NOTES ON THE POSSIBILITY OF RETARDON SOLUTIONS IN A DELAYED PARTIAL DIFFERENTIAL EQUATION

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Earlier, Rey and Mackey ["Bifurcations and traveling waves in a delayed partial differential equation", Chaos (1992) *2*, 231-244 and "Multistability and boundary layer development in a transport equation with delayed arguments", Can. Appl. Math. Quar. (1993) *1*, 1-21], Mackey and Rudnicki ["Global stability in a delayed partial differential equation describing cellular replication", submitted to J. Math. Biol.], Crabb, Losson, and Mackey ["Solution multistability in differential delay equations", Proc. Inter. Conf. Nonlin. Anal. (Tampa Bay, 1993, in press)] and Crabb, Mackey, and Rey [in preparation] have considered the solution behaviour of an unusual equation describing cellular replication and maturation.

Specifically, we studied the solution behaviour of

$$\frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} = -\delta u + \lambda u_\tau (1 - u_\tau), \tag{1}$$

where $u_\tau \equiv u(t-\tau, xe^{-r\tau})$, with the initial condition

$$u(t-\tau, xe^{-r\tau}) \equiv \varphi(x) \quad \text{for} \quad (t,x) \in [0,\tau] \times [0,1]. \tag{2}$$

Thus, the model is a nonlinear first order partial differential equation for the cell density $u(t,x)$ in which there is retardation in both temporal (t) and maturation variables (x), and contains three parameters.

We found that the solution behaviour depends on the initial function $\varphi(x)$ and a four component parameter vector $P = (\delta, \lambda, r, \tau)$. One of these parameters can eliminated by judicious scaling, so the parameter space reduces to three dimensions. For strictly positive initial functions, $\varphi(0) > 0$, there are three homogeneous solutions of biological (*i.e.*, nonnegative) importance: a trivial solution $u_t \equiv 0$, a positive stationary solution u_{st}, and a time periodic solution $u_p(t)$. For $\varphi(0) = 0$ there are a number of different asymptotic $(t \to \infty)$ solution types depending on P: the trivial solution u_t, a spatially inhomogeneous stationary solution $u_{nh}(x)$, a spatially homogeneous singular solution u_s, a traveling wave solution $u_{tw}(t,x)$, slow traveling waves $u_{stw}(t,x)$ and slow traveling chaotic waves $u_{scw}(t,x)$. We delineated the regions of parameter space in which these solutions exist and are locally stable, and studied them numerically. Though we never observed soliton like solutions to (1), both Rey and I suspected that they might well exist.

On Friday, 10 September, 1993, David Campbell (University of Illinois, Urbana) was giving a summary talk about the state of soliton theory at the Grand Finale of the Chaos, Order, and Patterns NATO programme in Como, Italy. He mentioned very interesting work on the behaviour of "compactons", specifically that of Rosenau and Hyman ["Compactons: Solitons with finite wavelength", Phys. Rev. Lett. (1993) *70*, 564-567], and I began to seriously think about the possibility of demonstrating soliton like solutions analytically in an equation analogous to (1). These notes indicate that such solutions, called retardons, may exist.

Specifically, first consider the modified equation

$$\frac{\partial u}{\partial t} + r\frac{\partial u}{\partial x} = \lambda\sqrt{u_\tau (1 - u_\tau)}, \tag{3}$$

where $u_\tau \equiv u(t-\tau, x)$. The changes between (3) and (1) are that we have a constant velocity r instead of rx, the loss term $-\delta u$ has been eliminated, and the nonlinearity on the right hand side has been modified as well as the definition of u_τ so that now there is a retardation in only the temporal variable without the nonlocal effect in the spatial variable. We take $(t,x) \in R^+ \times R$ and for the time being we will leave the initial function unspecified.

