Vacuum Fluctuations as a Stochastic Process

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Abstract

WRITE AN ABSTRACT WHEN WE ARE DONE
HELMUT: WE NEED TO SETTLE ON WHICH MEASURE-MENT SYSTEM TO USE, AND MAKE ALL OF THE FORMULAE CONSISTENT.

Contents

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1 Introduction

Planck (1914) postulated the average energy of a collection of one dimensional harmonic oscillators at temperature $\beta = 1/kT$ and frequency $\omega = 2\pi\nu$ to be

$$U(\omega, \beta) = \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{\exp(\beta\hbar\omega) - 1}. \qquad (1.1)$$

The first term in Equation 1.1 is now known as the zero point energy (since it is the only limiting term when $T = 0$) and attributed to vacuum fluctuations, while the second is due to ordinary Bose-Einstein statistics. The relation (1.1) is now enshrined in the foundations of quantum mechanics and quantum field theory Milonni (1994), and has played a seminal role in the development of much of the physics of the last 100 years. It obviously suffers from the prediction that the total integrated energy

$$
\begin{aligned}
\int_0^\infty U(\omega, T)d\omega &= \int_0^\infty \frac{1}{2}\hbar\omega\, d\omega + \int_0^\infty \frac{\hbar\omega}{\exp(\beta\hbar\omega) - 1}d\omega \\
&= \lim_{\omega_c \to \infty} \int_0^{\omega_c} \frac{1}{2}\hbar\omega\, d\omega + \frac{\pi^2}{6\beta^2\hbar}
\end{aligned}
$$

is infinite due to the presence of the zero point energy term. This problem is typically obviated in quantum field theory by invoking a cutoff frequency ω_c corresponding to the Planck length (Milonni, 1994), but this in turn leads directly to the infamous and vexing cosmological constant problem (Weinberg, 1989; Rugh and Zinkernagel, 2002; Peebles and Ratra, 2003; Bousso, 2008).

The concept of vacuum fluctuations really had its birth after the work of Planck, and in spite of the convergence difficulties mentioned above the concept has proved enormously successful in explaining a variety of experimental facts. For example it has been successfully invoked to explain X-ray scattering in solids (Debye, 1914); understand the Lamb shift between the s and p levels in hydrogen (Welton, 1948; Power, 1966); predict the Casimir effect (Casimir, 1948; Casimir and Polder, 1948); understand the nature of Van der Waals forces (Casimir, 1948; Boyer, 1969, 1972a,b, 1973b, 1974, 1975); offer an interpretation of the Aharonov-Bohm effect (Boyer, 1973a, 1987); and explain Compton scattering (Welton, 1948).

Some of these explanations invoking the concept of the concept of vacuum fluctuations have been indirect, while other have been much more direct. In our opinion one of the most direct and powerful demonstrations that vacuum fluctuations have the spectral properties included in Equation 1.1 by is based on the theoretical work of Callen and Welton (1951) and Koch
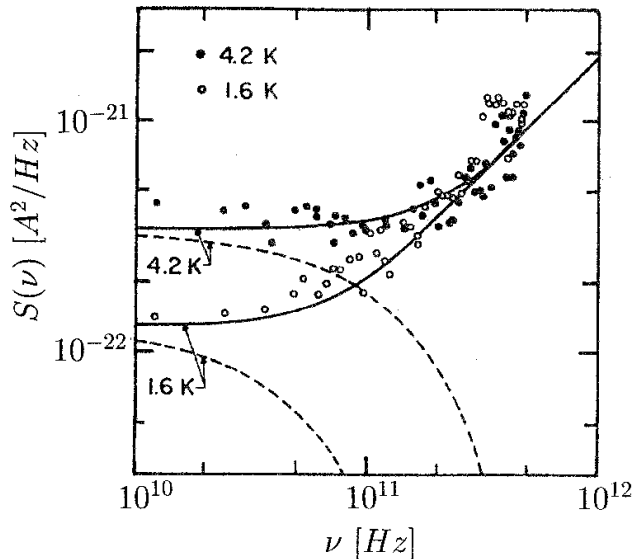
Figure 1.1: Spectral density of current noise as measured in the Koch et al. (1982) experiment for two different temperatures (note $T = 1/k\beta$ and $\nu = \omega/2\pi$). The solid lines are the prediction of Equation 1.2, whereas the dashed lines are given by $(4\bar{h}\omega/R)(\exp(\beta\bar{h}\omega)-1)^{-1}$. The shunting resistance in this case was $R = 0.7\,\Omega$ so the corresponding spectral density of the energy is given by $U(\omega,\beta) = S(\omega) \times R/4$. (Taken from Koch et al. (1982) with permission.)

et al. (1980). They showed, based on the concept of vacuum fluctuations, that in a Josephson junction shunted by a resistance R the junction noise current should have a spectral density given by

$$S(\omega) = \frac{4}{R}U(\omega,\beta) = \frac{4\hbar\omega}{R}\left(\frac{1}{2} + \frac{1}{\exp(\beta\hbar\omega) - 1}\right). \qquad (1.2)$$

In a beautiful experimental study, this predicted behaviour has been verified at two different temperatures up to frequencies of order 6×10^{11} Hz by Koch et al. (1982), see Figure 1.1. Experiments are currently underway (Barber and Blamire, 2006; Warburton, 2006) to extend these measurements to higher frequencies.

The central questions we address in this paper are: first, what is the density of the distribution of the vacuum and thermal fluctuations as a function of frequency ω and temperature T; and second, can one construct a stochastic process with the known properties of the spectrum 1.1 insofar as they have been measured?

