf{f}_n(\mathbf{y})\mathcal{O}(\mathbf{y})\, d\mathbf{y}, \tag{9}$$

where $\mathbf{f}_n(\mathbf{x})$ is the density of the variable $\mathbf{x}_n$, *i.e.* the probability $p(\mathbf{x}'_n)$ of finding $\mathbf{x}_n$ between $\mathbf{x}'_n$ and $\mathbf{x}'_n + \delta\mathbf{x}'_n$, is

$$p(\mathbf{x}'_n) = \int_{\mathbf{x}'_n}^{\mathbf{x}'_n+\delta\mathbf{x}'_n} \mathbf{f}_n(\mathbf{y})\, d\mathbf{y}.$$

All extrinsic functions which characterize the thermodynamic properties of a system are observables whose expectation values are defined by (9) since $\mathcal{O}$ was arbitrary. Therefore, the thermodynamic state of the CML $\mathcal{T}$ at time $n$ is completely characterized by the density function $\mathbf{f}_n$. Hence a complete description of the thermodynamics of $\mathcal{T}$ must focus on the behavior and properties of $\mathbf{f}_n$. To this end, we introduce the transfer operator associated with $\mathcal{T}$, denoted by $\mathcal{P}_{\mathcal{T}}$, which governs the time evolution of $\mathbf{f}_n$

$$\mathbf{f}_{n+1}(\mathbf{x}) = \mathcal{P}_{\mathcal{T}}\mathbf{f}_n(\mathbf{x}), \quad n = 0, 1, \cdots. \tag{10}$$

To draw an analogy with more familiar physical systems, the transfer operators discussed here describe the arbitrary dynamical system $\mathcal{T}$ much as the Liouville equation describes the ensemble dynamics of ODE's, the Fokker Planck equation those of the Langevin equation (which is a stochastic ODE), or the Perron-Frobenius operator (defined in Section 3.1) those of deterministic maps (*cf.* Table 1.1).

| *Description of the model* | *Description of ensemble dynamics* |
|---|---|
| Deterministic maps | The Perron-Frobenius operator |
| Stochastic maps | The transfer (Markov) operator |
| Deterministic ODE's | The generalized Liouville equation |
| Stochastic ODE's (white noise) | The Fokker Planck equation |
| Stochastic ODE's (non-white noise) | The Kramers- Moyal equation |
| Differential delay equations | The Hopf equation for the characteristic functional |

TABLE 1.1:
Brief summary of the probabilistic descriptions associated with various types of discrete and continuous-time models.

## 3.1 The Perron-Frobenius operator $\mathcal{P}_\Phi$

A discrete-time nonsingular transformation $\Phi : \mathbb{X} \longmapsto \mathbb{X}$ ($\mathbb{X} \subset \mathbb{R}^N$) induces an operator denoted $\mathcal{P}_\Phi$ which acts on probability densities, and which is defined implicitly by the relation

$$\int_{\mathbb{A}} \mathcal{P}_\Phi \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \int_{\Phi^{-1}(\mathbb{A})} \mathbf{f}(\mathbf{x})\, d\mathbf{x}, \quad \text{for all } \mathbb{A} \subset \mathbb{X}, \tag{11}$$

and all probability densities $\mathbf{f}$. $\mathcal{P}_\Phi$ is called the Perron-Frobenius operator induced by $\Phi$, and a study of its properties will be the cornerstone of our probabilistic description of deterministic CML's. If the transformation $\Phi$ is piecewise diffeomorphic, it is possible to give a more explicit definition of $\mathcal{P}_\Phi$ by performing a change of variable in the above definition.

Define $\Pi$ to be a partition of the phase space $\mathbb{X}$ which contains $s(\Pi)$ elements denoted $\pi_1, \pi_2, \cdots, \pi_{s(\Pi)}$. Let $\Phi_{|i}$ be the monotone restriction of $\Phi$ to the set $\pi_i \subset \mathbb{X}$, $i = 1, \cdots, s(\Pi)$ (with $\bigcup_{i=1}^{s(\Pi)} \pi_i = \mathbb{X}$). Let $\tilde{\pi}_i$ denote the image of the set $\pi_i$: $\tilde{\pi}_i \equiv \Phi_{|i}(\pi_i)$. The Perron-Frobenius operator induced by $\Phi$ can be written

$$\mathbf{f}_{n+1}(\mathbf{x}) \equiv \mathcal{P}_\Phi \mathbf{f}_n(\mathbf{x}) = \sum_{i=1}^{s(\Pi)} \frac{\mathbf{f}_n(\Phi_{|i}^{-1}(\mathbf{x}))}{\mathcal{J}_{\Phi_{|i}}(\Phi_{|i}^{-1}(\mathbf{x}))} \chi_{\tilde{\pi}_i}(\mathbf{x}), \tag{12}$$

where $\chi_{\tilde{\pi}_i}(\mathbf{x}) \equiv 1$ iff $\mathbf{x} \in \tilde{\pi}_i$, and 0 otherwise, and $\mathcal{J}_\tau(Z)$ is the absolute value of the Jacobian of transformation $\tau$, evaluated at $Z$. It should be clear from our presentation that the asymptotic properties of the sequence $\{\mathbf{f}_n\}$ of the iterates of an initial density $\mathbf{f}_0$ under the action of $\mathcal{P}_\Phi$ determine the thermodynamic behavior of the dynamical system $\Phi$. These asymptotic properties of $\{\mathbf{f}_n\}$ are themselves dependent on the spectral characteristics of the operator $\mathcal{P}_\Phi$, and our investigations of CML thermodynamics will in fact focus on the spectral properties of $\mathcal{P}_\Phi$.

There have been several attempts to use the Perron-Frobenius operator to describe the dynamics of CML's [32, 33, 42, 77], but these have all concentrated on the properties of an operator acting on one-dimensional densities. The "proper", or complete description is given instead by the $N$-dimensional operator, and it will be the object of our attention.

*Remark* 1 The invariant density $\mathbf{f}_*$ is implicitly defined by the relation

$$\mathbf{f}_* = \mathcal{P}_\Phi \mathbf{f}_*,$$

and it plays a special role in the thermodynamic description of any dynamical system, since it describes the state(s) of thermodynamic equilibrium(ia). Uniqueness of the invariant density implies uniqueness of the state of thermodynamic equilibrium for the system, and the approach of the sequence $\{\mathbf{f}_n\}$ to $\mathbf{f}_*$ describes the non-equilibrium behavior of the dynamical system.

## 3.2 The transfer operators $\mathcal{P}_{\Phi_{\mathbf{add}}}$ and $\mathcal{P}_{\Phi_{\mathbf{mul}}}$

When considering stochastic CML's like the ones introduced in Section 1.1.2, an operator governing the evolution of ensemble densities can be defined in analogy with the definition of the Perron-Frobenius operator of the previous section. The main difference in the derivation of this operator is that (11) does not hold since the system is no longer deterministic. This equality must be replaced by one which equates the expectation to be in a given preimage at a time $t$, with the expectation to be in the image at time $t + 1$. More precisely, we introduce an arbitrary bounded measurable function $\boldsymbol{h} : \mathbb{X} \longmapsto \mathbb{R}$ which can be written

$$\boldsymbol{h}(\mathbf{x}) = \prod_{i=1}^{N} h^{(i)}(x^{(i)}).$$

The expectation value of $\boldsymbol{h}(\mathbf{x}_{t+1})$ is given by

$$E(\boldsymbol{h}(\mathbf{x}_{t+1})) = \int_{\mathbb{X}} \boldsymbol{h}(\mathbf{x})\mathbf{f}_{t+1}(\mathbf{x})\, d\mathbf{x}. \tag{13}$$

