Time reversal invariance and Markov Operators: Thoughts coming from reading Uffink (file: `time reversal and MO's.tex`)

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1. Introduction

2. Preliminaries

Consider a system operating in a phase space X. On this phase space the temporal evolution of a system is described by a **dynamical law** S_t that maps points in the phase space X into new points, *i.e.*, $S_t : X \to X$, as time t changes. In general X may be a d-dimensional phase space, either finite or not, and therefore x is a d-dimensional vector. Time t may be either continuous ($t \in \mathbf{R}$) as, for example, it would be for a system whose dynamics were governed by a set of differential equations, or discrete (integer valued, $t \in \mathbf{Z}$) if the dynamics are determined by discrete time maps.

I only consider autonomous processes in which the dynamics S_t are not an explicit function of the time t so it is always the case that $S_t(S_{t'}(x)) = S_{t+t'}(x)$. Thus, the dynamics governing the evolution of the system are the same on the intervals $[0, t^{'}]$ and $[t, t + t^{'}]$.

Remark 1. *This is not a serious restriction since any non-autonomous system can always be reformulated as an autonomous one by the definition of new dependent variables.*

Two types of dynamics will be important. Consider a phase space X and a dynamics $S_t : X \to X$. For every initial point x^0, the sequence of successive points $S_t(x^0)$, considered as a function of time t, is called a **trajectory**.

Definition 1. Time reversal invariant *or* **reversible** *or* **invertible** *dynamics.*

In the phase space X, if the trajectory $S_t(x^0)$ is nonintersecting with itself, or intersecting but periodic, then at any given final time t_f such that $x^f = S_{t_f}(x^0)$ we could change the sign of time by replacing t by $-t$, and run the trajectory backward using x^f as a new initial point in X. Then our new trajectory $S_{-t}(x^f)$ would arrive precisely back at x^0 after a time t_f had elapsed: $x^0 = S_{-t_f}(x^f)$. Thus in this case we have a dynamics that may be reversed in time completely unambiguously. Dynamics with this character are known variously as **time reversal invariant** *Sachs (1987) or* **reversible** *Reichenbach (1957) in the physics literature, and as* **invertible** *Lasota and Mackey (1994) in the mathematics literature.*

Definition 2. *We formalize this by introducing the concept of a* **dynamical system** $\{S_t\}_{t \in \mathbf{R}}$ *(or, alternately, $t \in \mathbf{Z}$ for discrete time systems) on a phase space X, which is simply any group of transformations $S_t : X \to X$ having the properties:*

1. $S_0(x) = x$; *and*
2. $S_t(S_{t'}(x)) = S_{t+t'}(x)$ *for* $t, t' \in \mathbf{R}$ *or* $\mathbf{Z}$.

Remark 2. *Since, from the definition, for any* $t \in \mathbf{R}$ *or* $\mathbf{Z}$*, we have*

$$S_t(S_{-t}(x)) = x = S_{-t}(S_t(x)),$$

it is clear that dynamical systems are **invertible** *in the sense discussed above since they may be run either forward or backward in time. Systems of ordinary differential equations are examples of dynamical systems as are invertible maps. (See Examples 1 and 2 below). All of the equations of classical and quantum physics are invertible.*

Example 1. *Think about a simple discrete time system whose dynamics are given by*

$$S(x) = 2x,$$

so $S_n(x) = 2^n x$ *for* $n \in \mathbf{Z}$ *and let the system run until a final time* n_f *so* $S_{n_f}(x) = 2^{n_f} x$ *and let this be the initial condition. Now reverse the 'time'* $n \to -n$ *and run the system backwards so* $S_{n_f - 1}(x) = 2^{n_f - 1} x$ *and finally* $S_{n_f - n_f} = S_0(x) = x$*, the initial value.*

The reversed trajectory has identically traversed (in reverse time) the initial trajectory that was in forward time.