The outline of the paper is as follows. In Section 2 we derive the density of the distribution of the vacuum and thermal fluctuations and show that it is Gaussian at all frequencies ω. The following Section 3 uses a linear system driven by a white noise (Wiener process) to construct a filtered noise source with a spectrum approximating that observed in the Josephson junction experiments of Koch et al. (1982). The paper concludes with a brief discussion in Section 4.

2 Vacuum and thermal fluctuations are normally distributed

The $T = 0$ and $T \neq 0$ sections should be combined.

2.1 3D spectrum

Equation 1.1 gives the vacuum fluctuation spectrum for one degree of freedom. For three degrees of freedom, the relation $\rho(\omega, \beta) = 2\omega^2 \bar{U}(\omega, \beta)/\pi c^3$ between the energy $\bar{U}$ and the spectrum ρ, yields

$$\rho(\omega, \beta) = \frac{\hbar \omega^3}{\pi c^3} \left[1 + \frac{2}{\exp(\beta \hbar \omega) - 1} \right] \tag{2.3}$$

2.2 Zero temperature

This section provides a quantum theoretical calculation of the density $w(E)$ of the distribution of the electric field fluctuations in the vacuum state $|0>$. Here E denotes the value of an arbitrary component of the electric field. For the calculations in the following we choose the z component of E whose operator at a position x is given by

$$E_z = \sum_{\boldsymbol{k}, \lambda} e_z^{\boldsymbol{k}, \lambda} \sqrt{\frac{\hbar \omega(k)}{2\epsilon_0 V}} \; i \; \left[a_{\boldsymbol{k}, \lambda} e^{i \boldsymbol{k} \cdot \boldsymbol{x}} - a_{\boldsymbol{k}, \lambda}^+ e^{-i \boldsymbol{k} \cdot \boldsymbol{x}} \right] \tag{2.4}$$

Here $\omega(k) = c|\boldsymbol{k}|$, and $\boldsymbol{e}^{\boldsymbol{k}, \lambda}$ ($\lambda = 1, 2$) are two unit vectors for two polarization directions orthogonal to the wave vector $\boldsymbol{k}$, $a_{\boldsymbol{k}, \lambda}$ and $a_{\boldsymbol{k}, \lambda}^+$ are,

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respectively, the annihilation and creation operators for the mode $(\boldsymbol{k}, \lambda)$, and V is the volume to which the system is confined[1]. For a finite volume $V = L^3$ the allowed wave vectors are of the form

$$\boldsymbol{k} = \frac{2\pi}{L}(n_1, n_2, n_3), \qquad n_i = 0, \pm 1, \pm, \cdots, \qquad i = 1, 2, 3. \qquad (2.5)$$

Since the different modes $\boldsymbol{k}, \lambda$ do not interact, we can calculate the contribution of each term of (2.4) separately, and later combine these contributions to give the final result. We begin by considering a wave vector $\boldsymbol{k}$ orthogonal to the z direction. Then only the polarization in the z direction (say $\lambda = 1$) is relevant and the orthogonal polarization (say $\lambda = 2$) does not contribute. Suppressing the polarization index $\lambda = 1$ we have to consider the operator

$$\sqrt{\frac{\hbar\omega(k)}{2\epsilon_0 V}} \; i \; \left[a_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - a_{\boldsymbol{k}}^+ e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right]. \qquad (2.6)$$

We can easily obtain an expression for the distribution $w_{\boldsymbol{k}}(E)$ due to the term (2.6) through a comparison with the simple harmonic oscillator where $(a_{\boldsymbol{k}} + a_{\boldsymbol{k}}^+)$ and $i(a_{\boldsymbol{k}} - a_{\boldsymbol{k}}^+)$ are the position and momentum operators up to a multiplicative numerical factor. At first sight the space dependent factors $\exp(\pm i\boldsymbol{k}\boldsymbol{x})$ in the term (2.6) would seem to complicate things. However, because of the homogeneity of physical space the final results for the field fluctuations cannot depend on the position $\boldsymbol{x}$[2]. Therefore, we can choose the exponential factors so that we have the simple operators $(a_{\boldsymbol{k}} + a_{\boldsymbol{k}}^+)$ or $i(a_{\boldsymbol{k}} - a_{\boldsymbol{k}}^+)$.

In the first case we have to compare with the position operator of the harmonic oscillator and the probability for a position x in the ground state $|0>$,

$$| < 0|x > |^2 = \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\frac{m\omega}{\hbar}x^2}, \qquad (2.7)$$

and immediately obtain for a vector $\boldsymbol{k}$ orthogonal to the z direction the density of the Gaussian distribution

$$w_{\boldsymbol{k}}(E) = \sqrt{\frac{\epsilon_0 V}{\hbar\omega(k)\pi}} e^{-\frac{\epsilon_0 V}{\hbar\omega(k)}E^2}. \qquad (2.8)$$

The second case with the momentum operator gives the same result.

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[1]We will see later that our final results are independent of V.

[2]A more complicated calculation confirms this without resorting to this argument.