In the additive noise case, we also have

$$\begin{aligned} E(\boldsymbol{h}(\mathbf{x}_{t+1})) &= E(\boldsymbol{h}(\Phi_{\mathrm{add}}(\mathbf{x}_t))) \\ &= \int_{\mathbb{X}}\int_{\mathbb{X}} \mathbf{f}_t(\mathbf{y}) \prod_{i=1}^{N} h^{(i)}(\Phi^{(i)}(\mathbf{y}) + z^{(i)}) g(z^{(i)})\, d\mathbf{z} d\mathbf{y}, \end{aligned} \tag{14}$$

while in the multiplicative case,

$$\begin{aligned} E(\boldsymbol{h}(\mathbf{x}_{t+1})) &= E(\boldsymbol{h}(\Phi_{\mathrm{mul}}(\mathbf{x}_t))) \\ &= \int_{\mathbb{X}}\int_{\mathbb{X}} \mathbf{f}_t(\mathbf{y}) \prod_{i=1}^{N} h^{(i)}(z^{(i)}\Phi^{(i)}(\mathbf{y})) g(z^{(i)})\, d\mathbf{z} d\mathbf{y}. \end{aligned} \tag{15}$$

Using (14) and (15) in conjunction with the right hand side of (13), one can obtain the explicit expression for the transfer operators governing the evolution of ensemble densities in CML's perturbed by additive or multiplicative noise. For CML's perturbed as in (5), the expression is [55]:

$$\mathcal{P}_{\Phi_{\mathrm{add}}}\mathbf{f}_t(\mathbf{x}) \equiv \mathbf{f}_{t+1}(\mathbf{x}) = \int_{\mathbb{X}} \mathbf{f}_t(\mathbf{y})\mathbf{g}(\mathbf{x} - \Phi(\mathbf{y}))\, d\mathbf{y}, \quad n = 0, 1, \cdots. \tag{16}$$

For CML's perturbed as in (6), we have [62]

$$\begin{aligned} \mathcal{P}_{\Phi_{\mathrm{mul}}}\mathbf{f}_t(\mathbf{x}) &\equiv \mathbf{f}_{t+1}(\mathbf{x}) \\ &= \int_{x^{(N)}}^{1} \cdots \int_{x^{(1)}}^{1} \mathbf{f}_t(\mathbf{y}) \prod_{i=1}^{N} \left[ g\left( \frac{x^{(i)}}{\Phi^{(i)}(\mathbf{y})} \right) \frac{1}{\Phi^{(i)}(\mathbf{y})}. \right] d\mathbf{y} \end{aligned} \tag{17}$$

It is not difficult to show (*cf.* [55, 62]) that the operators defined in (16) and (17) are Markov, and defined implicitly by stochastic kernels. [recall that $\mathcal{P}$ is a Markov operator if it is linear, and if for all probability densities $f$ it satisfies (1) $\mathcal{P}f \geq 0$ for $f \geq 0$, (2) $\|\mathcal{P}f\|_{L^1} = \|f\|_{L^1}$].

In Section 5.2, these observations are used to gain insight into the thermodynamic properties of the corresponding CML's. It is useful at this point to recall some basic notions which will be needed as we proceed.

### 3.3 Ergodicity, mixing and asymptotic periodicity

Here we discuss the behavior of the sequence of densities $\{\mathbf{f}_n\}$ which is intimately linked to the equilibrium and nonequilibrium statistical properties of the CML. For example, $\mathcal{T}$ is ergodic if and only if the sequence is *weak Cesàro convergent* to the invariant density $\mathbf{f}_*(\mathbf{x})$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{\mathbb{X}} \mathbf{f}_k(\mathbf{x})\mathbf{q}(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{X}} \mathbf{f}_*(\mathbf{x})\mathbf{q}(\mathbf{x})\, d\mathbf{x}, \quad \text{for all } \mathbf{q} \in L^\infty(\mathbb{X}),$$

and all initial probability densities $\mathbf{f}_0(\mathbf{x})$. A stronger (and more familiar) property, mixing, is equivalent to the *weak convergence* of the sequence to $\mathbf{f}_*$:

$$\lim_{n\to\infty} \int_{\mathbb{X}} \mathbf{f}_n(\mathbf{x})\mathbf{q}(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{X}} \mathbf{f}_*(\mathbf{x})\mathbf{q}(\mathbf{x})\, d\mathbf{x}, \quad \text{for all } \mathbf{q} \in L^\infty(\mathbb{X})$$

and all initial probability densities $\mathbf{f}_0(\mathbf{x})$. An even stronger type of chaotic behavior, known as *exactness* (or asymptotic stability) is reflected by the *strong convergence* of the sequence $\{\mathbf{f}_n\}$ to the invariant density $\mathbf{f}_*$:

$$\lim_{n\to\infty} \|\mathcal{P}_{\mathcal{T}}\mathbf{f}_n - \mathbf{f}_*\|_{L^1} = 0$$

for all initial probability densities $\mathbf{f}_0(\mathbf{x})$. Exactness implies mixing and is interesting from a physical point of view because it is the only one of the properties discussed so far which guarantees the evolution of the thermodynamic entropy of $\mathcal{T}$ to a global maximum, irrespective of the initial condition $\mathbf{f}_0$ [65].

The hierarchy of chaotic behaviors

$$\text{Exactness} \implies \text{Mixing} \implies \text{Ergodicity}$$

is discussed here because it is shown in Sections 4 and 5.2 that some deterministic and many stochastic CML's are either exact, or possess a another dynamical property, known as *asymptotic periodicity*, of which exactness is a special case.

Asymptotic periodicity is a property of certain Markov operators which ensures that the density sequence $\{\mathbf{f}_n\}$ converges strongly to a periodic cycle.

**Definition 1** (Asymptotic Periodicity) A Markov operator $\mathcal{P}$ is asymptotically periodic if there exist finitely many distinct probability density functions $v_1, \cdots, v_r$ with disjoint supports, a unique permutation $\gamma$ of the set $\{1, \cdots, r\}$ and positive linear continuous functionals $\Gamma_1, \cdots, \Gamma_r$, on $L_1(\mathbb{X})$ such that, for almost all initial densities $\mathbf{f}_0$,

$$\lim_{n\to\infty} \left\| \mathcal{P}^n \left( \mathbf{f}_0 - \sum_{i=1}^{r} \Gamma_i[\mathbf{f}_0] v_i \right) \right\|_{L^1} = 0 \tag{18}$$

and

$$\mathcal{P} v_i = v_{\gamma(i)}, \quad i = 1, \cdots, r.$$

Clearly, if $\mathcal{P}$ satisfies these conditions with $r = 1$, it is exact (or *asymptotically stable*). If $r > 1$ and the permutation $\gamma$ is cyclical, asymptotic periodicity also implies ergodicity [65]. The early papers discussing asymptotically periodic Markov operators are [52, 51, 56, 57, 53]. A somewhat more intuitive presentation is given in [54]. $\Xi$

*Remark* 2 The phase space density $\mathbf{f}_n$ of an AP system at any (large) time $n$ is a linear combination of "basis states" (denoted $v_i$ above) with disjoint supports, and at every time step the coefficients ($\Gamma_i$) of this linear combination are permuted by $\gamma$. Therefore, the density evolution in such systems is periodic, with a period bounded above by $r!$, but with the exact cycle depending on the initial preparation since the $\Gamma_i$'s are functionals of the initial density (*cf.* (18)). A direct consequence of asymptotic periodicity is that the thermodynamic equilibrium of the system consists in a sequence of metastable states which are visited periodically. It was shown in [54] that AP systems are ergodic if and only if the permutation $\gamma$ is cyclical.

# 4 Deterministic CML's

As mentioned in Section 2, there are numerous motivations for investigating the dynamics of coupled map lattices. When these dynamics are temporally and/or spatially chaotic, it is natural to turn to a probabilistic description in terms of the Perron-Frobenius operator. For example, there is now ample evidence [67, 60] that CML's can possess different phases which correspond to qualitatively different behavior of statistical averages, and it is legitimate to try and understand the connection between the presence of these different phases and the properties of the Perron-Frobenius operator. Before proceeding with the analysis, it is instructive to numerically illustrate this multi-phase phenomenology in a relatively simple toy model.

