DERIVATION OF A BOLTZMANN-LIKE EQUATION FOR PARTICLES SUBJECT TO MULTIPLE COLLISIONS
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In reviewing the notes of 18 November, 1993 ["Density evolution for dynamics perturbed by a Poisson process", Lasota and Mackey, File: POISSON.TEX], it would appear that a straightforward extension of the procedure there will allow us to derive an interesting Boltzmann like equation for the density evolution in a gas whose particles suffer multiple collisions. Thus, we do not make the customary assumption that there are only binary collisions, but rather take the approach that in a given short interval of time a specific particle may suffer collisions with an integer number of particles between 0 and N. Each time there is a collision, there may be alterations in either the momentum and/or force. For convenience, the derivation is carried out for a one dimensional gas operating in a two dimensional phase space of position and momentum, but the extension to the customary six dimensional phase space should be straightforward.

We consider a dynamical system with dependent variables (x, p), where x is the position and p is the momentum, evolving according to the dynamics

$$\begin{aligned} \frac{dx}{dt} &= \mathcal{V}(x,p) \\ \frac{dp}{dt} &= \mathcal{F}(x,p). \end{aligned} \tag{1}$$

We assume in a short time interval Δt, with probability $\varphi(x,p)\Delta t$ there is a perturbation $f(x,p)$ to $\mathcal{V}$ and $g(x,p)$ to $\mathcal{F}$. To capture the essence of this scheme, we use an Euler approximation for (1) and write

$$\begin{aligned} x(t+\Delta t) &\simeq x(t) + \mathcal{V}(x(t),p(t))\Delta t + f(x(t),p(t))[y(t+\Delta t) - y(t)] \\ p(t+\Delta t) &\simeq p(t) + \mathcal{F}(x(t),p(t))\Delta t + g(x(t),p(t))[y(t+\Delta t) - y(t)], \end{aligned} \tag{2}$$

where the distribution of $z \equiv y(t+\Delta t) - y(t)$ is given by

$$\Phi_{xp}(dz) = \begin{cases} k & \text{with probability} \quad \varphi_k(x,p)\Delta t, \quad k = 1, \dots, N \\ 0 & \text{with probability} \quad \varphi_0(x,p)\Delta t = 1 - \sum_{k=1}^{N} \varphi_k(x,p)\Delta t. \end{cases} \tag{3}$$

If we examine an ensemble of test particles with dynamics described by (1) subject to these perturbations, then we wish to find the evolution equation for the density $u(t,x,p)$ defined by

$$\text{prob}\{x(t) \in \mathcal{X}, p(t) \in \mathcal{P}\} = \int_X \int_Y u(t,x,p)\,dx\,dp.$$

Derivation of the Boltzmann Equation.

Let $h(x,p) \in C_0^2(R)$ be an arbitrary function with compact support. The expected value of $h(x(t+\Delta t), p(t+\Delta t))$ is given by

$$E_{\Delta t} \equiv E(h(x(t+\Delta t), p(t+\Delta t))) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,p) u(t+\Delta t, x, p)\,dx\,dp \tag{4}$$

so

$$E_0 \equiv E(h(x(t), p(t))) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,p) u(t,x,p)\,dx\,dp. \tag{5}$$

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Defining

$$\begin{aligned} Q(x,p,z) &= x + \mathcal{V}(x,p)\Delta t + f(x,p)z \\ R(x,p,z) &= p + \mathcal{F}(x,p)\Delta t + g(x,p)z \end{aligned} \tag{6}$$

it is clear that we may also write

$$\begin{aligned} E_{\Delta t} &= E\left(h(Q(x(t),p(t),z),R(x(t),p(t),z))\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(Q(x,p,z),R(x,p,z))u(t,x,p)\Phi_{xp}(dz)dxdp. \end{aligned} \tag{7}$$