The generalization to arbitrary wave-vectors $\boldsymbol{k}$ is straightforward. We choose the polarization direction $\lambda = 1$ in the $(\boldsymbol{k}, z)$ plane, and the direction $\lambda = 2$ orthogonal to it. Then only $\lambda = 1$ contributes to E_z, namely with a factor of $\sin\theta(\boldsymbol{k})$ where $\theta(\boldsymbol{k})$ is the angle between $\boldsymbol{k}$ and z. Thus the generalization of Equation 2.8 is the Gaussian density

$$w_{\boldsymbol{k}}(E) = \sqrt{\frac{\epsilon_0 V}{\hbar\omega(k)\pi}} \cdot \frac{1}{\sin\theta(\boldsymbol{k})} e^{-\frac{\epsilon_0 V}{\hbar\omega(k)}\frac{1}{\sin^2\theta(\boldsymbol{k})}E^2}. \tag{2.9}$$

The total electric field component in the z direction is the sum of all contributions from the different modes $\boldsymbol{k}$. Since these are statistically independent variables the distribution of the sum is obtained easily, namely the characteristic function is a product and all cumulants are simply sums of the single contributions. This means, in our case of Gaussians, c.f (2.9), that we have also a Gaussian density for the total electric field:

$$w(E)) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{E^2}{2\sigma^2}}. \tag{2.10}$$

The second cumulant, the variance σ^2, is the vacuum expectation value of the square of Equation 2.4 with $e_z^{\boldsymbol{k}} = \sin\theta(\boldsymbol{k})$. This, in turn, is simply the sum of all the contributions from the different modes $\boldsymbol{k}$

$$
\begin{aligned}
\sigma^2 &= \sum_k \frac{\hbar\omega(k)}{2\epsilon_0 V} \sin^2\theta(\boldsymbol{k}) < 0|a_{\boldsymbol{k}}a_{\boldsymbol{k}}^+ + a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}|0 > \\
&= \frac{\hbar}{2\epsilon_0 V} \sum_k \omega(k)\sin^2\theta(\boldsymbol{k}).
\end{aligned} \tag{2.11}
$$

We approximate the summation by an integration using the (smeared) density of $\boldsymbol{k}$ states

$$\rho(\boldsymbol{k})d^3k = \frac{V}{(2\pi)^3}d^3k = \frac{V}{(2\pi)^3}k^2\sin\theta dk d\theta d\phi, \tag{2.12}$$

with the result

$$
\begin{aligned}
\sigma^2 &\simeq \frac{\hbar}{2\epsilon_0 V}\frac{V}{(2\pi)^3}\int \omega(k)k^2 dk \sin^3\theta d\theta d\phi \\
&= \frac{\hbar}{6\pi^2\epsilon_0}\int_0^\infty \omega(k)k^2 dk \\
&= \frac{\hbar}{6\pi^2\epsilon_0 c^3}\int_0^\infty \omega^3 d\omega.
\end{aligned} \tag{2.13}
$$

(Note that the pesky volume V has disappeared!) If we assume a cutoff frequency ω_c we obtain a Gaussian density of the electric field component with a variance given by

$$\sigma^2 = \frac{\hbar \omega_c^4}{2^6 \pi^2 \epsilon_0 c^3}. \tag{2.14}$$

2.3   Non-zero temperature

Now our aim is a quantum theoretical calculation of the distribution $w(E)$ of the electric field fluctuations in a thermodynamic equilibrium situation at temperature T (the vacuum case of $T = 0$ was already dealt with in the previous section).

At a temperature T the energy eigenstates $| \cdots n_{\boldsymbol{k}\lambda} \cdots >$ of the electromagnetic field characterized by a set of occupation numbers $n_{\boldsymbol{k},\lambda}$ for all modes $(\boldsymbol{k}, \lambda)$ are weighted according to a canonical distribution

$$p(\cdots n_{\boldsymbol{k},\lambda} \cdots) = \frac{\exp\left(-\beta \sum_{\boldsymbol{k},\lambda} n_{\boldsymbol{k}\lambda} \hbar \omega(k)\right)}{\sum_{\cdots n_{\boldsymbol{k}'\lambda'} \cdots} \exp\left(-\beta \sum_{\boldsymbol{k}',\lambda'} n_{\boldsymbol{k}'\lambda'} \hbar \omega(k')\right)} \tag{2.15}$$

with $\beta = 1/(k_B T)$. This is nothing more than the canonical Gibbs distribution. The sum in the exponent of the numerator runs over all modes $\boldsymbol{k}, \lambda$ and uses for the $n_{\boldsymbol{k},\lambda}$ the values fixed as an argument on the left hand side of (2.15). The denominator is the partition sum function Z which runs over all energy eignestates $| \cdots n_{\boldsymbol{k}\lambda} >$. That is it runs over all numbers $n_{\boldsymbol{k}',\lambda'}$ from 0 to ∞, which is analogous to going over all the states $(\cdots n_{\boldsymbol{k}',\lambda'} \cdots)$ of the system. These are parameterized by the occupation numbers $n_{\boldsymbol{k}',\lambda'}$ for all modes $\boldsymbol{k}', \lambda'$. The sum in the denominator is analogous to the exponential in the numerator. Namely it runs now over all the modes $\boldsymbol{k}', \lambda'$ and uses for the $n_{\boldsymbol{k}',\lambda'}$ the values of the chosen state $(\cdots n_{\boldsymbol{k}',\lambda'} \cdots)$.

