DERIVATION OF AN EVOLUTION EQUATION FOR THE DENSITY OF A TEST PARTICLE SUBJECT TO PERTURBATIONS BY A POISSON PROCESS NOTES PREPARED FOR PIOTR GARBACZEWSKI VERSION OF 8 SEPTEMBER, 1992 CORRECTIONS 14 DECEMBER, 1993 FILE: GARBACZ.TEX

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We consider a dynamical system with dependent variables (x, p) evolving according to the dynamics

$$\begin{aligned} \frac{dx}{dt} &= \mathcal{F}(x,p) \\ \frac{dp}{dt} &= \mathcal{G}(x,p). \end{aligned} \tag{1}$$

[One could think of a position x and momentum p for concreteness.] We assume in a short time interval Δt, with probability $\varphi(x,p)\Delta t$ there is a perturbation $f(x,p)$ to $\mathcal{F}$ and $g(x,p)$ to $\mathcal{G}$. To capture the essence of this scheme, we use an Euler approximation for (1) and write

$$\begin{aligned} x(t+\Delta t) &\simeq x(t) + \mathcal{F}(x(t),p(t))\Delta t + f(x(t),p(t))[y(t+\Delta t) - y(t)] \\ p(t+\Delta t) &\simeq p(t) + \mathcal{G}(x(t),p(t))\Delta t + g(x(t),p(t))[y(t+\Delta t) - y(t)], \end{aligned} \tag{2}$$

where the distribution of $z \equiv y(t+\Delta t) - y(t)$ is given by

$$\Phi_{xp}(dz) = \begin{cases} 1 & \text{with probability} \quad \varphi(x,p)\Delta t \\ 0 & \text{with probability} \quad 1 - \varphi(x,p)\Delta t. \end{cases} \tag{3}$$

If we examine an ensemble of test particles with dynamics described by (1) subject to these perturbations, then we wish to find the evolution equation for the density $u(t,x,p)$ defined by

$$\text{prob}\{x(t) \in \mathcal{X}, p(t) \in \mathcal{P}\} = \int_X \int_Y u(t,x,p)\,dx\,dp.$$

The following derivation of this evolution equation is an extension of a similar derivation by A. Lasota carried out in April, 1990.

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Let $h(x,p) \in C_0^2(R)$ be an arbitrary function with compact support. The expected value of $h(x(t+\Delta t), p(t+\Delta t))$ is given by

$$E_{\Delta t} \equiv E(h(x(t+\Delta t), p(t+\Delta t))) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,p)u(t+\Delta t, x, p)dxdp \tag{4}$$

so

$$E_0 \equiv E(h(x(t), p(t)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,p)u(t,x,p)dxdp. \tag{5}$$

Defining

$$\begin{aligned} Q(x,p,z) &= x + \mathcal{F}(x,p)\Delta t + f(x,p)z \\ R(x,p,z) &= p + \mathcal{G}(x,p)\Delta t + g(x,p)z \end{aligned} \tag{6}$$

it is clear that we may also write

$$\begin{aligned} E_{\Delta t} &= E\left(h(Q(x(t),p(t),z), R(x(t),p(t),z))\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(Q(x,p,z),R(x,p,z))u(t,x,p)\Phi_{xp}(dz)dxdp. \end{aligned} \tag{7}$$

Using the properties of the distribution Φ_{xp}, equation (7) can be rewritten in the form

$$\begin{aligned} E_{\Delta t} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(Q(x,p,0),R(x,p,0))u(t,x,p)[1-\varphi(x,p)\Delta t]dxdp \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(Q(x,p,1),R(x,p,1))u(t,x,p)\varphi(x,p)\Delta t dxdp \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(Q(x,p,0),R(x,p,0))u(t,x,p)dxdp \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{h(Q(x,p,1),R(x,p,1)) - h(Q(x,p,0),R(x,p,0))\} \\ &\qquad u(t,x,p)\varphi(x,p)\Delta t dxdp. \end{aligned} \tag{8}$$

Note that to $\mathcal{O}(\Delta t^2)$ we may write

$$\begin{aligned} h(Q(x,p,0),R(x,p,0)) &\simeq h(x+\mathcal{F}(x,p)\Delta t, p+\mathcal{G}(x,p)\Delta t) \\ &\simeq h(x,p) + \mathcal{F}(x,p)\frac{\partial h}{\partial x}\Delta t + \mathcal{G}(x,p)\frac{\partial h}{\partial p}\Delta t, \end{aligned} \tag{9a}$$

while

$$\begin{aligned} h(Q(x,p,1),R(x,p,1)) &\simeq h(x+\mathcal{F}(x,p)\Delta t + f(x,p), p+\mathcal{G}(x,p)\Delta t + g(x,p)) \\ &\simeq h(x+f,p+g) + \mathcal{F}(x,p)\frac{\partial h(x+f,p+g)}{\partial(x+f)}\Delta t \\ &\quad + \mathcal{G}(x,p)\frac{\partial h(x+f,p+g)}{\partial(p+g)}\Delta t. \end{aligned} \tag{9b}$$

Inserting the approximations (9a,b) into equation (8), we have

$$E_{\Delta t} \simeq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ h(x,p) + \mathcal{F}(x,p)\frac{\partial h}{\partial x}\Delta t + \mathcal{G}(x,p)\frac{\partial h}{\partial p}\Delta t \right\} u(t,x,p)dxdp + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[h(x+f,p+g) - h(x,p)\right] u(t,x,p)\varphi(x,p)\Delta t dxdp. \tag{10}$$