How should we deal with (3)? To start with, make the ansatz

$$u(t,x) \equiv \eta(x+rt) = \eta(\xi), \tag{4}$$

so equation (3) may be written in the alternate form

$$\frac{d\eta}{d\xi} = \frac{\lambda}{2r}\sqrt{\eta(\xi - r\tau)[1-\eta(\xi - r\tau)]}. \tag{5}$$

If we make the assumption that η is an $r\tau$ periodic function, i.e., $\eta(\xi) \equiv \eta(\xi - r\tau)$, then it is easy to obtain two particular solutions to (5):

$$\eta(\xi) = \begin{cases} \sin^2\left(\frac{\lambda}{4r}\xi\right) & 0 < \frac{\lambda}{4r}\xi < \pi \\ \cos^2\left(\frac{\lambda}{4r}\xi\right) & -\frac{\pi}{2} < \frac{\lambda}{4r}\xi < \frac{\pi}{2}, \end{cases} \tag{6}$$

whenever the parameters λ and τ satisfy

$$\lambda\tau = 4\pi m. \tag{7}$$

with m an integer (either positive or negative).

Remembering that $\xi = x + rt$, we thus have particular solutions to (3), namely

$$u(t,x) = \begin{cases} \sin^2\left[\frac{\lambda}{4r}(x+rt)\right] & 0 < x + rt < \frac{4\pi r}{\lambda} \\ \cos^2\left[\frac{\lambda}{4r}(x+rt)\right] & -\frac{2\pi r}{\lambda} < x + rt < \frac{2\pi r}{\lambda}. \end{cases} \tag{8}$$

These particular solutions to (3) have constant amplitude, and propagate at a speed r. We refer to them as **retardons**. Note that from (8) retardons are different from many soliton types in that they have compact support, as do compactons. However, there are a number of outstanding questions that I am just starting to think about:

(1) For what class of initial functions

$$u(t-\tau, x) \equiv \varphi(t', x') \quad \text{for} \quad (t', x') \in [0,\tau] \times R \tag{9}$$

will retardon solutions like (9) be reached, if from any? Will different classes of retardons result from generically different types of initial functions?

(2) How would the *numerical* retardon solutions of (3) behave? Will retardons collide elastically? Will collisions between retardons give birth to new retardons, as is the case with compactons?

28 November, 1993. In an attempt to provide an answer for question 1 above, consider the following.

Suppose we simply view equation (3)

$$\frac{\partial u}{\partial t} + r\frac{\partial u}{\partial x} = \lambda\sqrt{u_\tau(1-u_\tau)} \tag{3}$$

as (rather blindly, and following our nose–which means I don't know if the following is valid) a first order partial differential equation with an initial function

$$u(t', x') \equiv \varphi(t', x') \quad \text{for} \quad (t', x') \in [0,\tau] \times R. \tag{9}$$

If we try to solve this system (3) and (9) using the method of characteristics we have

$$\frac{dx}{dt} = r \implies x = rt + \mathcal{C}_x \implies \mathcal{C}_x = x - rt, \tag{10}$$

and

$$\frac{dz}{dt} = \sqrt{z_\tau(1 - z_\tau)}. \tag{11}$$

Again, if z is a τ periodic function so $z(t) \equiv z(t-\tau)$, then the solution of the second characteristic equation (11) is given by

$$z(t) = \mathcal{C}_z \begin{cases} \sin^2\left(\frac{\lambda}{2}t\right) & 0 < \frac{\lambda}{2}t < \pi \\ \cos^2\left(\frac{\lambda}{2}t\right) & -\frac{\pi}{2} < \frac{\lambda}{2}t < \frac{\pi}{2}. \end{cases} \tag{12}$$

In order to get a general solution to the whole problem, we must find a time $\bar{t}$ such that $u(\bar{t}, x) = z(\bar{t})$. For the $sin^2(\lambda t/2)$ solution in (12), pick $\bar{t} = 0$ so

$$u(0, x) = \varphi(0, x) = \varphi(0, \mathcal{C}_x) = \varphi(0, x - rt) = \mathcal{C}_z.$$

Thus in this case we conclude that the solution is $u(t, x) = \varphi(0, x - rt)sin^2(\lambda t/2)$. However, for this to satisfy the original problem (3), we must have $\varphi \equiv 1$. Exactly the same conclusion holds for the other case, which indicates that the solution will be independent of x and only arrived at with the special initial condition that is uniform for $(t', x) \in [-\tau, 0] \times R$. Something is screwy here. I think that some numerical work needs to be done to see if there is anything of substance in these ideas before proceeding analytically. Maybe A. Rey is interested?