FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAY
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I. STATEMENT OF THE PROBLEM.

Retarded functional differential equations with distributed delay have been investigated by Hale (1977), Kolmanovskii and Nosov (1980), Stepan (1989), Kuang (1993), Anderson (1991), Walther (1975), McDonald (1989). Let us review some known results.

For processes with aftereffect the case of distributed retardation is often encountered. It means that the derivative $\dot{x}(t)$ depends not on the values of unknown function at some fixed moments of time $t-\tau_i(t)$, but on all values $x(s)$ for s varying over a segment $a_t \le s \le b_t$. For example

$$\dot{x}(t) = f\left(t, x(t), \int_{a_t}^{b_t} K(t,s)x(s)ds\right)$$

Thus we come to consider integrodifferential equations.

The theory of differential equations with aftereffect is often set forth in such a way as to include both the case of concentrated and distributed delay by considering integrodifferential equations containing the Stieltjes integral. For example the theory of linear equations may be represented by equations of the form

$$\dot{x}(t) = \int_0^\infty x(t-s)d_sK(t,s) + f(t),$$

or for the system of equations

$$\dot{x}_i(t) = \sum_{j=1}^m \int_0^\infty x_j(t-s)d_sK_{ij}(t,s) + f_i(t),$$

as Myshkis (1966) has considered.

If the function $K(t,s)$ or $K_{ij}(t,s)$ with finite variation with respect to s is piecewise constant in s, then we will have linear equation (or system) with concentrated delay.

In particular, if the kernel $K(t,s)$ (or $K_{ij}(t,s)$) is independent of t, the equation is a generalization of a linear equation with constant coefficients, and particular solutions corresponding to their homogeneous equations may be sought in the form of exponential functions. For instance the equation

$$\dot{x}(t) = \int_0^\infty x(t-s)dK(s)$$

has a solution of the form e^{kt} where k satisfies the characteristic equation

$$k = \int_0^\infty e^{-ks}dK(s).$$

With respect to the kernel classification the following types of kernels were considered:

1. K(s) is nonincreasing and non-constant [Stepan, 1989, p.87, equation (3.36)].
2. K(s) is nondecreasing.
3. K(s) is convex.
4. Results are given in the terms of kernel's moments (Kolmanovskii).

e) ?

Following Stépán (1989, p.4), the general form of a linear autonomous (and homogeneous) retarded functional differential equation (RFDE) is

$$\dot{x}(t) = L(x_t) \tag{1}$$

where the functional $L : B \to R^n$ is a continuous and linear. Here B denotes the vector space of continuous and bounded functions mapping the interval $[-h, 0]$ into R^n. With the norm given by

$$||\phi|| = \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|, \qquad \phi \in B$$

B is a Banach space.

By x_t we understand $x_t = x(t+\theta), \theta \in [-\tau, 0]$. According to the Rietz Representation Theorem, the equation can be represented as

$$\dot{x}(t) = \int_{-\infty}^{0} [d\mu(\theta)] x(t+\theta) \tag{2}$$

where μ is n-dimensional matrix of functions of bounded variation on $(-\infty, 0]$ and the integral is a Riemann-Stieltjes one. Let C denote the set of complex numbers. By substitution $x(t) \simeq e^{\lambda t}$ into (2) or by means of Laplace transformation of (2) we can obtain the **characteristic equation**

$$D(\lambda) = \det(\lambda I - \int_{-\infty}^{0} e^{\lambda\theta} d\mu(\theta)), \qquad \lambda \in C \tag{3}$$

where I is the unit matrix.

Definition. *The characteristic function D of RFDE is called stable if*

$$\{\lambda \in C : Re\lambda \geq 0, D(\lambda) = 0\} = \emptyset$$

where D is given by (3).

The linear equation (2) can represent the case of finite delay as well. The delay has a finite length τ when μ or μ_0 are constants in $(-\infty, -\tau)$.

II. DIFFERENT KINDS OF KERNELS.

1. Kernel $\mu(s)$ is non-increasing and non-constant $\Longrightarrow$ kernel is decreasing.

Stépán (1989, p. 87, eq. (3.36)) investigated the **first order scalar** equation

$$\dot{x}(t) = \int_{-\infty}^{0} x(t+\theta) d\mu(\theta) \tag{4}$$

where $x \in R$, μ is a scalar function of bounded variation and it is satisfied the condition

$$\int_{-\infty}^{0} e^{-\nu\theta} d\mu(\theta) < +\infty.$$

Theorem 1. *(Theorem 3.28 in Stépán) Suppose that μ is a* **non-constant and non-increasing** *function in (4). The trivial solution of (4) is asymptotically exponentially stable if*

$$\int_{-\infty}^{0} \theta d\mu(\theta) < 1. \tag{5}$$

For proof see p. 87-88 in Stépán (1989).

