DERIVING THE SOLUTIONS OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS 27 OCTOBER, 1992 FILE:SOLNSPDE.TEX

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Using the techniques outlined in Kamke (*Differentialgleichungen Lösungsmethoden und Lösungen*, Chelsa Publishing Company, New York, 1974) it is easy to find the solutions to first order partial differential equations.

Homogeneous Equations. Consider the homogeneous equation

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(t, x_1, \dots, x_n) \frac{\partial u}{\partial x_k} = 0 \tag{1}$$

with the associated initial function

$$u(0, x_1, \dots, x_k) = v(x_1, \dots, x_k), \tag{2}$$

so we have a *Cauchy problem*. Associated with this equation we have n *characteristic equations*

$$\frac{d\bar{x}_k}{dt} = a_k(t, \bar{x}_1, \dots, \bar{x}_n). \tag{3}$$

Then $u(t, x_1, \dots, x_n)$ is a solution of equation 1 *if and only if* it is possible for the relation

$$u(t, \bar{x}_1, \dots, \bar{x}_n) = \text{Constant} \tag{4}$$

to be satisfied for some time t.

Example 1. Consider the equation

$$\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = 0 \tag{5}$$

with the initial condition

$$u(0, x) = v(x) \tag{6}$$

so $n = 1$ in equation 1.

The single characteristic equation is

$$\frac{dx}{dt} = x, \tag{7}$$

with the solution

$$x(t) = C_x e^t. \tag{8}$$

Applying condition 4 we must have

$$u(t, x) = \text{Constant} \tag{9}$$

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for some time t. Since we have information about the situation at $t = 0$ from the initial condition, pick $t = 0$ so

$$u(t = 0, x) = u(0, C_x) \equiv v(C_x) = u(t, x) \tag{10}$$

so our Constant in condition 4 is simply $v(C_x)$. However, note that we can rewrite C_x as

$$C_x = xe^{-t} \tag{11}$$

so the general solution of equations 1 and 2 is simply

$$u(t, x) = v(xe^{-t}). \tag{12}$$

It is easy to verify that this is a solution by differentiating with respect to both t and x and substituting back into equation 5.

Nonhomogeneous Equations. Now consider the nonhomogeneous equation

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(t, x_1, \ldots, x_n, u)\frac{\partial u}{\partial x_k} = f(t, x_1, \ldots, x_n, u) \tag{13}$$

with the associated initial function

$$u(0, x_1, \ldots, x_k) = v(x_1, \ldots, x_k). \tag{14}$$

Associated with this problem we have the $n + 1$ dimensional set of characteristic equations

$$\begin{aligned} \frac{dx_k}{dt} &= a_k(t, x_1, \ldots, x_n, z) \qquad k = 1, \ldots, n \\ \frac{dz}{dt} &= f(t, x_1, \ldots, x_n, z). \end{aligned} \tag{15}$$

Now, $u(t, x_1, \ldots, x_n)$ is a solution of equation 13 *if and only if*

$$u(t, x_1, \ldots, x_n) = z(t) \tag{16}$$

is satisfied for some time t.

Example 2. Consider the system

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \lambda u \tag{17}$$

with the initial condition

$$u(0, x) = v(x) \tag{18}$$

and the associated characteristic equations

$$\begin{aligned} \frac{dx}{dt} &= 1 \\ \frac{dz}{dt} &= \lambda z. \end{aligned} \tag{19}$$

The solutions to the characteristic equations are

$$x = t + C_x \tag{20a}$$

and

$$z(t) = C_z e^{\lambda t} \tag{20b}$$

so condition (16) becomes

$$u(t, x) \equiv u(t, t + C_x) = C_z e^{\lambda t}. \tag{21}$$

Now from the initial condition we have, at $t = 0$, that

$$u(0, C_x) = C_z = v(C_x) \tag{22}$$

so it is immediate that

$$u(t, x) = v(C_x)e^{\lambda t} \tag{23}$$

However, from (20a) we have also that $C_x = x - t$ so the general solution of equation 17 is given by

$$u(t, x) = v(x - t)e^{\lambda t} \tag{24}$$