QUANTUM MECHANICS AND HIDDEN VARIABLES
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MICHAEL C. MACKEY

Centre for Nonlinear Dynamics in Physiology and Medicine
and
Departments of Physiology, Physics and Mathematics
McGill University
Montreal, Quebec, Canada

When I was a student, I was very disturbed by quantum mechanics. Even more disturbing to me was the fact that neither my professor nor the other students seemed to have any difficulty with it. I later learned that I was not alone.

Almost from its very inception, the formalism and interpretation of quantum mechanics spawned a number of attempts to find alternative formulations that had interpretations more in accord with the notions of reality that physicists were accustomed to from their study of macroscopic behaviour (see Jammer [1989, **The Conceptual Development of Quantum Mechanics**, AIP; and 1974, **The Philosophy of Quantum Mechanics**, Wiley] for an excellent account of the history of this subject). Some of these came wearing the clothes of so called "hidden variables" theories.[1] These are interesting because of their connection with the operation of taking a trace of a dynamical system, and because they illustrate that taking a trace of a reversible system may not automatically lead to entropy evolution as time changes.

A few years ago I happened to read two papers written by the physicist David Bohm [Physical Review (1952a,b), **85**, 166-193] that excited me because they provided a simple example of a hidden variables theory that was completely consistent with the predictions of quantum mechanics. In my opinion, the original work of Bohm is one of the most interesting of the existing hidden variables theories (there are many) because of its simplicity, and because it was the first widely known[2] clear counter example that led to the discovery of the inapplicability of the famous "proof" by von Neumann (1932) that hidden variable representations of quantum mechanics were impossible.

Following Bohm[3], we consider the non-relativistic Schrödinger equation for a single particle with position x and mass m moving in a potential $V(x)$ (the argument carries through for many particle systems)

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi. \tag{1}$$

Bohm postulated that the wave function ψ is an objectively real quantity that is to be thought of as a field, satisfying a field equation (1) just like the electromagnetic field satisfies Maxwell's equations. Since ψ is generally complex, we let the amplitude of ψ be $\sqrt{f}$ and its phase be $\mathcal{S}$ (both f and $\mathcal{S}$ are real quantities) so

$$\psi = \sqrt{f}e^{i\mathcal{S}/\hbar}, \tag{2}$$

with $\psi^* = \sqrt{f}e^{-i\mathcal{S}/\hbar}$ and $\psi\psi^* = f$.

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[1] To my knowledge, the most penetrating analysis of hidden variable theories is by Belinfante (1973),**A Survey of Hidden-Variables Theories**, Pergamon Press, Oxford.

[2] de Broglie's "pilot wave" hypothesis in his 1927 thesis is mathematically identical with Bohm's theory, though with differing interpretations on various points.

[3] Bohm's hidden variable theory is what Belinfante calls a hidden variable theory of the first kind, *i.e.*, those in which deviations from the predictions of quantum mechanics will only occur in a non-equilibrium situation, whereas von Neumann's "proof" only applies to hidden variable theories of the zeroth kind.

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It is a simple series of calculations to show that f and $\mathcal{S}$ satisfy the pair of coupled partial differential equations

$$\frac{\partial f}{\partial t} + \nabla \cdot \left(f \frac{\nabla \mathcal{S}}{m} \right) = 0 \tag{3}$$

and

$$\frac{\partial \mathcal{S}}{\partial t} + \frac{(\nabla \mathcal{S})^2}{2m} + V(x) + V_Q(x, f) = 0, \tag{4}$$

where

$$V_Q = -\frac{\hbar^2}{2m} \frac{\nabla^2(\sqrt{f})}{\sqrt{f}}. \tag{5}$$

How should we interpret this rewriting of the Schrödinger equation? Bohm reasoned as follows. $\mathcal{S}$ is the solution of equation (4), and in the "classical" limit of $\hbar \to 0$ this equation is precisely the Hamilton-Jacobi equation for a particle moving in a potential V. If one considers a large number of particles moving according to (4), then from classical mechanics $\nabla \mathcal{S}/m$ is the velocity $v(x)$ of a particle at the point x. When $\hbar \neq 0$, Bohm interpreted (4) as the Hamilton–Jacobi equation for a single particle moving in a combined potential consisting of the classical potential V plus a quantum potential V_Q given by (5).