In [4, p.94] Kuang considered the more general case of a scalar equation

$$\dot{x}(t) = -ax(t) + b \int_{-\infty}^{0} x(t+\theta) d\eta(\theta), \tag{6}$$

$a, b > 0$, $\eta(\theta)$ is non-constant and non-increasing.

Theorem 2 (Theorem 5.3 in Kuang (1993). *The trivial solution of (6) is uniformly asymptotically stable if*

$$a \geq 0, \qquad b\int_{-\infty}^{0} \theta d\eta(\theta) < 1 \tag{7}$$

and there exists a constant $v > 0$ such that

$$\int_{-\infty}^{0} e^{-v\theta}|d\eta(\theta)| < +\infty.$$

For proof see Kuang [1993, p.94].

2. Kernel $\mu(s)$ is non-decreasing $\Longrightarrow$ is constant or increasing.

Stépán gives an example of a scalar equation (Theorem 3.30, p.89)

$$\dot{x}(t) = -\mu_0 x(t) + \int_{-\infty}^{0} x(t+\theta)d\mu(\theta) \tag{8}$$

where $\mu(\theta)$ is non-decreasing and satisfied the condition

$$\int_{-\infty}^{0} e^{-\nu\theta}|d\mu(\theta)| < +\infty.$$

Theorem 3. *The trivial solution of (8) is exponentially asymptotically stable if and only if*

$$\mu_0 > \int_{-\infty}^{0} d\mu. \tag{9}$$

For proof see Stépán [1989, p.89].

Kuang (1993) cites the result of Walther (1975) who studied the stability of the solution of the equation with distributed delay over the finite segment

$$\dot{x}(t) = -\alpha\int_{-\tau}^{0} x(t+\theta)d\eta(\theta), \qquad \alpha > 0, \qquad \tau > 0, \tag{10}$$

where $\eta : [-\tau, 0] \to R$ is **increasing** and has total variation $V(\eta)$ not exceeding unity. The following theorem is due to Walther (1975).

Theorem 4. *Let $\alpha > 0$ and $\tau > 0$. Assume that $\eta : [-\tau, 0] \to R$ is nonconstant increasing function such that $\eta(0) - \eta(-\tau) \leq 1$ and $\alpha\tau < \pi/2$. Then equation (9) is uniformly asymptotically stable.*

For proof see Kuang (1993, p.99 or original paper of Walther).

3. Kernel is convex.

R. Anderson (1991) referred to the results of Walther (1976) and Cushing (1977) that for certain kinds of dynamical systems the introduction of a distributed delay that has a convex density function preserves stability.

I DID NOT TAKE YET ORIGINAL RESULTS.

3. Stability results given in terms of the kernel's moments (Kolmanovskii).

Kolmanovskii (1981, p.) considers the equation

$$\dot{x}(t) = \int_{0}^{\infty} x(t-s)dK_0(s), \tag{11}$$

which characteristic equation is

$$\lambda - \int_{0}^{\infty} e^{-\lambda s}dK_0(s) = 0.$$

Denote

$$\beta_{00} = \int_{0}^{\infty} dK_0(s), \qquad \alpha_{10} = \int_{0}^{\infty} s|dK_0(s)|. \tag{12}$$

Lemma 1. *Let the conditions*

$$\beta_{00} < 0, \qquad \alpha_{10} < 1 \tag{13}$$

be satisfied. Then the characteristic polynomial does not have a root with $Re\lambda \geq 0$*).*

Lemma 2. *Let the conditions (13) be satisfied*

1. $K_0(s)$ *is monotonic, nonincreasing and* $\beta_{00} < 0$ (*i.e.* $\int_0^\infty dK_0(s) < 0$).
2. $K_0(s) = const$ *for* $s \geq h > 0$.
3. $h\alpha_{00} < \frac{\pi}{2}$. (*i.e.* $h\int_0^\infty |dK_0(s)| < \frac{\pi}{2}$.

Then the equation

$$\lambda - \int_0^\infty e^{-\lambda s} dK_0(s) = 0$$

does not have roots with $Re\lambda \geq 0$.