Example 2. *Alternately consider a continuous time system whose dynamics are described by the simple differential equation*

$$\frac{dx}{dt} = -\gamma x, \; x(t = 0) = x_0$$

so

$$x(t) = x_0 e^{-\gamma t}.$$

At a final time t_f *we have*

$$x(t_f) = x_0 e^{-\gamma t_f}.$$

If we pick this value of $x(t_f)$ *as the initial value and 'reverse the time' so now the dynamics are given by*

$$\frac{dx}{dt} = \gamma x, \; x_0 = x(t_f)$$

then

$$x(t) = x(t_f) e^{\gamma t},$$

and finally

$$x(t = t_f) = x(t_f)e^{\gamma t_f} = x_0 e^{-\gamma t_f} e^{\gamma t_f} \equiv x_0$$

and again the reversed trajectory has exactly traversed (in reverse time) the initial trajectory that was forward in time.

Definition 3. *To illustrate the second type of dynamics, consider a trajectory that intersects itself but is not periodic. Now starting from an initial point* x^0 *we find that the trajectory* $\{S_t(x^0)\}$ *eventually has one or more transversal crossings* $x^\perp$ *of itself. If we let* $t_\perp$ *be the time at which the first one of these crossings occurs, and choose our final time* $t_f > t_\perp$*, then picking* $x^f = S_{t_f}(x^0)$ *and reversing the sign of time to run the trajectory backward from* x^f *poses a dilemma once the reversed trajectory reaches* $x^\perp$ *because the dynamics give us no clue about which way to go! Situations like this are called* **irreversible** *in the physics literature, while mathematicians call them* **non-invertible**.

Definition 4. *Therefore, the second type of dynamics that is important to distinguish are those of* **semi-dynamical systems** $\{S_t\}_{t>0}$*, which is any semigroup of transformations* $S_t : X \to X$*, i.e.*

1. $S_0(x) = x$*; and*
2. $S_t(S_{t'}(x)) = S_{t+t'}(x)$ *for* $t, t' \in \mathbf{R}^+$ *(or* $\mathbf{Z}$*).*

Remark 3. *The difference between the definition of dynamical and semi-dynamical systems lies solely in the restriction of* t *and* t' *to values drawn from the positive real numbers, or the positive integers, for the semi-dynamical systems. Thus, in contrast to dynamical systems, semi-dynamical systems are* **non-invertible** *and may not be run backward in time in an unambiguous fashion. Examples of semi-dynamical systems are given by non-invertible maps, delay differential equations, and some partial differential equations.*

Example 3. *Now consider a slightly more complicated discrete time system with dynamics given by*

$$S(x) = 2x \mod 1.$$

Again it is quite easy to iterate this forward in time since $S_n(x) = 2^n x \mod 1$ *for* $n \in \mathbf{Z}$*. Again let the system run until a final time* n_f *so* $S_{n_f}(x) = 2^{n_f} x \mod 1$ *and let this be the initial condition.*

Now reverse the 'time' $n \to -n$ *and try to run the system backwards. One immediately comes to an impasse since* $S_{n_f - 1}(x)$ *is not unique and indeed there are two possible choices for the first backwards (in time) iteration.*

This means that for a given $S_{n_f}(x)$ there are 2^{n_f} possible values of x that could have yielded the same value.

This is an example of a non-invertible system.

Remark 4. *Often there is a certain confusion in the literature when the terms reversible and irreversible are used, and to avoid this* **we will always use the adjectives invertible and non-invertible**. *In spite of the enormous significance of distinguishing between dynamical and semi-dynamical systems later, at this point no assumption is made concerning the invertibility or non-invertibility of the system dynamics.*

3. Background for standard stuff: Setting the scene

Consider the stochastic perturbed differential equation

$$\frac{dx_i}{dt} = F_i(x) + \sum_{j=1}^{d} \sigma_{ij}(x)\xi_j, \qquad i = 1, \dots, d \tag{1}$$

with the initial conditions $x_i(0) = x_{i,0}$, where $\sigma_{ij}(x)$ is the amplitude of the stochastic perturbation and $\xi_j = \frac{dw_j}{dt}$ is a "white noise" term that is the derivative of a Wiener process $w(t)$. In matrix notation we can rewrite Eq. 1 as

$$dx(t) = F(x(t))dt + \Sigma(x(t))\, dw(t), \tag{2}$$

where $\Sigma(x) = [\sigma_{ij}(x)]_{i,j=1,\dots,d}$.