Consider a lattice of the form (2) with $S : [0:1] \longmapsto [0:1]$ given by the tent map $S(z) = \min(az, a(1-z))$, and with $p = 4$ (to mimick diffusive coupling in two spatial dimensions). As the slope of the local transformation is varied in the interval $(1, 2]$, for fixed $\varepsilon \in [0, 1]$, one observes a sequence of "bifurcations": on both sides of the bifurcation point, the lattice evolves chaotically in time, but the number of "bands", or simply connected subsets of $[0, 1]$ on which the activity of a site is supported, changes abruptly. This behavior is observed in the single tent map, where it can be shown to reflect a change in the degeneracy of the Perron-Frobenius operator's eigenvalues of unit modulus (*i.e.* a change in $r$ in (18); for a detailed discussion see [79, 106]). The extension of these results for one-dimensional maps to lattices with arbitrarily large numbers of elements has proven to be a major theoretical challenge, which has only been met in rather small regions of parameter space.

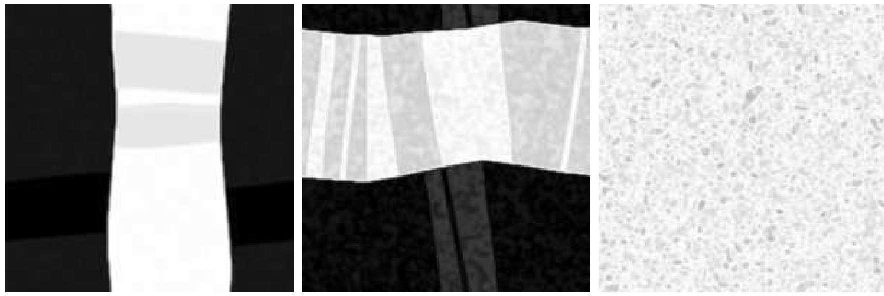
## 4.1 Phenomenology of the tent CML

Here we focus on the model (2) with local map defined by either the tent map, defined in the previous paragraph. For the tent map lattice two qualitatively different types of statistical behavior are evidenced in Figure 2. The first is characterized by the evolution of large scale patterns from the random initial conditions; this is the clustered, or ordered state $a = 1.1, \cdots, 1.5$. The panel $a = 1.3$ presents an interesting limiting case for which the "cluster" covers the entire area of the lattice; different initial conditions for such parameter values evolve to the more usual large scale patterns. Note that the lattices are not at equilibrium in the panels displayed in this figure. It is not possible to observe the true equilibrium because of the astronomically large transients typical of a lattice of 40000 elements. The point of our investigation is not to describe explicitly the presence, stability and asymptotic behavior of the patterns presented here, but to understand how the thermodynamics of these lattices should be investigated. Although the problem of pattern formation in CML's is fascinating, it is not the focus of our investigation, and we will therefore not spend more time discussing the pattern dynamics *per se*. The interesting observation from our point of view is that the pattern-forming behavior associated with small values of $a$ is also accompanied by statistical cycling in the lattice. This is illustrated by the behavior of various statistical quantifiers of the motion discussed below, rather than by the snapshots of Figure 2.

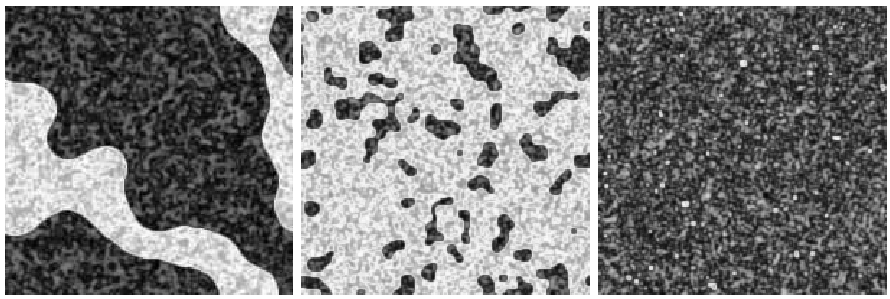
The second phase is described statistically by a unique invariant measure generated by almost all initial conditions. This corresponds to the spatiotemporally chaotic state described rigorously by Bunimovich and Sinai [9] in infinite lattices.

Before proceeding, we should note that this oscillatory behavior of macroscopic observable has also been observed in lattices of logistic maps, as well as in more complex, biologically motivated models [63]. In fact, in the recent literature [76, 14], this behavior has been referred to as *periodic collective behavior*, and understanding its origin in various spatially extended models is an on-going endeavour.

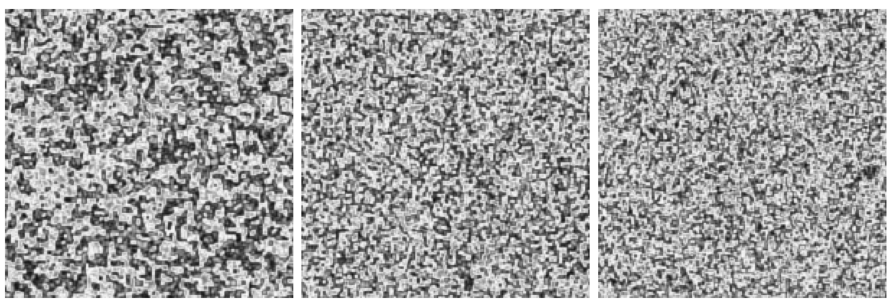
We propose as a possible mechanism that the Perron-Frobenius operator induced by lattices such as the tent CML in the statistically periodic regime are in fact asymptotically periodic and possess the cyclical spectral decomposition (18). At present, proving this statement is only possible in very limited cases, namely, in lattices perturbed by noise (as in Section 5), and in lattices of piecewise linear, expanding maps (see the contribution of Keller in this issue for more on this topic). Nevertheless, this working hypothesis is interesting because it provides some insight into the dynamics underlying periodic collective behavior (*cf* the last paragraph of the next section).



FROM LEFT TO RIGHT: $a = 1.1,\ a = 1.2,\ a = 1.3$



FROM LEFT TO RIGHT: $a = 1.4,\ a = 1.5,\ a = 1.6$



FROM LEFT TO RIGHT: $a = 1.7,\ a = 1.8,\ a = 1.9$

FIGURE 2:

*Snapshots of the activity at the surface of a* $200 \times 200$ *lattice of diffusively coupled tent maps when the coupling is constant (*$\varepsilon = 0.45$*) but the local slope is increased from* $a = 1.1$ *to* $a = 1.9$*. For all panels, the transient discarded is of length* $10^5$*. The* 256 *grey scales range from black when* $x^{i,j} = x_{min}$ *to white when* $x^{i,j} = x_{max}$ *where* $x_{min}$ *and* $x_{max}$ *are the lower and upper bounds of the attracting subinterval of* $[0, 1]$ *respectively. The initial values on the lattice were in all cases given by a random number generator yielding uniform distributions on the unit interval. The transition from statistical cycling to statistical equilibrium occurs between* $a = 1.5$ *and* $a = 1.6$ *for this value of the coupling. This observation is not made from Figure 2 but with the help of other the statistical quantifiers (cf. Figure 3 for example).*