III. RANDOM DELAY.

1. Concentrated delay. Consider the equation (MCM thinks that need to be careful cuz could give problems with consistency)

$$\dot{x}(t) = ax(t) + bx(t-u), \qquad u > 0.$$

Here we suppose that u is random with given mathematical expectation

$$Eu = \int_0^\infty uf(u)du < \infty,$$

where $f(u)$ denotes the probability density of u. Let us seek a solution $x(t) = Ae^{\lambda t}$ where A is random variable with finite mathematical expectation $EA < \infty$.

The solution is

$$x(t) = x(0) + a\int_0^t x(s)ds + b\int_0^t x(s-u)ds.$$

taking the mathematical expectation we have

$$Ex(t) = Ex(0) + a\int_0^t Ex(s)ds + b\int_0^t Ex(s-u)ds.$$

Substitute $x(t) = Ae^{\lambda t}$.

$$E\{Ae^{\lambda t}\} = E\{A\} + a\int_0^t E\{Ae^{\lambda s}\}ds + b\int_0^t E\{Ae^{\lambda s}e^{-\lambda u}\}ds.$$

$$e^{\lambda t}E\{A\} = E\{A\} + a\int_0^t E\{A\}e^{\lambda s} + b\int_0^t E\{A\}E\{e^{\lambda s}e^{-\lambda u}\}ds.$$

We have

$$e^{\lambda t} = 1 + a\int_0^t e^{\lambda s}ds + b\int_0^t e^{\lambda s}E\{e^{-\lambda u}\}ds.$$

Differentiating with respect to t we obtain the characteristic equation

$$\lambda = a + bE\{e^{-\lambda u}\}.$$

2. Distributed random delay.

Let us investigate the equation

$$dx(t) = ax(t) + b\int_{-\tau}^0 x(t+s)dK(s)$$

or

$$dx(t) = ax(t) + b\int_{-\infty}^0 x(t+s)dK(s).$$

Now let $dK(s) = f(s)ds$,

$$dx(t) = ax(t) + b\int_{-\infty}^{0} x(t+s)f(s)ds.$$

or we can rewrite this equation as

$$dx(t) = ax(t) + b\int_{0}^{\infty} x(t-s)f(s)ds.$$

Suppose $x(t) \sim e^{\lambda t}$. Then we have the characteristic equation

$$\lambda - a - b\int_{0}^{\infty} e^{-\lambda s}f(s)ds = 0.$$

Actually we have the same characteristic equation as for the case of concentrated random delay.

Suppose that $\lambda = i\omega$. Substituting into the characteristic equation gives us

$$i\omega - a - b\int_{0}^{\infty} \cos(\omega s)f(s)ds + ib\int_{0}^{\infty} sin(\omega s)f(s)ds = 0.$$

Separating real and imaginary parts we get

$$-a = b\int_{0}^{\infty} \cos(\omega s)f(s)ds,$$

$$\omega = -b\int_{0}^{\infty} sin(\omega s)f(s)ds.$$

For the second equation we can make an estimation

$$\omega = -b\int_{0}^{\infty} sin(\omega s)f(s)ds \geq -b\int_{0}^{\infty} \omega s f(s)ds.$$

From this we have that

$$\omega(1 + b\int_{0}^{\infty} sf(s)ds) > 0.$$

Consequently the condition

$$\int_{0}^{\infty} sf(s)ds < -\frac{1}{b}$$

which assumes $b < 0$. In terms of mathematical expectation we have the condition

$$Eu < -\frac{1}{b}$$

References.

Anderson R. (1991). Geometric and probabilistic stability criteria for delay systems. *Mathematical Biosciences*,**105**: 81-96.

El'sgol'ts, L.E. (1966). *Introduction to the Theory of Differential Equations with Deviating Arguments*, Holden-Day Inc., New York.

Hale, J.K. (1977). *Theory of Functional Differential Equations,* Springer-Verlag, New York.

Kolmanovskii, V.B., Nosov, V.R. (1986). *Stability of Functional Differential Equations*, Academic Press, New York.

Kuang Y. (1993) *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Inc.

Stépán G. (1989).*Retarded Dynamical Systems: Stability and Characteristic Functions*, Longman Scientific and Technical, UK.

Walther H.-O. (1975). Asymptotic stability for some functional differential equations. *Proc. Roy. Soc. Edinburgh* **74a**, 253-255.