LASOTA MIXING NOTES
AN ACCOUNT OF ANDY LASOTA'S IDEA OF HOW TO DEFINE MIXING FOR NON-MEASURE PRESERVING DYNAMICS (LIKE THE HENON MAP)
FILE: LASMIX.TEX

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Let X be a sigma finite measure space, and $S : X \to X$ a dynamics that is not necessarily measure preserving.

Definition LM1. *We say that S is* **Lasota Mixing 1** *if and only if $\forall\ A, B, C \subset X$ with $\mu(A), \mu(B), \mu(C)$ nonzero and finite we have*

$$\lim_{n\to\infty} \frac{\mu(S^{-n}(C)\cap A)}{\mu(S^{-n}(C)\cap B)} \to \frac{\mu(A)}{\mu(B)}. \tag{1}$$

OBSERVATIONS ABOUT LM1.

(1) This definition reduces to the normal definition of mixing if S preserves the measure μ. To see this consider $B = X$ and $\mu(X) = 1$ so

$$\begin{aligned} \mu(S^{-n}(C)\cap B) &= \mu(S^{-n}(C)\cap X) \\ &= \mu(S^{-n}(C)) \\ &= \mu(C), \end{aligned}$$

where the last two lines follow from the assumption that μ is preserved. Therefore we have

$$\lim_{n\to\infty} \mu(S^{-n}(C)\cap A) = \mu(A)\mu(C).$$

(2) In equation (1) the left hand side may not be defined for small n, i.e. n must be sufficiently large to permit adequate spreading. This does not, however, matter for the limit.

(3) As an example of Lasota Mixing 1, we might consider the baker transformation in which you have compression by a factor of 4 but only stretch by a factor of 2:

$$T(x,y) = \begin{cases} 2x, \frac{1}{4}y & x \in [0, \frac{1}{2}] \\ 2x-1, \frac{1}{4}y + \frac{1}{2} & x \in (\frac{1}{2}, 1] \end{cases}$$

(4) The definition of Lasota Mixing 1 is good for situations in which the contraction (or expansion) of a set by iteration is *independent of the set* (or its location).

In thinking about a second (and more general) definition of Lasota Mixing, let X be a sigma finite measure space, and $S : X \to X$ a dynamics that is not necessarily measure preserving.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Definition LM2. *We say that* S *is* **Lasota Mixing 2** *if and only if* $\forall\ A, B, C \subset X$ *with* $\mu(A), \mu(B), \mu(C)$ *nonzero and finite* $\exists$ *a finite* $\lambda > 0$, *independent of* C, $\ni$

$$\lim_{n\to\infty} \frac{\mu(S^{-n}(C)\cap A)}{\mu(S^{-n}(C)\cap B)} \to \lambda \tag{2}$$

In general, λ *depends on* A *and* B.

OBSERVATIONS ABOUT LM2.

(1) If $\mu(X) = 1$ and S is measure preserving, then LM2 is equivalent to normal (Strong or Hopf) mixing.

Proof. *Since* B *is arbitrary, take* $B = X$. *then we have*

$$\mu(X \cap S^{-n}(C)) = \mu(S^{-n}(C)) = \mu(C)$$

since S *is measure preserving. Thus from (2) we have*

$$\lim_{n\to\infty} \mu(S^{-n}(C)\cap A) = \lambda\mu(C)$$

since λ *only depends on* A. *This, in turn, implies that*

$$\lim_{n\to\infty} \mu(S^{-n}(X\setminus C)\cap A) = \lambda\mu(X\setminus C)$$

since λ *is independent of* C. *But we can rewrite the left hand side of this relation as*

$$\begin{aligned}\mu(S^{-n}(X\setminus C)\cap A) &= \mu((X\setminus S^{-n}(C)\cap A)\\ &= \mu(A\setminus S^{-n}(C))\\ &= \mu(A) - \mu(A\cap S^{-n}(C))\\ &\to \mu(A) - \lambda\mu(C)\end{aligned}$$

so we have

$$\begin{aligned}\mu(A) - \lambda\mu(C) &= \lambda\mu(X\setminus C)\\ &= \lambda[\mu(X) - \mu(C)]\\ &= \lambda[1-\mu(C)].\end{aligned}$$

This implies that $\lambda = \mu(A)$ *and thus*

$$\lim_{n\to\infty} \mu(S^{-n}(C)\cap A) = \mu(A)\mu(C)$$

(15 June, 1987: Brissac). If S is LM2, then S is ergodic.