This expression (2.15) can be factored with respect to the modes $(\boldsymbol{k}, \lambda)$ and written in the form

$$p(\cdots n_{\boldsymbol{k},\lambda} \cdots) = \prod_{\boldsymbol{k},\lambda} \frac{\exp\left(-\beta n_{\boldsymbol{k}\lambda} \hbar \omega(k)\right)}{\sum_{\tilde{n}_{\boldsymbol{k}\lambda}=0}^{\infty} \exp\left(-\beta \tilde{n}_{\boldsymbol{k}\lambda} \hbar \omega(k)\right)} = \prod_{\boldsymbol{k},\lambda} p(n_{\boldsymbol{k}\lambda}), \tag{2.16}$$

where

$$p(n_{\boldsymbol{k}\lambda}) = \frac{1}{Z_{\boldsymbol{k},\lambda}} \exp\left(-\beta n_{\boldsymbol{k}\lambda} \hbar \omega(k)\right) \tag{2.17}$$

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is the probability that there are $n_{\boldsymbol{k},\lambda}$ photons in the mode $(\boldsymbol{k},\lambda)$, and

$$Z_{\boldsymbol{k},\lambda} = \sum_{\tilde{n}_{\boldsymbol{k}\lambda}=0}^{\infty} \exp\left(-\beta\tilde{n}_{\boldsymbol{k}\lambda}\hbar\omega(k)\right) = \frac{1}{1 - \exp(-\beta\hbar\omega(k))} \qquad (2.18)$$

is the partition function for the mode $\boldsymbol{k},\lambda$. Physically this can be interpreted that each mode $\boldsymbol{k},\lambda$ is an independent harmonic oscillator. Equation 2.17 is simply the canonical Gibbs' distribution for the single mode $\boldsymbol{k},\lambda$.

As in Section 2.2 we first consider the contribution of each term $\boldsymbol{k},\lambda$ in the electric field operator (2.4) separately, and call this contribution $w_{\boldsymbol{k}}(E)$. Also, as before, only the polarization in the $(\boldsymbol{k},z)$ plane is important, and it is again taken as $\lambda = 1$. Henceforth this index will be suppressed.

In the case of non-zero temperature, for a mode $\boldsymbol{k}$ we have to take into account all states $|n_{\boldsymbol{k}} >$ with different photon numbers $n_{\boldsymbol{k}}$. As is usual in statistical thermodynamics, we obtain the distribution $w_{\boldsymbol{k}}$ as a thermal average over the contributions $w_{n_{\boldsymbol{k}}}$, each resulting from a state $|n_{\boldsymbol{k}} >$ of the mode $\boldsymbol{k}$:

$$\begin{aligned}
w_{\boldsymbol{k}}(E) &= \sum_{n_{\boldsymbol{k}}=0}^{\infty} p(n_{\boldsymbol{k}})w_{n_{\boldsymbol{k}}}(E) \\
&= \frac{1}{Z_{\boldsymbol{k}}} \sum_{n_{\boldsymbol{k}}=0}^{\infty} e^{-\beta n_{\boldsymbol{k}}\hbar\omega(k)} w_{n_{\boldsymbol{k}}}(E).
\end{aligned} \qquad (2.19)$$

The quantum theoretical probabilities $w_{n_{\boldsymbol{k}}}(E)$ are calculated as in Section 2.2 in analogy with the simple harmonic oscillator. In the special case that $\boldsymbol{k}$ is orthogonal to the z direction Equation 2.8 is generalized to

$$w_{n_{\boldsymbol{k}}}(E) = \sqrt{\frac{\epsilon_0 V}{\hbar\omega(k)\pi}} \cdot \frac{1}{2^{n_{\boldsymbol{k}}} n_{\boldsymbol{k}}!} H_{n_{\boldsymbol{k}}}^2\left(\sqrt{\frac{\epsilon_0 V}{\hbar\omega(k)}}E\right) e^{-\frac{\epsilon_0 V}{\hbar\omega(k)}E^2}, \qquad (2.20)$$

where the $H_n(\xi)$ are the Hermite polynomials, well known as parts of the eigenfunctions of the harmonic oscillator.

For arbitrary wave vectors $\boldsymbol{k}$ we have to include the same factor $\sin\theta(\boldsymbol{k})$ as in Equation 2.9 with the result

$$w_{n_{\boldsymbol{k}}}(E) = \sqrt{\frac{\epsilon_0 V}{\hbar\omega(k)\pi}} \frac{1}{2^{n_{\boldsymbol{k}}} n_{\boldsymbol{k}}!} \frac{1}{\sin\theta(\boldsymbol{k})} H_{n_{\boldsymbol{k}}}^2\left(\sqrt{\frac{\epsilon_0 V}{\hbar\omega(k)}} \frac{1}{\sin\theta(\boldsymbol{k})}E\right) e^{-\frac{\epsilon_0 V}{\hbar\omega(k)} \frac{1}{\sin^2\theta(\boldsymbol{k})}E^2}.$$

$$(2.21)$$

Together with the weights $p(n_{\boldsymbol{k}})$ [see Equations 2.17 and 2.19] we have

$$w_{\boldsymbol{k}}(E) = (1 - r) \sum_{n_{\boldsymbol{k}}=0}^{\infty} \frac{\alpha}{\sqrt{\pi} 2^{n_{\boldsymbol{k}}} n_{\boldsymbol{k}}!} H_{n_{\boldsymbol{k}}}^2(\alpha E) e^{-\alpha^2 E^2} r^{n_{\boldsymbol{k}}}, \qquad (2.22)$$

where $r = e^{-\beta \hbar \omega(k)}$ and $\alpha = \sqrt{\dfrac{\epsilon_0 V}{\hbar \omega(k)}} \cdot \dfrac{1}{\sin \theta(\boldsymbol{k})}$.