Remark 5. *Here it is always assumed that the Itô, rather that the Stratonovich, calculus, is used. For a discussion of the differences see Horsthemke and Lefever (1984), Lasota and Mackey (1994) and Risken (1984). In particular, if the σ_{ij} are independent of x then the Itô and the Stratonovich approaches yield identical results.*

The *Fokker-Planck equation* that governs the evolution of the density function $f(t, x)$ of the process $x(t)$ generated by the solution to the stochastic differential equation (2) is given by

$$\frac{\partial f}{\partial t} = -\sum_{i=1}^{d} \frac{\partial [F_i(x)f]}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 [a_{ij}(x)f]}{\partial x_i \partial x_j} \tag{3}$$

where

$$a_{ij}(x) = \sum_{k=1}^{d} \sigma_{ik}(x)\sigma_{jk}(x).$$

If $k(t, x, x_0)$ is the fundamental solution of the Fokker-Planck equation, i.e. for every x_0 the function $(t, x) \mapsto k(t, x, x_0)$ is a solution of the Fokker-Planck equation with the initial condition $\delta(x-x_0)$, then the general solution $f(t, x)$ of the Fokker-Planck equation (3) with the initial condition

$$f(x, 0) = f_0(x)$$

is given by

$$f(t, x) = \int k(t, x, x_0) f_0(x_0)\, dx_0. \tag{4}$$

From a probabilistic point of view $k(t, x, x_0)$ is a stochastic kernel (transition density) and describes the probability of passing from the state x_0 at time $t = 0$ to the state x at a time t. Define the Markov operators P^t by

$$P^t f_0(x) = \int k(t, x, x_0) f_0(x_0)\, dx_0, \quad f_0 \in L^1. \tag{5}$$

Then $P^t f_0$ is the density of the solution $x(t)$ of Eq. 2 provided that f_0 is the density of $x(0)$.

The steady state density $f_*(x)$ is the stationary solution of the Fokker Planck Eq. 3:

$$-\sum_{i=1}^{d} \frac{\partial [F_i(x) f]}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 [a_{ij}(x) f]}{\partial x_i \partial x_j} = 0. \tag{6}$$

If the coefficients a_{ij} and F_i are sufficiently regular so that a fundamental solution k exists, and $\int_X k(t, x, y)\, dx = 1$, then the unique generalized solution (4) to the Fokker-Planck equation (3) is given by Eq. 5.

4. The one-dimensional case

Now since what I want to do requires that I have an analytic expression for the evolution of densities in both forward and reversed time, I need to get away from the really general stuff to something where I can write down closed form expressions. That means pretty low-dimensional stuff.

4.1. Generalities

If we are dealing with a one dimensional system, $d = 1$, then the stochastic differential Eq. 1 simply becomes

$$\frac{dx}{dt} = F(x) + \sigma(x)\xi, \tag{7}$$

where again ξ is a (Gaussian distributed) perturbation with zero mean and unit variance, and $\sigma(x)$ is the amplitude of the perturbation. The corresponding Fokker-Planck equation (3) becomes

$$\frac{\partial f}{\partial t} = -\frac{\partial[F(x)f]}{\partial x} + \frac{1}{2}\frac{\partial^2[\sigma^2(x)f]}{\partial x^2}. \tag{8}$$

The Fokker-Planck equation can also be written in the equivalent form

$$\frac{\partial f}{\partial t} = -\frac{\partial S}{\partial x} \tag{9}$$

where

$$S = -\frac{1}{2}\frac{\partial[\sigma^2(x)f]}{\partial x} + F(x)f \tag{10}$$

is called the *probability current.*

When stationary solutions of (8), denoted by $f_*(x)$ and defined by $P_t f_* = f_*$ for all t, exist they are given as the generally unique (up to a multiplicative constant) solution of (6). In the case $d = 1$:

$$-\frac{\partial[F(x)f_*]}{\partial x} + \frac{1}{2}\frac{\partial^2[\sigma^2(x)f_*]}{\partial x^2} = 0. \tag{11}$$

Integration of Eq. 11 by parts with the assumption that the probability current S vanishes at the integration limits, followed by a second integration, yields the solution

$$f_*(x) = \frac{K}{\sigma^2(x)} \exp\left[\int_{x_0}^{x} \frac{2F(z)}{\sigma^2(z)}\, dz\right]. \tag{12}$$

This stationary solution f_* will be a density if and only if there exists a positive constant $K > 0$ such that f_* can be normalized.