Proof. *Assume* C *is an invariant set so*

$$C = S^{-1}(C) \qquad \Longrightarrow \qquad C = S^{-n}(C).$$

From the definition of LM2 we have

$$\frac{\mu(A\cap C}{\mu(B\cap C)} = \lambda$$

We want to show that λ *independent of* $C \implies c \subset X$ *is trivial.*
a. *If* $C = X$ *then* λ *is independent of* C *since*

$$\lambda = \frac{\mu(A)}{\mu(B)}.$$

b. A *is arbitrary so set* $A = X\setminus C \implies$

$$\mu(A\cap C) = 0 \tag{3}$$

But

$$\mu(A\cap C) = \lambda\mu(B\cap C)$$

so with (3) and $\lambda > 0$ *(definition of LM2), we must have* $\mu(B\cap C) = 0$. *Since* $\mu(B) > 0$ *by assumption, this* $\implies \mu(C) = 0$.
Therefore, all invariant subsets are trivial and ergodicity is proved.

with f_* $\forall$ $f \in \mathcal{D}$.

16 June, 1987: Brissac
Correlations and Lasota Mixing

See "Time's Arrow", Chapter 5 (Mixing), Section C (The Decay of Correlations) for the relevant computations reprinted below.

To understand the connection between mixing and the decay of correlations requires the introduction of a few concepts. If we have a time series $x(t)$ [either discrete or continuous], and two bounded integrable functions $\sigma, \eta : X \to R$, then the **correlation** of σ with η is defined as

$$R_{\sigma,\eta}(\tau) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sigma(x(t+\tau))\eta(x(t))$$

in the discrete time case, or

$$R_{\sigma,\eta}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sigma(x(t+\tau))\eta(x(t))\, dt$$

in the continuous case. The **average** of the function σ is just

$$< \sigma >= \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sigma(x(t)),$$

or

$$< \sigma >= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sigma(x(t))\, dt,$$

so it is clear that

(1) $R_{\sigma,\eta}(0) =< \sigma\eta >$; and
(2) $R_{\sigma^2}(0)R_{\eta^2}(0) \geq R_{\sigma,\eta}(\tau)$. This follows directly by writing out the expression $< [\alpha\sigma(t) + \beta\eta(t+\tau)]^2 >$ for real and nonzero α and β, and noting that it must be nonnegative.

The **covariance** of σ with η, $C_{\sigma,\eta}(\tau)$, is defined by

$$C_{\sigma,\eta}(\tau) = R_{\sigma,\eta}(\tau) - < \sigma >< \eta >,$$

while the **normalized covariance** $\rho_{\sigma,\eta}(\tau)$ is

$$\rho_{\sigma,\eta}(\tau) = \frac{R_{\sigma,\eta}(\tau) - < \sigma >< \eta >}{< \sigma\eta > - < \sigma >< \eta >}.$$

Clearly, $\rho_{\sigma,\eta}(0) = 1$.

Now assume we have an ergodic transformation S_t with consequent unique stationary density f_*, operating in a finite normalized phase space X, and that S_t is generating the sequence of values $\{x(t)\}$. Then the correlation of σ with η can be written in both the discrete and continuous time case as

$$R_{\sigma,\eta}(\tau) = \int_X \sigma(S_\tau(x))\eta(x)f_*(x)\, dx \tag{5.7}$$

by use of the extension of the Birkhoff Ergodic Theorem 4.6. Using the definition of the Koopman operator, along with the adjointness of the Frobenius-Perron and Koopman operators, equation (5.7) can be rewritten in the form

$$R_{\sigma,\eta}(\tau) =< \eta f_*, U^\tau \sigma >=< P^\tau(\eta f_*), \sigma > . \tag{5.8}$$