The computation of the sum in Equation 2.22 is accomplished through a result of Hille (1926, Eq. 39). Namely, it was shown that for $r \in (0, 1)$

$$\sum_{n=0}^{\infty} \frac{r^n}{2^n n!} H_n^2(x) e^{-x^2} = \frac{1}{\sqrt{1 - r^2}} \exp\left(-\frac{1 - r}{1 + r} x^2\right). \qquad (2.23)$$

Thus, with $x = \alpha E$ we simply have that the distribution of the vacuum and thermal fluctuations at all temperatures β is given by

$$w_{\boldsymbol{k}}(E) = \alpha \sqrt{\frac{1 - r}{\pi(1 + r)}} \exp\left(-\frac{1 - r}{1 + r} \alpha^2 E^2\right). \qquad (2.24)$$

This is a Gaussian with variance

$$\sigma_{\boldsymbol{k}}^2 = \frac{1}{2\alpha^2} \cdot \frac{1 + r}{1 - r} = \frac{\hbar \omega(k)}{2\epsilon_0 V} \sin^2 \theta(\boldsymbol{k}) \left(1 + \frac{2}{e^{\beta \hbar \omega(k)} - 1}\right). \qquad (2.25)$$

To combine all of the $w_{\boldsymbol{k}}(E)$ to obtain the density $w(E)$ for the resultant field we can use the same arguments as in Section 2.2 to again obtain a Gaussian

$$w(E) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{E^2}{2\sigma^2}} \qquad (2.26)$$

with variance

$$\sigma^2 = \frac{\hbar}{2\epsilon_0 V} \sum_{\boldsymbol{k}} \omega(k) \sin^2 \theta(\boldsymbol{k}) \left(1 + \frac{2}{e^{\beta \hbar \omega(k)} - 1}\right). \qquad (2.27)$$

After a change to a continuous $\boldsymbol{k}$ space and integration over θ and ϕ we have

$$\sigma^2 = <E_z^2> = \frac{\hbar}{3\epsilon_0 \pi^2 c^3} \int_0^{\infty} \omega^3 \left(\frac{1}{2} + \frac{1}{e^{\beta \hbar \omega} - 1}\right) d\omega. \qquad (2.28)$$

As in Equation 2.13, this will converge only if there is an upper cutoff of ω_c:

$$\sigma^2 = <E_z^2> = \frac{\hbar}{3\epsilon_0 \pi^2 c^3} \int_0^{\omega_c} \omega^3 \left(\frac{1}{2} + \frac{1}{e^{\beta \hbar \omega} - 1}\right) d\omega. \qquad (2.29)$$

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Notice the interesting connection with the Planck formula for the energy density U of a photon gas at temperature β:

$$U = \frac{\hbar}{\pi^2 c^3} \int \omega^3 \left(\frac{1}{2} + \frac{1}{e^{\beta \hbar \omega} - 1} \right) d\omega. \tag{2.30}$$

Integration of the second term gives the Stefan-Boltzmann law, and the first term is finite only if it is assumed that there is a finite cutoff frequency ω_c. This first term in the integral is the zero point energy, and the second term gives the additional energy due to a non-zero temperature.

Since

$$U = \left\langle \frac{1}{2}(\epsilon_0 \boldsymbol{E}^2 + \frac{1}{\mu_0} \boldsymbol{B}^2) \right\rangle, \tag{2.31}$$

by using the equality of the electric and magnetic term and isotropy of physical space, for the variance $\sigma^2 = < E_z^2 >$ of the distribution of the field fluctuations, we again arrive at Equation 2.28.

2.4 Autocorrelation function

As for the variance, we can calculate an autocorrelation function. In the Heisenberg picture

$$a_{\boldsymbol{k},\lambda}(t) = a_{\boldsymbol{k},\lambda} e^{-i\omega t} \qquad \text{and} \qquad a_{\boldsymbol{k},\lambda}^+(t) = a_{\boldsymbol{k},\lambda}^+ e^{i\omega t}, \tag{2.32}$$

where we write $a_{\boldsymbol{k},\lambda} \equiv a_{\boldsymbol{k},\lambda}(0)$ and $a_{\boldsymbol{k},\lambda}^+ \equiv a_{\boldsymbol{k},\lambda}^+(0)$. Furthermore, because the order of operation of non-commuting operators is important we need to use a symmetric autocorrelation function of the form

$$\phi(t) = \frac{1}{2} \langle E_z(0) E_z(t) + E_z(t) E_z(0) \rangle. \tag{2.33}$$

Using Equations 2.4 and 2.32 and remembering that for each $\boldsymbol{k}$ only one polarization contributes so that we can suppress the polarization index λ, we have

$$
\begin{aligned}
\phi(t) &= \sum_{\boldsymbol{k}} \frac{\hbar \omega(k)}{2\epsilon_0 V} \sin^2 \theta \sum_{n_{\boldsymbol{k}}=0}^{\infty} p(n_{\boldsymbol{k}}) \frac{1}{2} \langle n_{\boldsymbol{k}} | a_{\boldsymbol{k}} a_{\boldsymbol{k}}^+ + a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}} | n_{\boldsymbol{k}} \rangle (e^{i\omega t} + e^{-i\omega t}) \\
&= \sum_{\boldsymbol{k}} \frac{\hbar \omega(k)}{2\epsilon_0 V} \sin^2 \theta \sum_{n_{\boldsymbol{k}}=0}^{\infty} p(n_{\boldsymbol{k}}) \left(\frac{1}{2} + n_{\boldsymbol{k}} \right) \cos \omega t \\
&= \sum_{\boldsymbol{k}} \frac{\hbar \omega(k)}{2\epsilon_0 V} \sin^2 \theta \left(\frac{1}{2} + \frac{1}{e^{\beta \hbar \omega(k)} - 1} \right) \cos \omega t. \tag{2.34}
\end{aligned}
$$