4.2. An Ornstein-Uhlenbeck process

Examples of $\sigma(x)$ and $F(x)$ for which one can determine the *time dependent* solution $f(t,x)$ of Eq. 8 are few and far between. One solution that is known is for an Ornstein-Uhlenbeck process.

Since it is an Ornstein-Uhlenbeck process, which was historically developed in thinking about perturbations to the velocity of a Brownian particle, we denote the dependent variable by v so we have $\sigma(v) \equiv \sigma > 0$ a constant, and $F(v) = -\gamma v$ with $\gamma \geq 0$.

In this case, Eq. 7 becomes

$$\frac{dv}{dt} = -\gamma v + \sigma \xi, \tag{13}$$

or

$$dv(t) = -\gamma v(t)dt + \sigma dw(t), \tag{14}$$

with the corresponding Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \frac{\partial[\gamma v f]}{\partial v} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial v^2}. \tag{15}$$

The unique stationary solution is

$$f_*(v) = \frac{e^{-\gamma v^2/\sigma^2}}{\int_{-\infty}^{+\infty} e^{-\gamma v^2/\sigma^2} dv} = \sqrt{\frac{\gamma}{\pi\sigma^2}} e^{-\gamma v^2/\sigma^2}. \tag{16}$$

Further, from Risken (1984, Eq. 5.28) the fundamental solution $k(t, v, v_0)$ is given by

$$k(t, v, v_0) = \frac{1}{\sqrt{2\pi b(t)}} \exp\left\{-\frac{(v - \exp(-\gamma t)v_0)^2}{2b(t)}\right\}, \tag{17}$$

where

$$b(t) = \frac{\sigma^2}{2\gamma}\left[1 - e^{-2\gamma t}\right] \geq 0 \quad \forall\gamma. \tag{18}$$

Remark 6. *Note that for a given y and t the function $k(t,\cdot,y)$ is a Gaussian density with mean $v_0 \exp(-\gamma t)$ and variance $b(t)$. b is always non-negative regardless of the sign of γ*

Now let f_0 be an initial Gaussian density of the form

$$f_0(v) = \frac{1}{\sqrt{2\pi b_0}} \exp\left\{-\frac{(v - m_0)^2}{2b_0}\right\}, \tag{19}$$

where the initial variance $b_0 > 0$ and mean $m_0 \in \mathbf{R}$. Since

$$P^t f_0(v) = \int_{\mathbf{R}} k(t, v, v_0) f_0(v_0)\, dv_0,$$

we obtain by direct calculation using Eq. 17

$$P^t f_0(v) = \frac{1}{\sqrt{2\pi(b_0 e^{-2\gamma t} + b(t))}} \exp\left\{ -\frac{\left(v - m_0 e^{-\gamma t}\right)^2}{2(b_0 e^{-2\gamma t} + b(t))} \right\}. \qquad (20)$$

Alternately we write

$$P^t f_0(v) = \frac{1}{\sqrt{2\pi \mathrm{Var}(t)}} \exp\left\{ -\frac{\left(v - m_0 e^{-\gamma t}\right)^2}{2\mathrm{Var}(t)} \right\} \qquad (21)$$

where

$$\mathrm{Var}(t) = (b_0 e^{-2\gamma t} + b(t)) = \frac{\sigma^2}{2\gamma} + \left(b_0 - \frac{\sigma^2}{2\gamma} \right) e^{-2\gamma t}. \qquad (22)$$