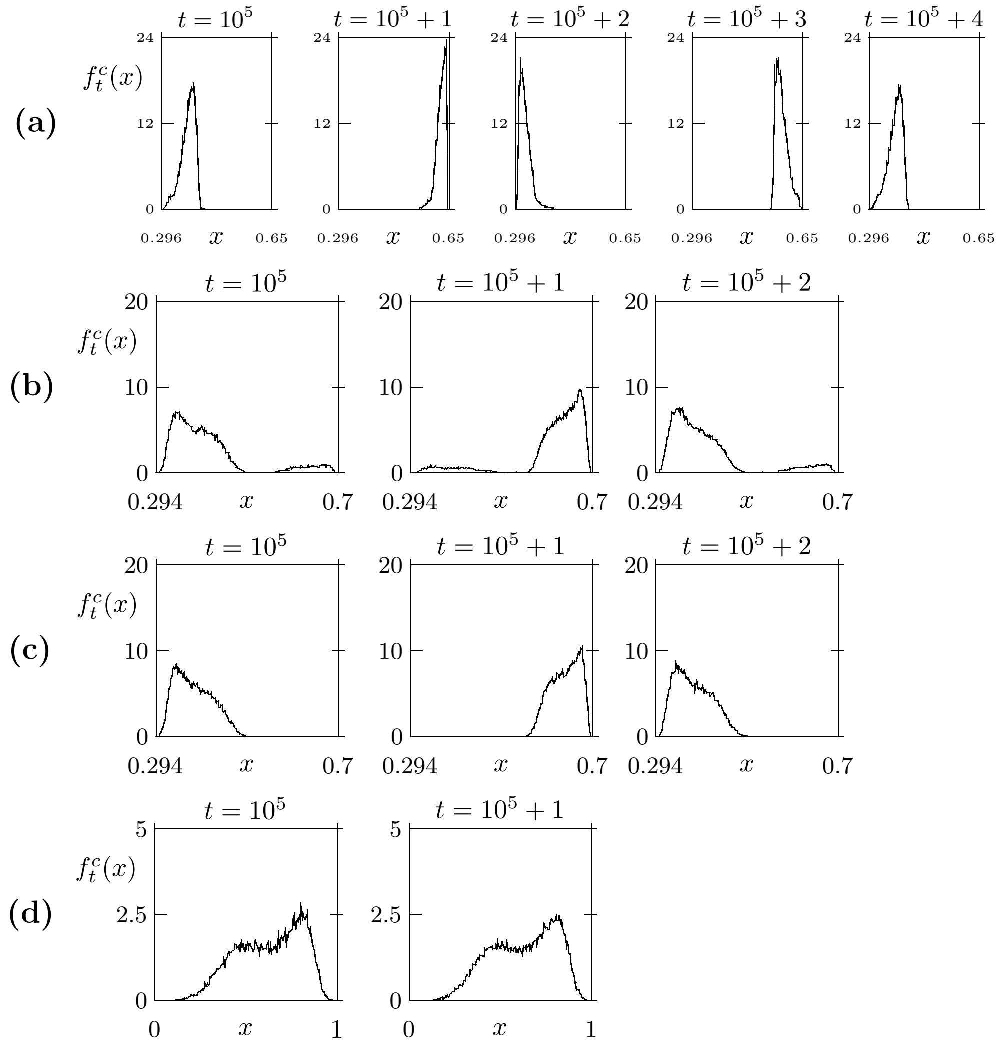



FIGURE 3:

*The collapsed density $f_t^c$ for a $200 \times 200$ lattice of diffusively coupled tent maps with $\varepsilon = 0.45$. The first $10^5$ iterations were discarded as transients. In a) the cycle is of period 4, and $a = 1.3$. The initial density was uniformly distributed on $[0.3 : 0.4]$. In b) the cycle is of period 2, $a = 1.4$ and the initial density was uniformly distributed on $[0 : 1]$. In c), the parameters are as in b) but the initial density was supported on $[0.39 : 0.43]$. This illustrates the dependence of the density cycle on the initial density. d) the slope of the map is $a = 1.99$ and the initial density is uniform on $[0 : 1]$. This density is numerically reached for all densities.*

### 4.2 Discussion

Until now, most attempts at constructing the statistical mechanics of high dimensional chaotic dynamical systems have followed two broad and intersecting paths. One is the extension of the so-called *thermodynamic formalism* of Ruelle [83], Bowen [5] and Sinai [86] to high dimensional hyperbolic dynamical systems, which has led to various proofs of existence of Gibbs measures describing spatio-temporal chaos in such models [9]. The other is an operator-theoretic approach which focuses on the Perron-Frobenius operator (11) as an operator acting on some suitably defined function space [6, 50]. The kinds of results that are sought are again the existence and uniqueness of invariant measures which are associated with fixed points of the Perron-Frobenius equation, and the properties of the system's relaxation to equilibrium when it is started out of equilibrium. Given that the Perron-Frobenius operator is a Markov operator, this information is given by its spectral properties. Obviously, these depend on the function space on which $\mathcal{P}_\Phi$ operates. We always consider here $\mathcal{P}_\Phi$ acting either on $L^1(\mathbb{X})$ ($\mathbb{X} \subset \mathbb{R}^n$), or on a subspace of $L^1(\mathbb{X})$. So far most investigations have actually focused on Banach spaces "properly" embedded in $L^1(\mathbb{X})$, where "properly" means here that one can then apply a now-famous theorem of Ionescu-Tulcea and Marinescu [94] to study the spectral properties of $\mathcal{P}_\Phi$ acting on the embedded space. Examples of such spaces are $BV(\mathbb{X})$, the space of functions of bounded variation (discussed in some detail in Chapter 5 of [16]), and the related $GH(\mathbb{X})$, the space of Generalized Hölder continuous functions (described in [64]).

If this second approach is followed, the objective is to place conditions on the parameters of the CML such that the conditions of the Ionescu-Tulcea and Marinescu theorem are satisfied. If one considers $\mathcal{P}_\Phi : BV(\mathbb{X}) \longmapsto BV(\mathbb{X})$, then it is possible to obtain conditions on the parameters of an expanding $\Phi$ such that $\mathcal{P}_\Phi$ satisfies (18) (see for example [28, 50, 61]). The weakness of this approach is that the conditions are usually extremely complicated, and they require very detailed geometrical knowledge of the transformation $\Phi$ (for example one needs to know a lot about the geometry of sets on which $\Phi$ is strictly monotone).

For a description of the operator-theoretic approach, and a discussion of its applications and limitations, the interested reader is referred to the contribution of Keller (this issue). One important step in this theoretical analysis is that the CML's which can be studied effectively with this operator-theoretic approach are product dynamical systems, which are close, in some clearly defined sense, to a direct product of independent low dimensional dynamical subsystems. This indicates that CML's which display periodic collective behavior, far from "synchronizing" in some loosely defined, and poorly understood manner, in fact *desynchronize* into a collection of statistically independent "clusters", each containing a few degrees of freedom. If this hypothesis holds, the statistical properties of a single CML's trajectory then approximate the ensemble properties of those low dimensional clusters. This in turn motivates the analysis of the cluster's Perron-Frobenius operator. The programme outlined in this paragraph, which is a direct consequence of the "cluster working hypothesis" is the subject of ongoing research [59].

We now turn our attention to the probabilistic description of CML's whose evolution is stochastically perturbed.

# 5 The statistics of stochastic CML's

The transfer operators for the stochastic CML's (5) and (6) were introduced in Section 3.2. The study of these operators is greatly simplified by the observation that they are Markov operators defined by stochastic kernels [54]. Before proceeding to their analysis, we will briefly describe some of the observed phenomenology in models like (5) and (6), since these are less frequently described than their deterministic counterparts in the literature.

## 5.1 Some numerical observations

Here we focus on the effects of additive noise on a piecewise linear toy model originally introduced by Keener [49]. The purpose is not to give an overview of the effects of stochastic perturbations on the dynamics of CML's, but to illustrate with a simple example that sometimes the presence of a little noise can have dramatic consequences which can, at first glance, seem rather counterintuitive.