Writing the defining relation for mixing transformations as in the proof of Theorem 5.1, it is clear that for general functions η and σ we have

$$\lim_{t\to\infty} < P^t(\eta f_*), \sigma >=< \eta f_*, 1 >< f_*, \sigma >,$$

so (5.8) yields

$$\lim_{\tau\to\infty} R_{\sigma,\eta}(\tau) =< \eta >< \sigma >$$

when S_t is mixing. Thus we have the following result connecting mixing with the limiting behaviour of the normalized covariance. Namely,

Theorem 5.3. *S_t is f_* mixing if and only if*

$$\lim_{\tau\to\infty} \rho_{\sigma,\eta}(\tau) = 0.$$

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Both Krylov and Ma emphasize decay of correlations to zero in a finite time as being important for the approach of the entropy to its maximum. Many exact transformations have this property, so we have the following conjectures.

Conjecture 1. *$S : X \to X$ measure preserving (is it necessary that measure be finite). S is exact if and only if $\exists$ finite $\tau_0 > 0 \ni$*

$$\rho_{f,g}(\tau) = 0 \qquad \forall\, \tau > \tau_0.$$

Corollary 1. *$\exists$ finite $n_0(f) > 0 \ni$*

$$H(P^n f) = 0 \qquad \forall\, n > n_0(f).$$

Conjecture 2. *$S : X \to X$ ergodic with stationary density f_*. S is asymptotically stable if and only if $\exists$ finite $\tau_0 > 0 \ni$*

$$R_{f,g}(\tau) = < f, 1 >< f_*, g) \qquad f \in L^1,\ g \in L^\infty$$

$\forall\, \tau > \tau_0$.

Corollary 2. *$\exists$ finite $n_0(f) > 0 \ni$*

$$H(P^n f | f_*) = 0 \qquad \forall\, n > n_0(f).$$

2 July, 1987: Brissac
Weak Lasota Mixing

In the normal definition of weak mixing we have $S : X \to X$ on a normalized measure space, S measure preserving $\iff f_* = 1$ is the only stationary density. Then S is **weak mixing** if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(A \cap S^{-k}(B)) - \mu(A)\mu(B)| = 0 \qquad A, B \in \mathcal{A}.$$

An obvious extension to a non measure preserving S is given by

Definition. *S is **Lasota Weak Mixing** if $\forall\, A, B, C \subset \mathcal{A}$ $\exists$ finite positive λ, independent of C, $\ni$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \frac{\mu(A \cap S^{-k}(C))}{\mu(B \cap S^{-k}(C))} - \lambda \right| = 0 \qquad \forall A, B \subset \mathcal{A}.$$

Generalizations of K-Automorphism

In the normal case we define K-automorphism in the following way. We have $S^n(\mathcal{A}) = \{S^n(A) : A \subset \mathcal{A}\}$. $(X, \mathcal{A}, \mu)$ is normalized and $S : X \to X$ is invertible and $\ni S$ and S^{-1} are measurable and measure preserving. S is a K automorphism is $\exists$ a sigma algebra $\mathcal{A}_0 \subset \mathcal{A} \ni$

(1) $S^{-1}(\mathcal{A}_0) \subset \mathcal{A}_0$;
(2) The sigma algebra $\cap_{n=0}^{\infty} S^{-n}(\mathcal{A}_0)$ is trivial (only consists of sets of measure 0 or 1); and
(3) The smallest sigma algebra containing $\cup_{n=0}^{\infty} S^{-n}(\mathcal{A}_0)$ is $\mathcal{A}$.

How to generalize this definition to non measure preserving S so that the new property $\iff$ Lasota Mixing? $\exists$ two questions here:

(1) Generalize K automorphism definition to non measure preserving S. **Hint**: Look at the proof that the baker transformation is a K automorphism (E4.5.1, pp 74-5) for clue of how to generalize.
(2) Show that the generalization $\iff$ Lasota Mixing. **Hint**: Look at proof of Theorem 4.5.2 (every K auto is mixing) for the clue. See also Walters (1982) **An Introduction to Ergodic Theory**, Springer; and Parry (1981) **Topics in Ergodic Theory**, Cambridge University Press.