Suppose that we now let the Ornstein-Uhlenbeck process (14) be started off with an initial density $f_0(v)$ given by (19) and let it run for a period of time $t_f > 0$ until the density $f(t_f, v) = P^{t_f} f_0(v)$ is given by

$$f(t_f, v) = P^{t_f} f_0(v) = \frac{1}{\sqrt{2\pi \mathrm{Var}(t_f)}} \exp\left\{ -\frac{(v - m_f)^2}{2\mathrm{Var}(t_f)} \right\}, \qquad (23)$$

where

$$\mathrm{Var}(t_f) = (b_0 e^{-2\gamma t_f} + b(t_f)) = \frac{\sigma^2}{2\gamma} + \left(b_0 - \frac{\sigma^2}{2\gamma} \right) e^{-2\gamma t_f} \qquad (24)$$

and

$$m_f = m_0 e^{-\gamma t_f}. \qquad (25)$$

4.3. Reversing the time in an Ornstein-Uhlenbeck process: Doing it rather blindly

Now, playing God (or something like that), we magically reverse time $t \to \tau = (-t)$ so the dynamics (14) are now transformed to

$$dv(t) = \gamma v(t)dt + \sigma dw(t), \tag{26}$$

with the corresponding Fokker-Planck equation

$$\frac{\partial f}{\partial t} = -\frac{\partial[\gamma v f]}{\partial v} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial v^2}. \tag{27}$$

Remark 7. *Note that the effect of the change in time $t \to -t$ is to change the sign of γ which is not reflected in a change of sign of σ in* (26)*.*

Remark 8. *I am not really sure that $w(-t) = w(t)$ which was assumed in arriving at* (26)*.*

Remark 9. *Is Equation 27 really the correct evolution equation for the density $f(t, v)$ in the case that the time is reversed?*

The reason that I wonder is the following. If we change $t \to -t$ in the Fokker Planck equation (15) *then we end up with something different. Namely we have*

$$\frac{\partial f}{\partial t} = -\frac{\partial[\gamma v f]}{\partial v} - \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial v^2}, \tag{28}$$

so we have a reversal of the sign of γ but also a reversal of the sign of σ^2!

Note that the stationary solution of (28) *is exactly the same as the stationary solution* (16)*! Interesting, and certainly different from what follows below.*

WHAT IS GOING ON HERE?

The stationary solution of (27) is

$$f_*(v) = \mathcal{C}e^{\gamma v^2/\sigma^2} \tag{29}$$

but it is not a density since it is not integrable. But that minor problem is not the point here.

Remark 10. *The solution trajectory for $v(t)$ given by* (26) *is not time reversal invariant in the sense defined earlier in Definition 1 because of the presence of the Wiener process $w(t)$.*

However, the question that we want to answer is whether or not the densities $P^t f_0(x)$ are.

That is, if we

1. *start off the Ornstein-Uhlenbeck process at time* $t = 0$ *with density* $f_0(v)$
2. *let the density evolve until a time* t_f *when the density has become* $f_{t_f}(v) = P^{t_f} f_0(v)$
3. *now reverse the time* $t \to -t$ *which is equivalent to* $\gamma \to -\gamma$ *and keep* $f_{t_f}(v)$ *as the initial condition and*
4. *let the density evolve (effectively backwards) until a time* t_f *when the density has become* $P^{t_f} f_{t_f}(v)$
5. *then ask the question of whether* $f_0(v) = P^{t_f} f_{t_f}(v)$*?*

Eq. (17) holds for both positive and negative γ (Risken, 1984, page 100), so rewriting explicitly for the case of negative γ we have

$$\bar{k}(t, v, v_0) = \frac{1}{\sqrt{2\pi \bar{b}(t)}} \exp\left\{ -\frac{(v - \exp(\gamma t) v_0)^2}{2\bar{b}(t)} \right\}, \tag{30}$$

where now

$$\bar{b}(t) = \frac{\sigma^2}{2\gamma} \left[e^{2\gamma t} - 1 \right]. \tag{31}$$