Equating (4) and (10), we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,p)u(t+\Delta t,x,p)dxdp = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ h(x,p) + \mathcal{F}(x,p)\frac{\partial h}{\partial x}\Delta t + \mathcal{G}(x,p)\frac{\partial h}{\partial p}\Delta t \right\} u(t,x,p)dxdp + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[h(x+f,p+g) - h(x,p)\right] u(t,x,p)\varphi(x,p)\Delta t dxdp \tag{11}$$

Rearranging the terms in (11), dividing through by Δt, taking the limit as $\Delta t \to 0$ in the result yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,p)\frac{\partial u}{\partial t}dxdp + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \mathcal{F}(x,p)\frac{\partial h}{\partial x} + \mathcal{G}(x,p)\frac{\partial h}{\partial p} \right\} u(t,x,p)dxdp = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[h(x+f,p+g) - h(x,p)\right] u(t,x,p)\varphi(x,p)dxdp \tag{12}$$

Using integration by parts on the left hand side of (12), and remembering that h has compact support, we arrive at

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,p)\left\{ \frac{\partial u}{\partial t} + \frac{\partial(\mathcal{F}u)}{\partial x} + \frac{\partial(\mathcal{G}u)}{\partial p} \right\} dxdp = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[h(x+f,p+g) - h(x,p)\right] u(t,x,p)\varphi(x,p)dxdp \tag{13}$$

We are almost there! All we have to do is change the variables in the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x+f,p+g)u(t,x,p)\varphi(x,p)dxdp.$$

Define new variables $v = x + f(x,p)$ and $w = p + g(x,p)$ so the pair (v,w) is given by the transformation $(v,w) = T(x,p)$. Assume that T is invertible so $(x,p) = T^{-1}(v,w)$, and denote the Jacobian of T^{-1} by $J^{-1}(v,w)$. Then we can write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x+f,p+g)u(t,x,p)\varphi(x,p)dxdp = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(v,w)u(t,T^{-1}(v,w))\varphi(T^{-1}(v,w))J^{-1}(v,w)dvdw. \tag{14}$$

From (14) and the fact that the function h was arbitrary, it is immediate from equation (13) that u satisfies the evolution equation

$$\frac{\partial u}{\partial t} + \frac{\partial(\mathcal{F}u)}{\partial x} + \frac{\partial(\mathcal{G}u)}{\partial p} = - u(t,x,p)\varphi(x,p) + u(t,T^{-1}(x,p))J^{-1}(x,p)\varphi(T^{-1}(x,p)). \quad (15)$$

To proceed to investigate equation (15), assume for simplicity that the perturbations f and g are both independent of x and p. Then our evolution equation (15) takes the form

$$\frac{\partial u}{\partial t} + \frac{\partial(\mathcal{F}u)}{\partial x} + \frac{\partial(\mathcal{G}u)}{\partial p} = -u(t,x,p)\varphi(x,p) + u_{f,g}(t,x,p)\varphi_{f,g}(x,p), \quad (16)$$

where we have used the notation $u_{f,g}(t,x,p) = u(t,x-f,p-g)$. Assume further that the pair (f,g) is distributed with density $\sigma(f,g)$. Multiplying (16) by $\sigma(f,g)$, and integrating we obtain

$$\frac{\partial u}{\partial t} + \frac{\partial(\mathcal{F}u)}{\partial x} + \frac{\partial(\mathcal{G}u)}{\partial p} = -u\varphi + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_{f,g}\varphi_{f,g}\sigma(f,g)df dg. \quad (17)$$

Example 1. To proceed further, expand the product $u(t,x-f,p-g)\varphi(x-f,p-g)$ about the point(x,p) to give

$$u_{f,g}(t,x,p)\varphi_{f,g}(x,p) = u(t,x,p)\varphi(x,p) + f\frac{\partial\left[u(t,x,p)\varphi(x,p)\right]}{\partial x} + g\frac{\partial\left[u(t,x,p)\varphi(x,p)\right]}{\partial p}$$
$$+ \frac{1}{2}\left\{f^2\frac{\partial^2\left[u(t,x,p)\varphi(x,p)\right]}{\partial x^2} + fg\frac{\partial^2\left[u(t,x,p)\varphi(x,p)\right]}{\partial x\partial p} + g^2\frac{\partial^2\left[u(t,x,p)\varphi(x,p)\right]}{\partial p^2}\right\} + \cdots . \quad (18)$$

Inserting the expansion (18) into (17) and carrying out the indicated integrations we obtain

$$\frac{\partial u}{\partial t} + \frac{\partial(\mathcal{F}u)}{\partial x} + \frac{\partial(\mathcal{G}u)}{\partial p} =< f > \frac{\partial\left[u\varphi\right]}{\partial x} + < g > \frac{\partial\left[u\varphi\right]}{\partial p}$$
$$+ \frac{1}{2}\left\{< f^2 > \frac{\partial^2\left[u\varphi\right]}{\partial x^2} + < fg > \frac{\partial^2\left[u\varphi\right]}{\partial x\partial p} + < g^2 > \frac{\partial^2\left[u\varphi\right]}{\partial p^2}\right\} + \cdots , \quad (19)$$

where

$$< f^n >= \int_{-\infty}^{\infty} f^n\sigma(f,g)df dg$$

and the other moments are defined in an obvious way.