In the continuous limit this gives

$$
\begin{aligned}
\phi(t) &= \frac{\hbar}{3\epsilon_0 \pi^2 c^3} \int_0^\infty \omega^3 \left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1} \right) \cos\omega t \, d\omega &\quad (2.35) \\
&= \frac{\hbar}{6\epsilon_0 \pi^2 c^3} \int_{-\infty}^\infty \omega^2 |\omega| \left(\frac{1}{2} + \frac{1}{e^{\beta\hbar|\omega|} - 1} \right) \cos\omega t \, d\omega &\quad (2.36) \\
&= \frac{\hbar}{6\epsilon_0 \pi^2 c^3} \int_{-\infty}^\infty \omega^2 |\omega| \left(\frac{1}{2} + \frac{1}{e^{\beta\hbar|\omega|} - 1} \right) e^{i\omega t} \, d\omega, &\quad (2.37)
\end{aligned}
$$

since the sine integral vanishes.

Finally we can write this in the form

$$
\phi(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(\omega) e^{i\omega t} d\omega \tag{2.38}
$$

with the interpretation taken from the Wiener-Khinchine theorem that

$$
S(\omega) = \frac{\hbar}{3\epsilon_0 \pi^2 c^3} \omega^2 |\omega| \left(\frac{1}{2} + \frac{1}{e^{\beta\hbar|\omega|} - 1} \right) \tag{2.39}
$$

is the spectral density of the vacuum plus thermal fluctuations.

3 Constructing a stochastic process with a non-white spectrum

To construct a stochastic process with a spectrum analogous to that measured by (Koch et al., 1982) is, in principle, straightforward. We could consider a multidimensional Ornstein-Uhlenbeck process

$$
\frac{dx}{dt} = Ax + \Sigma\xi, \tag{3.40}
$$

where A and Σ are $d \times d$ matrices and ξ is a d dimensional vector of Gaussian distributed, delta correlated processes. These could simply be given as Wiener processes, or could alternately be due to the operation of a totally deterministic system with these properties, c.f. Mackey and Tyran-Kamińska (2006). We then determine the character of F and Σ required to give the corresponding required spectrum as given in Equation 2.39 over the range of frequencies studied in Koch et al. (1982). It is important to remember that if ξ is a Gaussian distributed white noise (delta correlated) then the filtered response is still Gaussian distributed (Zakai and Snyders, 1970; Erickson,

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1971; Mackey and Tyran-Kamińska, 2006). However, it is equally important to realize that it is *impossible* to construct a spectrum like (2.39) that will be valid for all frequencies $\omega \in [0, \infty)$ from any linear system [**What is the justification for this statement? I am convinced it is true, how to prove it?**]. **Can I carry out the spectral derivation in the general d-dimensional case? Should I?**

However, this is too general and in fact is is sufficient (I hope) to consider the simpler second order system

$$\begin{aligned} \frac{dx_1}{dt} &= a_{1,1}x_1 + a_{1,2}x_2 + \sigma_1\xi_1 \\ \frac{dx_2}{dt} &= a_{2,1}x_1 + a_{2,2}x_2 + \sigma_2\xi_2. \end{aligned} \tag{3.41}$$

so

$$x = (x_1, x_2)^T, \; \xi = (\xi_1, \xi_2)^T, \; A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}, \text{ and } \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tag{3.42}$$

To derive the spectral density of x_1 if ξ is a white noise process (Gaussian distributed delta correlated noise), we first of take the Laplace transformation of both equations in (3.41) to give ($\tilde{x}_i(s) = \mathfrak{L}[x_i(t)]$)

$$\begin{aligned} s\tilde{x}_1 &= a_{1,1}\tilde{x}_1 + a_{1,2}\tilde{x}_2 + \sigma_1\tilde{\xi}_1 \\ s\tilde{x}_2 &= a_{2,1}\tilde{x}_1 + a_{2,2}\tilde{x}_2 + \sigma_2\tilde{\xi}_2. \end{aligned} \tag{3.43}$$

Thus the transfer function is given by

$$W_1(s) = \frac{\tilde{x}_1(s)}{\tilde{\xi}_2(s)} = \frac{1}{\Delta(s)} \begin{vmatrix} \sigma_1 & s - a_{1,2} \\ \sigma_2 & s - a_{2,2} \end{vmatrix}, \tag{3.44}$$

where

$$\Delta(s) = \begin{vmatrix} s - a_{1,1} & -a_{1,2} \\ -a_{2,1} & s - a_{2,2} \end{vmatrix}. \tag{3.45}$$

The spectral density $S(\omega)$ is defined by the Fourier transform of the correlation function

$$S(\omega) = \mathfrak{F}[\phi(t)] \tag{3.46}$$

and in our case it is a standard result from linear systems analysis that

$$S_1(\omega) = |W_1(j\omega)|^2 S_{\xi_2}(\omega). \tag{3.47}$$

Since the spectral density of a white noise process is constant we conclude that the spectral density Z of x_1 is given by

$$Z(\omega) \equiv \frac{S_1(\omega)}{S_{\xi_2}(\omega)} = |W_1(j\omega)|^2. \tag{3.48}$$