Bohm took the classical results as a justification for identifying the velocity v with $\nabla \mathcal{S}$ through the relation

$$v = \frac{\nabla \mathcal{S}}{m}, \tag{6}$$

thereby explicitly assuming that even in a quantum mechanical situation particles are real objects (not manifestations of wave functions that collapse upon measurement) and we can attach objective meaning to both particle position x and the wave function ψ.

With (6), equations (3) and (4) can be rewritten in the form

$$\frac{\partial f}{\partial t} + \nabla \cdot (f v) = 0 \tag{7}$$

and

$$\frac{\partial \mathcal{S}}{\partial t} + \frac{1}{2} m v^2 + V(x) + V_Q(x, f) = 0. \tag{8}$$

Mathematically, $f = \psi\psi^*$ is a density since $f \geq 0$ and $\int \psi\psi^* \, dx = \int f \, dx = 1$. The fact that (7) is of the form of a classical conservation, or continuity, equation for the density, much like the Liouville equation, was used by Bohm to motivate his final major assumption that $f = \psi\psi^*$ is a physical probability density of an ensemble of particles. Precisely the same point of view was taken by Schrödinger [1978, **Wave Mechanics**, Chelsea Pub. Co., New York] in his original development of the Schrödinger equation.

It is worth noting here that if, in analogy with the classical situation, we set

$$\mathcal{H} = \frac{1}{2} m v^2 + V + V_Q \tag{9}$$

then (8) is equivalent to

$$\frac{\partial \mathcal{S}}{\partial t} = -\mathcal{H} \tag{10}$$

or to

$$\begin{aligned} \frac{dx}{dt} &= v = \frac{1}{m} \frac{\partial \mathcal{H}}{\partial v} \\ m \frac{dv}{dt} &= \frac{d}{dt} \nabla \mathcal{S} = -\nabla \mathcal{H}. \end{aligned} \tag{11}$$

Noting further that

$$\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) = -v \cdot \nabla \mathcal{H} = -\frac{d}{dt}(V + V_Q),$$

we have

$$\mathcal{H} = \frac{1}{2} m v^2 + V + V_Q = \text{constant}, \tag{12}$$

so $\mathcal{H}$ plays the role of a conserved energy for the system.

Equations (11) are clearly reversible (and even of Hamiltonian form), so from the results of Mackey [1991, **Time's Arrow: The Origins of Thermodynamic Behavior**, Springer-Verlag, New York], if we identify $f(t,x) = P^t f(x)$ then it is clear that the conditional entropy H_c satisfies

$$H_c(P^t f | f_*) \equiv H_c(f | f_*)$$

for all initial densities. That there is no entropy evolution in quantum systems described by the Schrödinger equation is well known, and has been proved by Wehrl [Reviews of Modern Physics (1978), **50**, 221-260] using different techniques.

It is perhaps unfortunate that Bohm's work has acquired the label of a hidden variable theory. Since he is attaching objective significance to both particle position and the wave function, what are the "hidden variables" in Bohm's reinterpretation of quantum mechanics? It seems that there are two. First, the Schrödinger equation only deals with particle position x as the velocity v does not enter. Secondly, and more importantly, is the appearance of the quantum potential V_Q in Bohm's interpretation. Since equations (7) and (8) differ from their classical counterparts only in the appearance of V_Q, any attempt to deduce quantum mechanics from some more comprehensive theory must explain the origin of the quantum potential. We will return to this point in Chapter 11 when we discuss Nelsons derivation of the Schrödinger equation.

The quantum potential V_Q is a strange bird for at least two reasons.

(1) V_Q is quite different from any form of potential function we are accustomed to in classical physics because of its dependence on the probability density function $f(t,x)$ at all points in configuration space, and its lack of explicit diminution as the distance between a point and a particle increases. The former is the origin of the "nonlocal" nature of quantum mechanics, with its apparent conflict with relativistic theory, though Bohm and Hiley (1984) have argued that this conflict is more apparent than real.

(2) The second peculiar aspect of the quantum potential V_Q arises through the way in which it depends on the probability density function u, viz.

$$V_Q \sim \frac{\nabla^2(\sqrt{f})}{\sqrt{f}}.$$

From this it is clear that the quantum potential at a particular point in space is independent of the magnitude of the probability density function at that point, depending only on the shape of f.