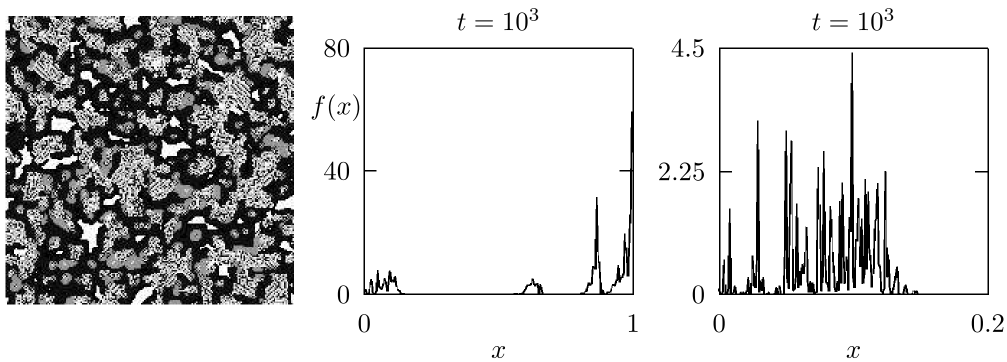



FIGURE 4:
*Left: Grey scale representation of the state of a $200 \times 200$ lattice of diffusively coupled Keener maps without noise, at time $t = 10^3$ with $a = 0.5$, $b = 0.571$ and $\varepsilon = 0.45$. The 256-level grey scale is such that if $x_{10^3}^{(kl)} = 0$, it is represented by a black pixel, while if $x_{10^3}^{(kl)} = 1$ it is represented by a white pixel. Center panel: 200-bin histogram of the state of the lattice displayed on the left panel. The fractal nature of the support of this distribution is suggested by the right panel, which was obtained from a larger lattice ($10^6$ sites) and a larger number of bins on $[0,1]$ ($10^3$).*

The example considered here is a perturbation of a two-dimensional lattice of diffusively coupled "Keener maps"

$$x_{t+1}^{(i,j)} = (1-\varepsilon)S(x_t^{(i,j)}) + \frac{\varepsilon}{4}\Big[S(x_t^{(i+1,j)}) + S(x_t^{(i-1,j)}) + S(x_t^{(i,j+1)}) + S(x_t^{(i,j-1)})\Big], \tag{19}$$

where the local map (a slight generalization of the r-adic map) was considered by Keener [49]

$$S(z) = (az + b) \bmod 1 \ , \quad a, b \in (0,1), \quad x \in [0,1]. \tag{20}$$

Before considering the dynamics of the lattice, it is useful to recall some basic properties of the single map. There exists a range of values for the parameters $a$ and $b$ such that the trajectories are chaotic in the sense that they attracted to a subset of $[0,1]$ of zero measure (a Cantor set) [49]. Numerically, this is reflected by the fact that if the histogram along a trajectory is constructed, the number of histogram peaks will increase as the bin size decreases. In this case, the Perron Frobenius operator does not possess a fixed point in the space of probability densities. In fact, it asymptotically transforms almost all initial probability densities into generalized functions. A rigorous treatment of such operators is possible, and studying the nonequilibrium statistical properties of the corresponding CML's involves the reformulation of the problem in terms of the evolution of measures. Figure 4 shows that the fractal nature of the attractor of the single map survives diffusive coupling.

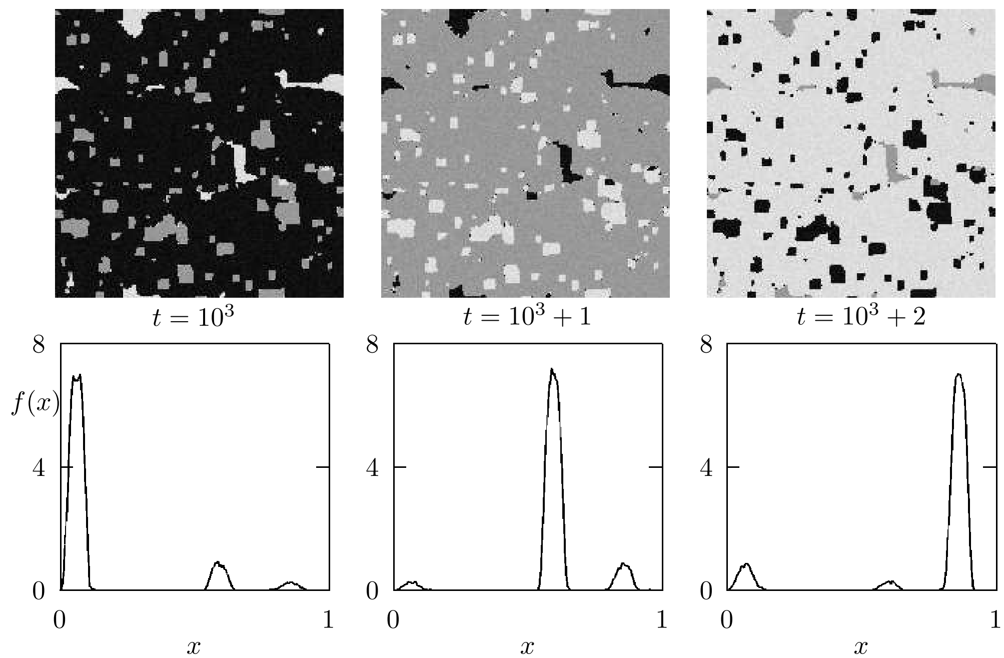



FIGURE 5:
*Noise induced statistical cycling in a lattice of* $200 \times 200$ *noisy "Keener maps" (21), with* $a = 0.5$, $b = 0.571$, $\varepsilon = 0.1$ *and* $\xi$ *uniformly supported on* $[0, 0.05]$. *The top panels display three successive iterations, and the bottom panels display the corresponding histograms (produced with 200 bins). The grey scale for the top row is the same as in Figure 4.*

This picture is greatly simplified when the local transformation is replaced by

$$S_\xi(z) = (az + b + \xi) \bmod 1 \ , \quad a, b \in (0,1), \quad x \in [0,1] \tag{21}$$

where $\xi$ is a random variable distributed with density $g$. Figure 5 displays the remarkable behavior of the lattice (19) when the map $S$ is replaced by $S_\xi$. Note that the noise present in (21) is multiplicative. The activity of the lattice no longer seems to be supported on a set of measure zero, and furthermore, it appears that the evolution of the histogram of activity on the lattice is periodic with period 3. To understand the origin of this simplification of the dynamics as a result of stochastic perturbations, one must focus on the properties of the transfer operators defined in Section 3.2.

## 5.2 Analytic results

In this section it is shown that under rather general circumstances, the transfer operators induced by stochastic coupled map lattices are asymptotically periodic.

### 5.2.1 Additive noise

Consider the CML (5), for which the density of the noisy perturbation satisfies

$$\mathbf{g}(\boldsymbol{\xi}) = \prod_{i=1}^{N} \chi_{[b,c]}(\xi^{(i)}), \quad 0 \le b < c \le 1, \tag{22}$$

where the indicator function $\chi$ is defined by $\chi_{[b,c]}(x) = (c-b)^{-1}$ if $x \in [b,c]$ and $\chi_{[b,c]}(x) = 0$ otherwise. This form for $g$ is chosen here to simplify the statement of the proof of Theorem 1 below, but more general forms can be treated in exactly the same way.

**Theorem 1** *If the CML $\Phi_{_{add}}$ is written in the form (5), where the density of the perturbation $\boldsymbol{\xi}$ is given by (22), and the local map $S$ of (3) is bounded and nonsingular then $\mathcal{P}_{\Phi_{add}}$ defined by (16) is asymptotically periodic.*

The proof consists in showing that $\mathcal{P}_{\Phi_{\rm add}}$ is a Markov operator defined by a stochastic Kernel which satisfies the conditions of theorem 5.7.2 of [55]. It is discussed in [62]. This is a general result. The two main assumptions which are necessary for its derivation are that $S$ be nonsingular and bounded. This generic nature of the result explains the ubiquitous presence of statistical cycling which has been reported in stochastic CML's elsewhere [62].

### 5.2.2 Multiplicative noise

Here the transformation $\Phi_{_{\rm mul}}$ is given by (6). In this section it is proved that multiplicative noise induces the spectral decomposition (18) in a large class of CML's. Our presentation is inspired by the treatment of one-dimensional maps perturbed by parametric noise given by Horbacz [31].