Some algebra leads to the more explicit expression

$$Z(\omega) = \frac{p^2 + \sigma^2\omega^2}{(\gamma - \omega^2)^2 + \alpha^2\omega^2} \tag{3.49}$$

wherein

$$\alpha = a_{1,1} + a_{2,2} \tag{3.50}$$
$$\gamma = a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \tag{3.51}$$
$$p = a_{1,2}\sigma_2 - a_{2,2}\sigma_1 \tag{3.52}$$
$$\sigma = \sigma_1 - \sigma_2, \tag{3.53}$$

and the dependence on the temperature T is to be determined. We identify this spectral density with the spectral energy density in Equation 1.1:

$$Z(\omega, T) \equiv U(\omega, T).$$

This spectral density has the property that

$$Z(\omega = 0, T) = \frac{p^2}{\gamma^2}, \tag{3.54}$$

$$\lim_{\omega \to \infty} Z_1(\omega, T) = 0. \tag{3.55}$$

To approximate the data of Koch et al. (1982) as shown in Figure 1.1 we note that in Equation 1.1, for $\beta\hbar\omega \ll 1$ we have

$$U \simeq \frac{1}{2}\hbar\omega + \frac{1}{\beta}, \tag{3.56}$$

and thus we must have

$$Z(0, T) = \frac{p^2}{\gamma^2} = \frac{1}{\beta}. \tag{3.57}$$

Hence

$$\left(\frac{p^2}{\gamma^2}\right)_{T)=1.6^\circ K} = 0.14\,\text{meV}$$

$$\left(\frac{p^2}{\gamma^2}\right)_{T)=4.2^\circ K} = 0.36\,\text{meV}. \tag{3.58}$$

To assess the success with which Equation 3.49 can account for the existing data (Koch et al., 1982) we consider two cases.

3.1 Case 1. $\sigma^2 = 0$

In this case it is easy to show that the spectrum (3.49) has a single maximum at an angular frequency

$$\omega_{max} = \sqrt{\gamma - \frac{\alpha^2}{2}} \tag{3.59}$$

with a value of

$$Z_1(\omega_{max}, T) = \frac{p^2}{\alpha^2 \left[\gamma - \dfrac{\alpha^2}{4}\right]}. \tag{3.60}$$

In order for this maximum to exist and be finite we must have $(\alpha^2/2\gamma) \in (0, 1]$.

In Figure 3.2, we show the fit of the Koch et al. (1982) data to Equation 3.49 under the constraint that $Z(0, T) = 1/\beta$ (Equation 3.57).

3.2 Case 2: $\sigma^2 > 0$

In Figure 3.3, we show the fit of the Koch et al. (1982) data to Equation 3.49 under the constraint that $Z(0, T) = 1/\beta$ (Equation 3.57).

4 Discussion

Acknowledgments

References

Barber, Z. and Blamire, M. (2006). Externally-shunted high-gap josephson junctions: Design, fabrication and noise measurements. *EPSRC (U.K.) grant*, EP/D029872/1.
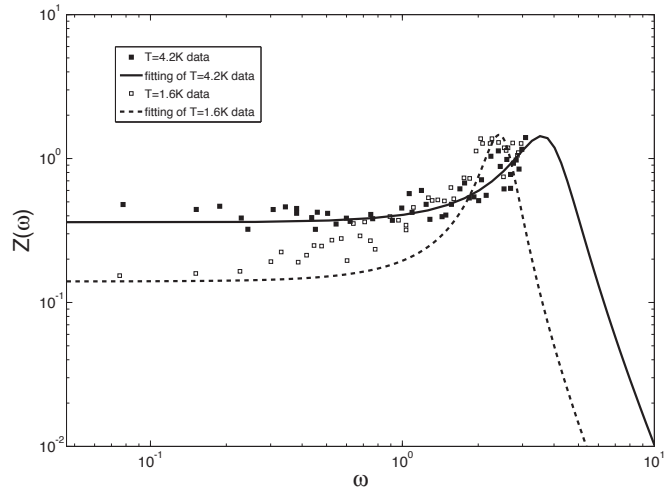
Figure 3.2: Fit of the spectrum as given by Equation 3.49 at two different temperatures and comparison with the data of the experiment of Koch et al. (1982). The best fit values of (α, γ, p) for $T = 1.6\text{K}$ are $(\alpha = 0.77901, \gamma = 6.1774, p = 2.3114)$ and for $T = 4.2\text{K}$ $(\alpha = 1.9932, \gamma = 14.7769, p = 8.8662)$. α is in units of 10^{12} radians, γ in units of 10^{24} radians, and p in units of 10^{24} rad-meV$^{1/2}$. The spectral energy $Z(\omega)$ is in units of meV and the angular frequency ω is in units of 10^{12} radians.
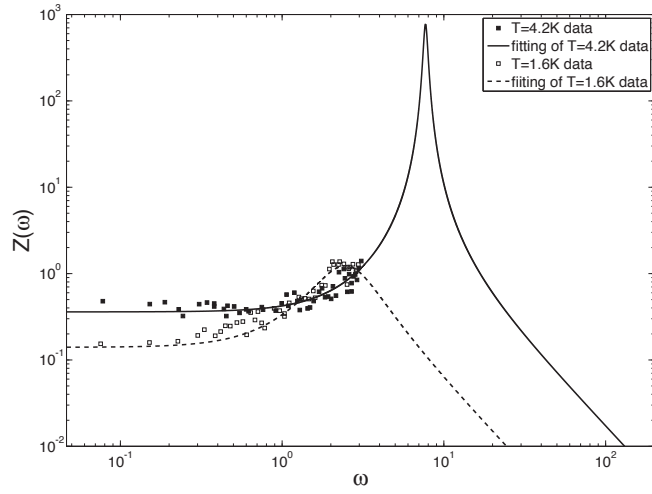
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Figure 3.3: Fit of the spectrum as given by Equation 3.49 at two different temperatures and comparison with the data of the experiment of Koch et al. (1982). The best fit values of $(\alpha, \gamma, p, \sigma)$ for $T = 1.6\text{K}$ are $(\alpha = 2.3502, \gamma = 6.4063, p = 2.397, \sigma = 2.4253)$ and for $T = 4.2\text{K}$ $(\alpha = 0.5, \gamma = 59.3082, p = 35.5849, \sigma = 13.0625)$. All units as in Figure 3.2 and σ has the units 10^{12} meV$^{1/2}$.