Using the properties of the distribution Φ_{xp}, equation (7) can be rewritten in the form

$$\begin{aligned} E_{\Delta t} = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(Q(x,p,0),R(x,p,0))u(t,x,p)\left[1-\sum_{k=1}^{N}\varphi_k(x,p)\Delta t\right]dxdp \\ &+\sum_{k=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(Q(x,p,k),R(x,p,k))u(t,x,p)\varphi_k(x,p)\Delta t dxdp \\ = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(Q(x,p,0),R(x,p,0))u(t,x,p)dxdp \\ &+\sum_{k=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{h(Q(x,p,k),R(x,p,k))-h(Q(x,p,0),R(x,p,0))\} \\ &\qquad u(t,x,p)\varphi_k(x,p)\Delta t dxdp. \end{aligned} \tag{8}$$

Note that to $\mathcal{O}(\Delta t^2)$ we may write

$$\begin{aligned} h(Q(x,p,0),R(x,p,0)) &\simeq h(x+\mathcal{V}(x,p)\Delta t, p+\mathcal{F}(x,p)\Delta t) \\ &\simeq h(x,p)+\mathcal{V}(x,p)\frac{\partial h}{\partial x}\Delta t+\mathcal{F}(x,p)\frac{\partial h}{\partial p}\Delta t, \end{aligned} \tag{9a}$$

while

$$\begin{aligned} h(Q(x,p,k),R(x,p,k)) &\simeq h(x+\mathcal{V}(x,p)\Delta t+kf(x,p), p+\mathcal{F}(x,p)\Delta t+kg(x,p)) \\ &\simeq h(x+kf,p+kg)+\mathcal{V}(x,p)\frac{\partial h(x+kf,p+kg)}{\partial(x+kf)}\Delta t \\ &\quad +\mathcal{F}(x,p)\frac{\partial h(x+kf,p+kg)}{\partial(p+kg)}\Delta t. \end{aligned} \tag{9b}$$

Inserting the approximations (9a,b) into equation (8), we have

$$\begin{aligned} E_{\Delta t} \simeq &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{h(x,p)+\mathcal{V}(x,p)\frac{\partial h}{\partial x}\Delta t+\mathcal{F}(x,p)\frac{\partial h}{\partial p}\Delta t\right\}u(t,x,p)dxdp \\ &+\sum_{k=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[h(x+kf,p+kg)-h(x,p)\right]u(t,x,p)\varphi_k(x,p)\Delta t dxdp. \end{aligned} \tag{10}$$

Equating (4) and (10), we have

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,p)u(t+\Delta t,x,p)dxdp = \\ &\quad\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{h(x,p)+\mathcal{V}(x,p)\frac{\partial h}{\partial x}\Delta t+\mathcal{F}(x,p)\frac{\partial h}{\partial p}\Delta t\right\}u(t,x,p)dxdp \\ &\qquad+\sum_{k=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[h(x+kf,p+kg)-h(x,p)\right]u(t,x,p)\varphi_k(x,p)\Delta t dxdp \end{aligned} \tag{11}$$

Rearranging the terms in (11), dividing through by Δt, taking the limit as $\Delta t \to 0$ in the result yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,p)\frac{\partial u}{\partial t}dxdp + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\mathcal{V}(x,p)\frac{\partial h}{\partial x} + \mathcal{F}(x,p)\frac{\partial h}{\partial p}\right\}u(t,x,p)dxdp$$
$$= \sum_{k=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[h(x+kf,p+kg) - h(x,p)\right]u(t,x,p)\varphi_k(x,p)dxdp \quad (12)$$

Using integration by parts on the left hand side of (12), and remembering that h has compact support, we arrive at

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,p)\left\{\frac{\partial u}{\partial t} + \frac{\partial(\mathcal{V}u)}{\partial x} + \frac{\partial(\mathcal{F}u)}{\partial p}\right\}dxdp$$
$$= \sum_{k=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[h(x+kf,p+kg) - h(x,p)\right]u(t,x,p)\varphi_k(x,p)dxdp \quad (13)$$

We are almost there! All we have to do is change the variables in the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x+kf,p+kg)u(t,x,p)\varphi_k(x,p)dxdp.$$