Further, the initial density is now given by (23) so the solution of the time reversed Ornstein-Uhlenbeck process is given by

$$P^t f_{t_f}(v) = \int_{\mathbf{R}} \bar{k}(t, v, v_0) f_{t_f}(v_0)\, dv_0.$$

By direct calculation we obtain

$$P^t f_{t_f}(v) = \frac{1}{\sqrt{2\pi(\mathrm{Var}(t_f) e^{2\gamma t} + \bar{b}(t))}} \exp\left\{ -\frac{(v - m_f e^{\gamma t})^2}{2(\mathrm{Var}(t_f) e^{2\gamma t} + \bar{b}(t))} \right\} \tag{32}$$

so

$$P^{t_f} f_{t_f}(v) = \frac{1}{\sqrt{2\pi(\mathrm{Var}(t_f) e^{2\gamma t_f} + \bar{b}(t_f))}} \exp\left\{ -\frac{(v - m_f e^{\gamma t_f})^2}{2(\mathrm{Var}(t_f) e^{2\gamma t_f} + \bar{b}(t_f))} \right\}. \tag{33}$$

with

$$\bar{b}(t_f) = \frac{\sigma^2}{2\gamma} \left[e^{2\gamma t_f} - 1 \right]. \tag{34}$$

Now if you make the appropriate substitutions to figure out what $P^{t_f} f_{t_f}(v)$ really looks like you end up with

$$P^{t_f} f_{t_f}(v) = \frac{1}{\sqrt{2\pi \mathrm{Var}_-(t_f)}} \exp\left\{ -\frac{(v - m_0)^2}{2\mathrm{Var}_-(t_f)} \right\} \tag{35}$$

with

$$\mathrm{Var}_-(t_f) = \mathrm{Var}(t_f)e^{2\gamma t_f} + \bar{b}(t_f) = b_0 + 2 \cdot \frac{\sigma^2}{2\gamma}[e^{2\gamma t_f} - 1]. \tag{36}$$

Remark 11. *So what is the bottom line?*

1. *Now it is clear from* (35) *that* $P^{t_f} f_{t_f}(v) \neq f_0(v)$ *as given in* (19)*.*
2. $P^{t_f} f_{t_f}(v)$ *has the same mean value* (m_0) *as* $f_0(v)$ *but the variance is greater.*
3. *If you keep repeating (iterating) this procedure, then the variance will just keep increasing with each successive reversal of time.*
4. *Is this because Uffink is wrong? He (Uffink, 2007, page 1062) states without equivocation that:*

   > *"I conclude that irreversible behaviour is not built into the Markov property, or in the non-invertibility of the transition probabilities, (or in the repeated randomness assumption, or in the Master equation or in the semigroup property). Rather the appearance of irreversible behaviour is due to the choice to rely on the forward transition probabilities, and not the backward."*

5. *Or because I misinterpreted what he was saying?*
6. *Or because there is an approximation in the derivation of the Fokker Planck equation that introduces irreversibility that is not really there (i.e. irreversibility is a mathematical artifact)?*
7. *The reason that it is important to understand the source of (my) confusion is because of the results that are contained in Mackey and Tyran-Kamińska (2006a,b) on the evolution of the conditional entropy in systems with noise.*

4.4. Reversing the time in an Ornstein-Uhlenbeck process: Doing it via moments from the stochastic equation

Going back to (13), remember that we had

$$\frac{dv}{dt} = -\gamma v + \sigma\xi, \qquad v(0) = v_0,$$

where v is a scalar and the coefficients γ and σ are constant. Let's try to consider this in a different light to see if the results are different.

By definition, the solution of (13) satisfies

$$v(t) = -\gamma \int_0^t v(s)\, ds + \sigma \int_0^t dw(s) + v_0$$

or, using the fact (Lasota and Mackey, 1994, Example 11.3.1) that

$$\int_0^T dw(t) = w(T),$$

we have instead

$$v(t) = -\gamma \int_0^t v(s)\, ds + \sigma w(t) + v_0. \tag{37}$$

Equation 37 is easy to deal with since it does not contain an Itô integral, and, since the one integral that does appear exists for almost ω taken separately, we may use the usual rules of calculus.