**Theorem 2** *Let $\mathbf{K} : \mathbb{X}\times\mathbb{X} \longmapsto \mathbb{R}$ be a stochastic kernel and $\mathcal{P}$ be the Markov operator defined by*

$$\mathcal{P}\mathbf{f}(\mathbf{x}) = \int_{x^{(N)}}^{1} \cdots \int_{x^{(1)}}^{1} \mathbf{K}(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{y})\, d\mathbf{y}. \tag{23}$$

*Assume that there is a nonnegative* $\lambda < 1$ *such that for every bounded* $\mathbb{B} \subset \mathbb{X}$ *there is a* $\delta = \delta(\mathbb{B}) > 0$ *for which*

$$\int_{\mathbb{A}} \mathbf{K}(\mathbf{x}, \mathbf{y})\, d\mathbf{x} \leq \lambda \qquad \textit{for } \mu(\mathbb{A}) < \delta, \quad y \in \mathbb{B}, \quad \mathbb{A} \subset \mathbb{B}. \tag{24}$$

*Assume further there exists a Lyapunov function* $\boldsymbol{V} : \mathbb{X} \longmapsto \mathbb{R}$ *such that*

$$\int_{\mathbb{X}} V(\mathbf{x}) \mathcal{P}\mathbf{f}(\mathbf{x})\, d\mathbf{x} \leq \alpha \int_{\mathbb{X}} V(\mathbf{x}) \mathbf{f}(\mathbf{x})\, d\mathbf{x} + \beta, \quad \alpha \in [0, 1), \ \beta > 0 \tag{25}$$

*for every density* $\mathbf{f}$*. Then* $\mathcal{P}$ *is asymptotically periodic, and therefore admits the representation (18).* [ *Recall that a nonnegative function* $V : \mathbb{X} \longmapsto \mathbb{R}$ *is known as a Lyapunov function if it satisfies* $\lim_{|\mathbf{x}| \to \infty} V(\mathbf{x}) = \infty$.]

The proof of the theorem is based on demonstrating that the operator defined by (23) is *contractive.* This property, in turn, was shown by Komorník [51] to imply asymptotic periodicity. The complete proof is given in [62].

The connection with stochastic CML's of the form (6) should now be clear: the operator $\mathcal{P}_{\Phi_{\text{mul}}}$ can easily be shown to satisfy the conditions of Theorem 2 under rather general circumstances. More precisely, we have the following corollary:

**Corollary 1** *A CML of the form (6), perturbed by the noise term* $\boldsymbol{\xi_t}$ *distributed with density (4) will induce a transfer operator* $\mathcal{P}_{\Phi_{mul}}$ *defined by (17). If the deterministic part of the transformation (denoted* $\Phi$*) is bounded and nonsingular, then* $\mathcal{P}_{\Phi_{mul}}$ *is asymptotically periodic.*

In light of this result, we can interpret the statistical cycling displayed in Figure 5 as an illustration of the cyclical spectral decomposition (18) of the transfer operator $\mathcal{P}_{\Phi_{\text{mul}}}$. In addition, the presence of asymptotic periodicity in this model, and in the large class of models which satisfy the conditions of Theorem 1 or Corollary 1, has some interesting applications for the construction of statistical mechanics for these models. Before exploring these, we briefly note that the results presented in this section do not allow us to predict the periodicity observed numerically in the evolution of histograms of activity (*cf.* Figure 5). The periodicity of these density cycles stems from stable period 3 orbits of the isolated Keener maps for certain values of the parameters (*cf* [80] for details).

# 6 Conclusion

This chapter has described the statistical dynamics of CML's by focusing on the properties of the transfer operators induced by these models. For deterministic CML's, the spectral characteristics of the Perron-Frobenius operator are investigated using some well-known bounded-variation techniques. When the CML's are perturbed by noise, the transfer operators are Markov and defined by stochastic kernels. This allows us to treat them using some basic results of the theory of Markov operators, and we show that in many cases of interest they are asymptotically periodic.

Asymptotic periodicity is an intriguing dynamical property which has several important implications for the construction of statistical mechanics for these high dimensional dynamical systems. The first one is that when the period of the density cycle in (18) is greater than one, the asymptotic ensemble statistics of the CML depend on the initial ensemble. This is due to the dependence of the functionals $\Gamma_i$ on $\mathbf{f}_0$ in (18), and it generalizes the usual dependence of trajectory dynamics on the initial conditions, to the evolution of ensemble probability densities. Another consequence of the presence of AP is the possible presence of phase transitions in the system: If the period of a density cycle changes as a control parameter is tuned, then the model undergoes a qualitative change in the behavior of its statistical quantifiers.

More importantly, some of the usual misconceptions concerning the true meaning of ergodicity are exacerbated when supposed consequences of ergodicity are violated by systems which turn out to be asymptotically periodic. Observations of the coherent behavior of globally coupled and some locally coupled CML's reported by Kaneko [43, 44] and Perez *et al.* [76] have led to a controversy in the recent literature concerning an apparent violation of the law of large numbers in these models. In fact, since the law of large numbers is a theorem, it cannot be violated, but its verification for CML's must performed with care. As explained by Pikovsky and Kurths [77], it is important when considering this law for ergodic systems which are non mixing, to compute the relevant averages with respect to ensemble densities, and not with respect to densities constructed from trajectories. Even if a system is ergodic, the two constructions will not in general be equivalent when it comes to verifying the law of large numbers. This is because the type of convergence to equilibrium guaranteed by ergodicity (*cf.* Section 3.3) is not strong enough to imply the equality of the two types of averages (trajectory *vs.* ensemble) when the system is started out of equilibrium. In most circumstances this would seem like a technical mathematical objection, not of great relevance to the practicing physicist, because one would nevertheless expect the system to relax to a state described by the invariant density. But if it is asymptotically periodic, a system will almost surely *not* converge to equilibrium, and in that case, the verification of the law of large numbers must necessarily be performed with ensemble densities.

# 7 Acknowledgments

The authors would like to thank Prof. G. Keller (Mathematics, Erlangen), and Prof. G. Nicolis (Chimie Physique, Bruxelles) for fruitful discussions and constructive criticism. J. Losson acknowledges support from the EEC and from *Fonds FCAR* (Québec). M.C. Mackey acknowledges support from NSERC (Canada) and the *Fonds FCAR*.

# References

[1] Andersen, P. and S. A. Andersson. *Physiological basis of the* alpha *rhythm.* Appleton Century Crofts, New York, 1968.

[2] Aranson, I., D. Golomb, and H. Sompolinski. Spatial coherence and temporal chaos in macroscopic systems with asymmetrical coupling. *Phys. Rev. Lett.*, **68**:3495–3498, 1992.

[3] Barkley, D.. A model for fast computer simulation of waves in excitable media. *Physica D*, **49**:61–70, 1991.

[4] Biferale, L., A. Crisanti, M. Falcioni, and A. Vulpiani. One-dimensional asymmetrically coupled maps with defects. *J. Phys. A*, **26**:L923–L928, 1993.

[5] Bowen, R.. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Springer, New York, 1975. Lecture notes in mathematics **470**.

[6] Bricmont, J. and A. Kuppiainen. Coupled analytic maps. , , 1995. Preprint.

[7] Brooks, M.J. and B.K.P. Horn. Shape and source from shading. In *Proc. Int. Joint Conf. Artificial Intel.*, pages 932–936, 1985.

[8] Brooks, M.J. and B.K.P. Horn. *Shape from shading.* MIT Press, Cambridge, MA, 1989.

[9] Bunimovich, L.A. and Ya.G. Sinai. Spacetime chaos in coupled map lattices. *Nonlinearity*, **1**:491–516, 1988.

[10] Caianiello, E.R.. Outline of a theory of thought processes and thinking machines. *J. Theor. Biol.*, **2**:204–235, 1961.

[11] Chaté, H. and P. Manneville. Transition to turbulence *via* spatiotemporal intermittency. *Phys. Rev. Lett.*, **58**:112–115, 1987.

[12] Chaté, H. and P. Manneville. Spatio-temporal intermittency in coupled map lattices. *Physica D*, **32**:409–422, 1988.

[13] Chaté, H. and P. Manneville. Structure of clusters generated by spatiotemporal intermittency and directed percolation in two space dimensions. *Phys. Rev. A*, **38**:4351–4354, 1988.

[14] Chaté, H. and P. Manneville. Collective behaviors in spatially extended systems with local interactions and synchronous updating. *Prog. Theor. Phys.*, **87**:1–60, 1992.

[15] Cocho, G. and G. Martínez-Mekler. On a coupled map lattice formulation of the evolution of genetic sequences. *Physica D*, **51**:119–130, 1991.