Bousso, R. (2008). The cosmological constant. *Gen. Relativ. Gravit.*, 40:607–637.

Boyer, T. (1969). Recalculations of long range van der Waals potentials. *Phys. Rev.*, 180(1):19–24.

Boyer, T. (1972a). Asymptotic retarded van der Waals forces derived from classical electrodynamics with classical electromagnetic zero point radiation. *Phys. Rev. A.*, 5(4):1799–1802.

Boyer, T. (1972b). Unretarded London-van der Waals forces derived from classical electrodynamics with classical electromagnetic zero point radiation. *Phys. Rev. A.*, 6(1):314–319.

Boyer, T. (1973a). Classical electromagnetic deflections and lag effects associated with quantum interference pattern shifts: Considerations related to the Aharonov-Bohm effect. *Phys. Rev. D.*, 8(6):1679–1693.

Boyer, T. (1973b). Retarded van der Waals forces at all distances derived from classical electrodynamics with classical electromagnetic zero point radiation. *Phys. Rev. A.*, 7(6):1832–1840.

Boyer, T. (1974). Van der Waals forces and zero point energy for dielectric and permeable materials. *Phys. Rev. A.*, 9(5):2078–2084.

Boyer, T. (1975). Temperature dependence of Van der Waals forces in classical electrodynamics with classical electromagnetic zero point radiation. *Phys. Rev. D.*, 11(5):1650–1663.

Boyer, T. (1987). Proposed Aharonov-Casher effect: Another example of an Aharonov-Bohm effect arising from a classical lag. *Phys. Rev. A.*, 36(10):5083–5086.

Callen, H. and Welton, T. (1951). Irreversibility and generalized noise. *Phys. Rev.*, 83(1):34–40.

Casimir, H. (1948). On the attraction between two perfectly conducting plates. *Kon. Ned. Akad. Wetensch.*, 51B:793–796.

Casimir, H. and Polder, D. (1948). The influence of retardation on the London-van der Waals forces. *Phys. Rev.*, 73:360–372.

Debye, P. (1914). Interferenz von Röntgenstrahlen und Wäarmebewegung. *Ann. d. Phys.*, 43:49–??

Erickson, R. V. (1971). Constant coefficient linear differential equations driven by white noise. *Ann. Math. Statist.*, 42:820–823.

Hille, E. (1926). A class of reciprocal functions. *Ann. of Math. (2)*, 27(4):427–464.

Koch, R. H., van Harlingen, D., and Clarke, J. (1980). Quantum-noise theory for the resistively shunted Josephson junction. *Phys. Rev. Lett.*, 45(26):2132–2135.

Koch, R. H., van Harlingen, D., and Clarke, J. (1982). Measurements of quantum noise in resistively shunted Josephson junctions. *Phys. Rev. B.*, 26(1):74–87.

Mackey, M. C. and Tyran-Kamińska, M. (2006). Deterministic Brownian motion: The effects of perturbing a dynamical system by a chaotic semi-dynamical system. *Phys. Rep.*, 422(5):167–222.

Mackey, M. C. and Tyran-Kamińska, M. (2006). Noise and conditional entropy evolution. *Physica A*, 365:360–382.

Milonni, P. W. (1994). *The Quantum Vacuum: An Introduction to Quantum Electrodynamics*. Academic Press Inc., Boston.

Peebles, P. and Ratra, B. (2003). The cosmological constiant and dark energy. *Rev. Mod. Phys.*, 75:559–606.

Planck, M. (1914). *The Theory of Heat Radiation*. P. Blakistons's Son and Co.

Power, E. (1966). Zero point energy and the Lamb shift. *Am. J. Phys.*, 34:516–518.

Rugh, S. and Zinkernagel, H. (2002). The quantum vacuum and the cosmological constant problem. *Stud. Hist. Philo. Mod. Phys.*, 33:663–705.

Warburton, P. (2006). Externally-shunted high-gap Josephson junctions: Design, fabrication and noise measurements. *EPSRC (U.K.) grant*, EP/D029783/1.

Weinberg, S. (1989). The cosmological constant problem. *Rev. Mod. Phys.*, 61(1):1–23.

Welton, T. (1948). Some observable effects of the quantum mechanical fluctuations of the electromagnetic field. *Phys. Rev.*, 74(9):1157–1167.

Zakai, M. and Snyders, J. (1970). Stationary probability measures for linear differential equations driven by white noise. *J. Differential Equations*, 8:27–33.