Setting

$$z(t) = \int_0^t v(s)\, ds, \tag{38}$$

(37) becomes, for almost all ω,

$$\frac{dz}{dt} = -\gamma z(t) + \sigma w(t) + v_0.$$

For fixed ω, this is an ordinary differential equation and, thus,

$$z(t) = \int_0^t e^{-\gamma(t-s)}(\sigma w(s) + v_0)\, ds. \tag{39}$$

Combining equations 37 through (39) after some manipulation, yields

$$v(t) = v_0 e^{-\gamma t} - \gamma\sigma \int_0^t e^{-\gamma(t-s)} w(s)\, ds + \sigma w(t).$$

Using the integration by parts formula (Lasota and Mackey, 1994, Equation 11.4.8),

$$\int_\alpha^\beta f(t)\,dw(t) = -\int_\alpha^\beta f'(t)w(t)\,dt + f(\beta)w(\beta) - f(\alpha)w(\alpha), \tag{40}$$

$f\colon [\alpha,\beta] \to R$ differentiable with a continuous derivative f', this becomes

$$v(t) = v_0 e^{-\gamma t} + \sigma \int_0^t e^{-\gamma(t-s)} dw(s). \tag{41}$$

From (Lasota and Mackey, 1994, Proposition 11.4.1), if $f\colon [\alpha,\beta] \to R$ is a continuous function, then we have

$$E\left(\int_\alpha^\beta f(t)\,dw(t)\right) = 0 \tag{42}$$

and

$$D^2\left(\int_\alpha^\beta f(t)\,dw(t)\right) = \int_\alpha^\beta [f(t)]^2\,dt. \tag{43}$$

From (42) and (43), it follows that

$$E(v(t)) = e^{-\gamma t} E(v_0) \tag{44}$$

and, taking note of the independence of v_0 and $w(t)$,

$$D^2(v(t)) = e^{-2\gamma t} D^2(v_0) + \sigma^2 D^2\left(\int_0^t e^{-\gamma(t-s)} dw(s)\right).$$

With (43), this finally reduces to

$$\begin{aligned} D^2(v(t)) &= e^{-2\gamma t} D^2(v_0) + \sigma^2 \int_0^t e^{-2\gamma(t-s)} ds \\ &= e^{-2\gamma t} D^2(v_0) + \frac{\sigma^2}{2\gamma}[1 - e^{-2\gamma t}] \end{aligned} \tag{45}$$

Remark 12. *What I want to do now is:*

1. *Start the system off with an initial mean value of* m_0 *at* $t = 0$ *and variance* $b_0 = D^2(v_0)$
2. *let the system run for a period of time* t_f *until it has a mean value from* (44)

   $$E(v(t)) = e^{-\gamma t_f} E(v_0) = m_0 e^{-\gamma t_f}$$

   and a variance

   $$D^2(v(t_f)) = e^{-2\gamma t_f} D^2(v_0) + \frac{\sigma^2}{2\gamma}[1 - e^{-2\gamma t_f}].$$

3. *and then reverse the time $t \to -t$ (which is equivalent to $\gamma \to -\gamma$) to see what the mean and variance are after a period of time t_f*

OK so with this as a programme we first of all have for the reversed process that

$$E_-(v(t)) = e^{\gamma t} E(v(t_f)) \tag{46}$$

and

$$D^2_-(v(t)) = e^{2\gamma t} D^2(v(t_f)) + \frac{\sigma^2}{2\gamma}[e^{2\gamma t} - 1]. \tag{47}$$

Inserting the appropriate initial conditions into these expressions we have at a time t_f

$$E_-(v(t_f)) = E(v(t_0)) \tag{48}$$

and

$$D^2_-(v(t_f)) = D^2(v_0)) + 2\frac{\sigma^2}{2\gamma}[e^{2\gamma t_f} - 1], \tag{49}$$

which is precisely the same result that I obtained in the previous section by looking at the solutions of the Fokker-Planck equation.

4.5. Reversing the time in an Ornstein-Uhlenbeck process: The view from electrical engineering

Realizing that I was totally lost I started to search the literature and discovered that there is quite a bit about time reversed diffusions. A lot of it is from finance, and I really do not understand what motivates it–something to do with a bridge. Also people are interested in Brownian motions with a fixed start and end point.

