Recurrent questions in statistical mechanics

Harvey R. Brown

Faculty of Philosophy, University of Oxford, Oxford, U.K.

Michael C. Mackey

Departments of Physiology, Physics & Mathematics, McGill University, 3655 Promenade Sir William Osler, Montreal, QC, CANADA, H3G 1Y6

Abstract

WRITE AN ABSTRACT WHEN DONE

Keywords:

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1. What does probability mean in physics?

This is something that HB is concerned about, specifically questions about the validity of equating probability with frequencies.

MCM doe not understand exactly what he means, nor what is concerning him.

2. Which entropy? Boltzmann, Gibbs, or other?

3. The issue of temporal change: Why does it happen?

Added is the issue of why there seems to be a universally preferred direction.

HB cites the case of the Boltzmann equation in which apparently fully time symmetric invariant laws give rise to temporally changing behaviours.

MCM says that this is because of the fact that in the derivation of the Boltzmann equation from the Liouville equation an irreversible assumption was snuck in when the collision integral is written. I maintain that this is the case in the "molecular chaos" assumption because in writing the two particle

Email addresses: harvey.brown@philosophy.ox.ac.uk (Harvey R. Brown), michael.mackey@mcgill.ca (Michael C. Mackey)

distributions $f^2(x,v,t)$ for the pre-collisional and post-collisional situation, call them $f^2_{pre}(x,v,t)$ and $f^2_{post}(x,v,t)$ the molecular chaos assumption is invoked (Chapman and Cowling, 1970; Cercignani, 1975) stating that

$$f^2_{pre}(x,v,t) = f^{1,a}_{pre}(x,v,t) f^{1,b}_{pre}(x,v,t) \tag{1}$$

and

$$f^2_{post}(x,v,t) = f^{1,a}_{post}(x,v,t) f^{1,b}_{post}(x,v,t). \tag{2}$$

4. Is the direction of time an issue (except for aging questioners)?

5. The issue of reversibility for Markov operators

Consider the stochastic perturbed differential equation

$$\frac{dx_i}{dt} = F_i(x) + \sum_{j=1}^{d} \sigma_{ij}(x)\xi_j, \qquad i = 1, \dots, d \tag{3}$$

with the initial conditions $x_i(0) = x_{i,0}$, where $\sigma_{ij}(x)$ is the amplitude of the stochastic perturbation and $\xi_j = \dfrac{dw_j}{dt}$ is a "white noise" term that is the derivative of a Wiener process. In matrix notation we can rewrite Eq. 3 as

$$dx(t) = F(x(t))dt + \Sigma(x(t))\, dw(t), \tag{4}$$

where $\Sigma(x) = [\sigma_{ij}(x)]_{i,j=1,\dots,d}$. Here it is always assumed that the Itô, rather that the Stratonovich, calculus, is used. For a discussion of the differences see Horsthemke and Lefever (1984), Lasota and Mackey (1994) and Risken (1984). In particular, if the σ_{ij} are independent of x then the Itô and the Stratonovich approaches yield identical results.

The *Fokker-Planck equation* that governs the evolution of the density function $f(t,x)$ of the process $x(t)$ generated by the solution to the stochastic differential equation (4) is given by

$$\frac{\partial f}{\partial t} = -\sum_{i=1}^{d} \frac{\partial [F_i(x) f]}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 [a_{ij}(x) f]}{\partial x_i \partial x_j} \tag{5}$$

where

$$a_{ij}(x) = \sum_{k=1}^{d} \sigma_{ik}(x)\sigma_{jk}(x).$$

If $k(t, x, x_0)$ is the fundamental solution of the Fokker-Planck equation, i.e. for every x_0 the function $(t, x) \mapsto k(t, x, x_0)$ is a solution of the Fokker-Planck equation with the initial condition $\delta(x-x_0)$, then the general solution $f(t, x)$ of the Fokker-Planck equation (5) with the initial condition

$$f(x, 0) = f_0(x)$$

is given by

$$f(t, x) = \int k(t, x, x_0) f_0(x_0)\, dx_0. \tag{6}$$

From a probabilistic point of view $k(t, x, x_0)$ is a stochastic kernel (transition density) and describes the probability of passing from the state x_0 at time $t = 0$ to the state x at a time t. Define the Markov operators P^t by

$$P^t f_0(x) = \int k(t, x, x_0) f_0(x_0)\, dx_0, \quad f_0 \in L^1. \tag{7}$$

Then $P^t f_0$ is the density of the solution $x(t)$ of Eq. 4 provided that f_0 is the density of $x(0)$.

The steady state density $f_*(x)$ is the stationary solution of the Fokker Planck Eq. (5):

$$-\sum_{i=1}^{d} \frac{\partial [F_i(x) f]}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 [a_{ij}(x) f]}{\partial x_i \partial x_j} = 0. \tag{8}$$

If the coefficients a_{ij} and F_i are sufficiently regular so that a fundamental solution k exists, and $\int_X k(t, x, y)\, dx = 1$, then the unique generalized solution (6) to the Fokker-Planck equation (5) is given by Eq. 7.

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