[16] Cocho, G., A. Gelover Santiago, and G. Martínez-Mekler. An interplay between local and global dynamics in biological networks: The case of genetic sequences. In N. Boccara, E. Goles, S. Martínez, and P. Picco, editors, *NATO ASI Series C (Vol. 396): Cellular Automata and Cooperative Systems*, pages 85–100. Kluwer, 1993.

[17] Cocho, G., A. Gelover Santiago, G. Martínez-Mekler, and A. Rodin. Nonlinear modeling of the AIDS virus genetic sequence evolution. *Int. J. Modern Phys. C*, **5**:321–324, 1994.

[18] Cosenza, M.G. and R. Kapral. Coupled maps on fractal lattices. *Phys. Rev. A.*, **46**:1850–1858, 1992.

[19] Cosenza, M.G. and R. Kapral. Spatiotemporal intermittency on fractal lattices. *CHAOS*, **4**:99–104, 1994.

[20] Deissler, R.J.. One dimensional strings, random fluctuations and complex chaotic structures. *Phys. Lett. A*, **100**:451–454, 1984.

[21] Deissler, R.J.. Spatially growing waves, intermittency and convective chaos in an open-flow system. *Physica D*, **25**:233–260, 1987.

[22] Denman, E.D.. Coupled mode theory. *J. Math. Anal. Appl.*, **21**:242–248, 1968.

[23] Destexhe, A.. *Aspect nonlinéaire de l'activité rythmique du cerveau.* PhD thesis, Universié Libre de Bruxelles, March 1992.

[24] Kaneko, K. (Editor), editor. *Theory and applications of coupled map lattices.* John Wiley & Sons, New York, 1993.

[25] Fabiny, L. and K. Wiesenfeld. Clustering behavior of oscillator arrays. *Phys. Rev. A*, **43**:2640–2648, 1991.

[26] Franke, J.E. and A.-A. Yakubu. Extinction in systems of bobwhite quail type populations. *Can. Appl. Math. Quart.*, 1994. In press.

[27] Golomb, D. and J. Rinzel. Clustering in globally coupled inhibitory neurons. *Physica D*, **72**:259–282, 1994.

[28] Góra, P. and A. Boyarski. Absolutely continous invariant measures for piecewise expanding $C^2$ transformations in $\mathbb{R}^n$. *Isr. J. Math.*, **67**:272–285, 1989.

[29] Hadley, P. and M.R. Beasley. Dynamical states and stability of linear arrays of Josephson junctions. *Appl. Phys. Lett.*, **50**:621–623, 1987.

[30] Herz, A.V.M.. Global analysis of recurrent neural networks. In E. Domany, J. L. van Hemmen, and K. Schulten, editors, *Models of Neural Networks,* Volume 3, page 44 pages. To appear in 1994. We refer more specifically to the discussion of Section 3.4.

[31] Horbacz, K.. Dynamical systems with multiplicative perturbations. *Ann. Polon. Math.*, **50**:93–102, 1989.

[32] Houlrik, J.M. and I. Webman. Critical behaviour and mean-field theory of coupled map lattices. *Physica Scripta*, **33**:189–192, 1989.

[33] Houlrik, J.M., I. Webman, and M.H. Jensen. Mean-field theory and critical behavior of coupled-map lattices. *Phys. Rev. E*, **41**:4210–4222, 1990.

[34] Ivanov, A.F. and A. N. Sharkovskiĭ. Oscillations in singularly perturbed delay equations. *Dynamics Reported*, **3**:165, 1991. Springer Verlag (edited by H. O. Walter and U. Kirchgraber).

[35] James, G.E., E. M. Harrell II, and R. Roy. Intermittency and chaos in intracavity doubled lasers. *Phys. Rev. A*, **41**:2778–2790, 1990.

[36] Jensen, M.H.. Boundary layer instability in a coupled map model. *Physica D*, **32**:203–207, 1989.

[37] Jensen, M.H.. Fluctuations and scaling in a model of boundary-layer-induced turbulence. *Phys. Rev. Lett.*, **62**:1361–1363, 1989.

[38] Kaneko, K.. Editor of a CHAOS focus issue on coupled map lattices, **2**:279-367 (1992).

[39] Kaneko, K.. Period doubling of kink-antikink patterns, quasiperiodicity of antiferro-like structures and spatial intermittency in coupled logistic lattice - Toward a prelude of a field-theory of chaos. *Prog. Theor. Phys.*, **72**:480–486, 1984.

[40] Kaneko, K.. Lyapunov analysis and information flow in coupled map lattices. *Physica D*, **23**:436–447, 1986.

[41] Kaneko, K.. Pattern dynamics in spatio-temporal chaos. *Physica D*, **34**:1–41, 1989.

[42] Kaneko, K.. Towards a thermodynamics of spatiotemporal chaos. *Prog. Theor. Phys. Supplement*, **99**:263–287, 1989.

[43] Kaneko, K.. Globally coupled chaos violates the law of large numbers but not the Central-Limit Theorem. *Phys. Rev. Lett.*, **65**:1391–1394, 1990.

[44] Kaneko, K.. Mean field fluctuations of a network of chaotic elements. *Physica D*, **55**:368–384, 1992.

[45] Kaneko, K. and T. Ikegami. Homeochaos: dynamic stability of a symbiotic network with population dynamics and evolving mutation rate. *Physica D*, **56**:406–429, 1992.

[46] Kapral, R.. Chemical waves and coupled map lattices. In K. Kaneko, editor, *Theory and applications of coupled map lattices*, pages 135–168. J. Wiley & Sons, 1993.

[47] Kapral, R., R. Livi, G. L. Oppo, and A. Politi. Dynamics of complex interfaces. *Phys. Rev. E*, **49**:2009–2022, 1994.

[48] Keeler, J.D. and J. D. Farmer. robust space-time intermittency and $1/f$ noise. *Physica D*, **23**:413–435, 1986.

[49] Keener, J.P.. Chaotic behavior in piecewise continuous difference equations. *Trans. Amer. Math. Soc.*, **261**:589–604, 1980.

[50] Keller, G. and M. Künzle. Transfer operators for coupled map lattices. *Ergod. Theor. Dyn. Sys.*, **12**:297–318, 1992.

[51] Komornik, J.. Asymptotic periodicity of the iterates of weakly contrictive Markov operators. *Tohoku Math. J.*, **38**:15–27, 1986.

[52] Komornìk, J. and A. Lasota. Asymptotic decomposition of Markov operators. *Bull. Polish Acad. Sci.*, **35**:321–327, 1987.

[53] Lasota, A., T. Y. Li, and J. A. Yorke. Asymptotic periodicity of the iterates of Markov operators. *Trans. Am. Math. Soc.*, **286**:751–764, 1984.

[54] Lasota, A. and M. C. Mackey. *Chaos, Fractals and Noise: Stochastic Aspects of Dynamics*. Springer Verlag, New York, 1994.

[55] Lasota, A. and M.C. Mackey. Noise and statistical periodicity. *Physica D*, **28**:143–154, 1987.

[56] Lasota, A. and J. Socała. Asymptotic properties of constrictive Markov operators. *Bull. Polish Acad. Sci.*, **35**:71–76, 1987.

[57] Lasota, A. and J. A. Yorke. Statistical periodicity of Markov operators. *Časopis pro pěstováni matematiky*, **111**:1–11, 1986. February 1986 issue.

[58] Levine, H. and W.N. Reynolds. Coupled map lattice techniques for simulating interfacial phenomena in reaction diffusion systems. *CHAOS*, **2**:337–342, 1992.

[59] Losson, J. and H. Chaté. Nontrivial collective behavior in coupled map lattices: A Perron-Frobenius perspective. *Physica D*, , 1995. In Press.

[60] Losson, J. and M.C. Mackey. Statistical cycling in coupled map lattices. *Phys. Rev. E*, **50**:843–856, 1994.

[61] Losson, J. and M.C. Mackey. Coupled map lattices as models of deterministic and stochastic differential delay equations. *Phys. Rev. E*, **52**:115–128, 1995.

[62] Losson, J. and M.C. Mackey. The evolution of probability densities in stochastic coupled map lattices. *Phys. Rev. E.*, **52**:1403–1417, 1995.