However I seemed to have a bit more luck in the electrical engineering literature, specifically with some papers from Ljung and Kailath (1976), Lindquist and Picci (1979), and Anderson and Kailath (1979) and Anderson (1982). As with finance I am still not totally sure what motivates them, but the following quote from Ljung and Kailath (1976) starts to give some of the flavor for what is going on.

> "Now it is clear that by choosing the representation (1) a certain asymmetry in the direction of time is introduced which is not present in the basic problem. While the covariance function $R(\cdot,\cdot)$ gives no preference to any direction of time, the model (1) is constructed as a state-space (or *Markovian*, as it is often called) representation only for increasing t (hence the subscript f), because of the condition that $u_f(s)$ be uncorrelated with $x_f(r)$. Although this asymmetry can be (partially) overcome by solving (1) backwards from $t = T$, the corresponding model is no longer suitable to use in estimation problems. The reason is that $u_f(s)$ is now correlated with the initial condition $x_f(T)$. Therefore, even though the process $x_f(\cdot)$ indeed has the Markovian property in either time-direction, the representation (1) cannot be used directly in estimation formulas evolving backwards in time, and hence we have essentially fixed the direction of time in (1)." (Ljung and Kailath, 1976, page 488, RH column, bottom)

I can go a bit further by showing an example drawn from Anderson (1982) that illustrated his main result, (Anderson, 1982, Theorem, page 317).

Example 4. *Namely if you look at (Anderson, 1982, Section 6.3) he considers, for scalar x*

$$dx = f(x)dt + g(x)dw. \tag{50}$$

Here, $f(\cdot)$ and $g(\cdot)$ are smooth and confined to the second and fourth quadrants, each lying in a cone whose boundaries are strictly within the quad-

rants. It follows easily that from his main theorem that

$$\pi(x) = \frac{k}{g^2(x)} \exp\left\{ \int_0^x \frac{2f(\sigma)}{g^2(\sigma)} d\sigma \right\}, \tag{51}$$

and then

$$d\bar{w} = dw + \left[\frac{2f(x)}{g(x)} - g'(x) \right] dt, \tag{52}$$

and the reverse-time equation becomes

$$dx = -f(x)dt + g(x)d\bar{w}. \tag{53}$$

Remark 13. *First note that* (53) *is identical to my equation 26 which I got by proceeding rather blindly in the Ornstein-Uhlenbeck case that $g = \sigma$ is a constant.*

However, if g is not a constant then the results are NOT the same.

Remark 14. *The mystery starts to become somewhat clearer/murkier if we write* (53) *in the form*

$$dx = -f(x)dt + g(x)d\bar{w} \tag{54}$$

$$= -[f(x) + \frac{1}{2}(g^2(x))']dt + g(x)dw \tag{55}$$

and the Fokker Planck equation corresponding to (53), *which is explicitly*

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[\left(f(x) + \frac{1}{2}(g^2(x))' \right) p \right] + \frac{g^2(x)}{2}\frac{\partial^2 p}{\partial x^2} \tag{56}$$

The terms on the RHS of (56) *is vaguely like the term that would be obtained in going from an Ito to a Stratonovich calculus in dealing with these problems (remember that I started off assuming an Ito calculus), but it is not quite right since there is a difference in sign in front of that first term.*

Remark 15. *So to conclude it looks like the Anderson (1982) time-reversal treatment proscribes the following:*

1. *In $dx = f(x)dt + g(x)dw$*
2. *Change the time $t \to -t$ to give $dx = -f(x)dt + g(x)d\bar{w}$*
3. *Where $\bar{w}(t) = w(-t)$ is given by*

$$d\bar{w}(t) = dw(-t) = dw(t) + \left[\frac{2f(x)}{g(x)} - g'(x) \right] dt$$

4. *To finally give*

$$dx = -[f(x) + \frac{1}{2}(g^2(x))']dt + g(x)dw$$

5. *With a corresponding Fokker Planck equation*

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[\left(f(x) + \frac{1}{2}(g^2(x))'\right)p\right] + \frac{g^2(x)}{2}\frac{\partial^2 p}{\partial x^2}.$$

6. *Frankly, at the moment this does not make sense to me.*

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