Define new variables $v = x + kf(x,p)$ and $w = p + kg(x,p)$ so the pair (v,w) is given by the transformation $(v,w) = T_k^{-1}(x,p)$. Assume that T_k^{-1} is invertible so $(x,p) = T_k(v,w)$, and denote the Jacobian of T_k by $J_k(v,w)$. Then we can write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x+kf,p+kg)u(t,x,p)\varphi_k(x,p)dxdp = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(v,w)u(t,T_k(v,w))\varphi_k(T_k(v,w))J_k(v,w)dvdw. \quad (14)$$

From (14) and the fact that the function h was arbitrary, it is immediate from equation (13) that u satisfies the evolution equation

$$\frac{\partial u}{\partial t} + \frac{\partial(\mathcal{V}u)}{\partial x} + \frac{\partial(\mathcal{F}u)}{\partial p} = -\sum_{k=1}^{N}u(t,x,p)\varphi_k(x,p) + \sum_{k=1}^{N}u(t,T_k(x,p))J_k(x,p)\varphi_k(T_k(x,p)). \quad (15)$$

Perturbations Independent of the State Variables x and p.

To proceed to investigate equation (15), assume for simplicity that the perturbations f and g are both independent of x and p. Then

$$T_k(x,p) = (x - kf, p - kg),$$

and our evolution equation (15) takes the form

$$\frac{\partial u}{\partial t} + \frac{\partial(\mathcal{V}u)}{\partial x} + \frac{\partial(\mathcal{F}u)}{\partial p} = -\sum_{k=1}^{N}u(t,x,p)\varphi_k(x,p) + \sum_{k=1}^{N}u_{f,g}(t,x,p)\varphi_{f,g;k}(x,p), \quad (16)$$

where we have used the notation $u_{f,g}(t,x,p) = u(t,x-kf,p-kg)$. Assume further that the pair (f,g) is distributed with density $\sigma(f,g)$. Multiplying (16) by $\sigma(f,g)$, and integrating we obtain

$$\frac{\partial u}{\partial t} + \frac{\partial(\mathcal{V}u)}{\partial x} + \frac{\partial(\mathcal{F}u)}{\partial p} = -\sum_{k=1}^{N}u\varphi_k + \sum_{k=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_{f,g}\varphi_{f,g;k}\sigma(f,g)dfdg. \quad (17)$$

Further, expand the product $u(t,x-kf,p-kg)\varphi_k(x-kf,p-kg)$ about the point(x,p) to give

$$u_{f,g}(t,x,p)\varphi_{f,g}(x,p)$$
$$= u(t,x,p)\varphi(x,p) + kf\frac{\partial\left[u(t,x,p)\varphi_k(x,p)\right]}{\partial x} + kg\frac{\partial\left[u(t,x,p)\varphi_k(x,p)\right]}{\partial p}$$
$$+ \frac{k^2}{2}\left\{f^2\frac{\partial^2\left[u(t,x,p)\varphi_k(x,p)\right]}{\partial x^2} + fg\frac{\partial^2\left[u(t,x,p)\varphi_k(x,p)\right]}{\partial x\partial p} + g^2\frac{\partial^2\left[u(t,x,p)\varphi_k(x,p)\right]}{\partial p^2}\right\} + \cdots . \quad (18)$$

Inserting the expansion (18) into (17) and carrying out the indicated integrations we obtain

$$\frac{\partial u}{\partial t} + \frac{\partial (\mathcal{V}u)}{\partial x} + \frac{\partial (\mathcal{F}u)}{\partial p} = k < f > \frac{\partial \left[u\varphi_k\right]}{\partial x} + k < g > \frac{\partial \left[u\varphi_k\right]}{\partial p} + \frac{k^2}{2}\left\{< f^2 > \frac{\partial^2 \left[u\varphi_k\right]}{\partial x^2} + < fg > \frac{\partial^2 \left[u\varphi_k\right]}{\partial x \partial p} + < g^2 > \frac{\partial^2 \left[u\varphi_k\right]}{\partial p^2}\right\} + \cdots , \quad (19)$$

where

$$< f^n >= \int_{-\infty}^{\infty} f^n \sigma(f,g) df dg$$

and the other moments are defined in an obvious way.