[63] Losson, J., J.G. Milton, and M.C. Mackey. Phase transitions in networks of chaotic elements with short and long range interactions. *Physica D*, **81**:177–203, 1995. An important *erratum* regarding this article can be obtained from the first author.

[64] Loviscach, J.. *Probabilistic models of chaotic maps.* PhD thesis, Universität Bielefeld, July 1993.

[65] Mackey, M.C.. *Time's Arrow: The Origin of Thermodynamic Behavior.* Springer-Verlag, Berlin, 1992.

[66] Manneville, P.. *Dissipative structures and weak turbulence.* Academic Press, San Diego, CA, 1990.

[67] Miller, J. and D.A. Huse. Macroscopic equilibrium from microscopic irreversibility in a chaotic coupled-map lattice. *Phys. Rev.*, **48E**:2528–2535, 1993.

[68] Milton, J., T. Mundel, U. an der Heiden, J.P. Spire, and J. Cowan. Travelling activity waves, 1994. Preprint, University of Chicago.

[69] Milton, J.G.. Private communication.

[70] Milton, J.G. and J.Bélair. Chaos, noise and extinction in models of population growth. *Theor. Pop. Biol.*, **37**:273–290, 1990.

[71] Molgedey, L., J. Schuchhardt, and H. G. Schuster. Suppressing chaos in neural networks by noise. *Phys. Rev. Lett.*, **69**:3717–3719, 1992.

[72] Nichols, S. and K. Wiesenfeld. Mutually destructive fluctuations in globally coupled arrays. *Phys. Rev. E*, **49**:1865–1868, 1994.

[73] Nozawa, H.. A neural network model as a globally coupled map and applications based on chaos. *CHAOS*, **2**:377–386, 1992.

[74] Oono, Y. and S. Puri. Study of phase separation dynamics by use of cell dynamical systems. 1. Models. *Phys. Rev. A*, **38**:434–453, 1988.

[75] Parekh, N. and S. Puri. Velocity selection in coupled-map lattices. *Phys. Rev. E*, **47**:1415–1418, 1993.

[76] Perez, G., S. Sinha, and A. Cerdeira. Order in the turbulent phase of globally coupled maps. *Physica D*, **63**:341–349, 1993.

[77] Pikovsky, A.. Do globally coupled maps really violate the law of large numbers. *Phys. Rev. Lett.*, **72**:1644–1646, 1994.

[78] Price, C.B., P. Wambacq, and A. Oosterlinck. The plastic coupled map lattice: A novel image-processing paradigm. *CHAOS*, **2**:351–366, 1992.

[79] Provatas, N. and M. C. Mackey. Asymptotic periodicity and banded chaos. *Physica D*, **53**:295–318, 1991.

[80] Provatas, N. and M. C. Mackey. Noise-induced asymptotic periodicity in a piecewise linear map. *J. Stat. Phys.*, **63**:585–612, 1991.

[81] Puri, S., R.C. Desay, and R. Kapral. Coupled map model with a conserved order parameter. *Physica D*, **50**:207–230, 1991.

[82] Redheffer, R.M.. Difference equations and functional equations in transmission line theory. In *Modern mathematics for the engineer (Chapter XII).* McGraw-Hill, New York, 1961.

[83] Ruelle, D.. *Thermodynamic formalism* in Encyclopedia of Mathematics and its Applications. Addison-Wesley, New York, 1978.

[84] Schuster, H.G., editor. *Nonlinear dynamics and Neural Networks.* VCH, Weinheim, 1991.

[85] Silber, M., L. Fabiny, and K. Wiesenfeld. Stability results for in-phase and splay-phase states of solid-state laser arrays. *J. Opt. Soc. Am. B*, **10**:1121–1129, 1993.

[86] Sinai, Ya.G.. Construction of Markov partitions. *Funct. Anal. Appl.*, **2**:245–2, 1968.

[87] Solé, R.V., J. Bascompte, and J. Valls. Nonequilibrium dynamics in lattice ecosystems: Chaotic stability and dissipative structures. *CHAOS*, **2**:387–395, 1992.

[88] Solé, R.V. and J. Valls. Order and chaos in a 2d Lotka-Volterra coupled map lattice. *Phys. Lett. A*, **153**:330–336, 1991.

[89] Sompolinsky, H., H. J. Sommers, and A. Crisanti. Chaos in random neural networks. *Phys. Rev. Lett.*, **61**:259–262, 1988.

[90] Steriade, M and M. Deschênes. The thalamus as a neuronal oscillator. *Brain Res. Rev.*, **8**:1–63, 1984.

[91] Swift, J., S. H. Strogatz, and K. Wiesenfeld. Averaging of globally coupled oscillators. *Physica D*, **55**:239–250, 1992.

[92] Tandri, S. and Y. H. Yang. Comparison of two shape-from-shading algorithms. *Patt. Recogn. Lett.*, **11**:637–642, 1990.

[93] Torcini, A., A. Politi, G. P. Pucioni, and G. D'Alessandro. Fractal dimension of spatially extended systems. *Physica D*, **53**:85–101, 1991.

[94] Ionescu, C.T. Tulcea and G. Marinescu. Théorie ergodique pour des classes d'opérations non complètement continues. *Ann. de Math.*, **52**:140–147, 1950.

[95] Turing, A.M.. Chemical basis of morphogenesis. *Phil. Trans. Roy. Soc. Lond. B*, **237**:37–72, 1952.

[96] Vandermeer, J.. The qualitative behavior of coupled predator-prey oscillations as deduced from simple circle maps. *Ecological modelling*, **73**:135–148, 1994.

[97] Vega, O.E. and Y. H. Yang. Shading logic: A heuristic approach to recover shape from shading. *IEEE Trans. Patt. Anal. Machine Intel.*, **15**:592–597, 1993.

[98] Waller, I. and R. Kapral. Spatial and temporal structures in systems of coupled nonlinear oscillators. *Phys. Rev. A*, **30**:2047–2055, 1984.

[99] Waller, I. and R. Kapral. Synchronization and chaos in coupled nonlinear oscillators. *Phys. Lett. A*, **105**:163–168, 1984.

[100] Wiesenfeld, K. and P. Hadley. Attractor crowding in oscillator arrays. *Phys. Rev. Lett.*, **62**:1335–1338, 1989.

[101] Willeboordse, F.H. and K. Kaneko. Bifurcations and spatial chaos in an open flow model. *Phys. Rev. Lett.*, **73**:533–536, 1994.

[102] Wilson, H.R. and J. D. Cowan. Excitatory and inhibitory interactions in localized populations of model neurons. *Biophysical J.*, **12**:1–24, 1972.

[103] Wilson, H.R. and J. D. Cowan. A theory of the functional dynamics of cortical and thalamic nervous tissue. *Kybernetik*, **13**:55–79, 1973.

[104] Wright, J.J. and R. R. Kydd. The electroencephalogram and cortical neural networks. *Network*, **3**:341–362, 1992.

[105] Yanagita, T.. Coupled map lattice model for boiling. *Phys. Lett. A*, **165**:405–408, 1992.

[106] Yoshida, T., H. Mori, and H. Shigematsu. Analytic study of chaos of the tent map: Band structures, power spectra and critical behaviors. *J. Stat. Phys.*, **31**:279, 1983.

[107] Yu, N., R.K. DeFreez, D. J. Bossert, R. A. Elliott, H. G. Winful, and D. F. Welch. Observation of sustained self-pulsation in CW operated flared Y-coupled laser arrays. *Electron. Lett*, **24**:1203–1204, 1988.

[108] Ziemer, W.P.. *Weakly differentiable functions*. Springer Verlag, Berlin, 1989.

*J. Losson, Service de Chimie Physique, Université Libre de Bruxelles,*
*Campus Plaine C.P.231, Bld. du Triomphe, Bruxelles 1050*
*M.C. Mackey, Center for Nonlinear Dynamics and Departments of Physiology,*
*Physics and Mathematics, McGill University,*
*3655 Drummond, Montréal H3G-1Y6 Canada*