QUANTIZED MOTION WITH NON-INSTANTANEOUS ACTION AT A DISTANCE

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These notes summarize a number of calculations that have been done over the past years in which I was trying to examine the effect of having forces acting that were not instantaneous, i.e., ones that either

(1) interact with a constant delay, or with a delay that is proportional to the distance over which the force acts. This latter situation has been extensively considered from a mathematical point of view by R.D. Driver and his co-workers[1]; or
(2) depend on both the positions of the interacting particles in the past *as well as* in the future (the so called half-advanced/half-retarded interaction case). This situation, in which causality is violated, is justified by the observation that if it is required that the laws of physics should be time reversal invariant, then the existence of retarded potentials implies the corresponding existence of advanced potentials. This notion, apparently contrary to common experience, has been raised a number of times over the past decades[2].

These theoretical investigations into the potential significance of retarded and/or advanced interactions have assumed even more significance in light of the recent experimental results of Sandoghdar et al.[3], so elegantly summarized in a historical context by Levy[4].

OK, enough of the blah-blah introduction. Now its time to do a

1. Warm Up Exercise.

[1]See R.D. Driver, Ann. Phys. (N.Y.) (1963) *21*, 122; R.D. Driver and M.J. Norris, *ibid.* (1967) *42*, 347; R.D. Driver, Phys. Rev. (1969) *178*, 2051; V.I. Zhdanov, "Convergence of iteration method in the relativistic two-body problem, taking into account the retardation of interactions", J. Phy. A: Math. Gen. (1991) *24*, 5011-5027.

[2]Namely in the work of K. Schwarzschild [Göttinger Nachrichten (1903) *128*, 132], H. Tetrode ["Über den Wirkungszusammenhang der Welt: Eine Erweiterung der Klassichen dynamik", Zeits. für Physik (1922) *10*, 317-328], A.D. Fokker [Zeits. für Physik (1929) *58*, 386; Physica (1929) *9*, 33; "Theorie relativiste de l'interaction de deux particules chargees", *ibid* (1932) *12*, 145-152], L. Page ["Advanced potentials and their application to atomic models", Phys. Rev. (1924) *24*, 296-305] J.A. Wheeler and R.P Feynman ["Interaction with the absorber as the mechanism of radiation", Rev. Mod. Phys. (1945) *17*, 157-181; "Classical electrodynamics in terms of direct interparticle action", *ibid* (1949) *21*, 425-433], C.M. Anderson and H.C. von Baeyer ["Almost circular orbits in classical action at a distance electrodynamics", Phys. Rev. D. (1972) *5*, 802; "Solutions of the two-body problem in classical action at a distance electrodynamics: Straight line motion", *ibid*, 2470-2476], R.D. Driver ["Can the future influence the present", Phys. Rev. D. (1979) *19*, 1098-1107], J.T. Hoag and R.D. Driver ["A delayed-advanced model for the electrodynamics two-body problem", Nonlin. Anal. (1990) *15*, 165-183]

[3]V. Sandoghdar, C.I. Sukenik, and E.A. Hinds, "'Direct measurement of the van der Waals interaction between an atom and its images in a micro- sized cavity", Phys. Rev. Lett. (1992), *68*, 3432-3435

[4]B.G. Levy, "New evidence confirms old predictions of retarded forces", Phys. Today (1993) *April, 1993*, 18-20.

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According to my notes, on September 18, 1986, Helmut Schwegler and I considered a slightly modified harmonic oscillator

$$\begin{aligned} m\frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -k\left[\alpha x(t-\tau)+\beta x(t+\tau)\right] \end{aligned} \tag{1.1}$$

with mass m, position x, velocity v, and spring constant k. The thing that is different about this harmonic oscillator is that we have allowed for the possibility of both retarded and advanced forces, where τ is the retardation and/or advancement. There are two free constants in this force, α and β. We assume that $\alpha, \beta \in [0,1]$ and that $\alpha+\beta=1$. Thus, if $(\alpha,\beta)=(1,0)$ then the force is purely retarded, while if $(\alpha,\beta)=(\frac{1}{2},\frac{1}{2})$ then we have the half-advanced/half-retarded situation that has received so much interest in the past (cf. references of the introduction).

Combining the equations (1.1) gives

$$m\frac{d^2x}{dt} = -k\left[\alpha x(t-\tau)+\beta x(t+\tau)\right]. \tag{1.2a}$$

From now on I will use the convention that $x_{-\tau} \equiv x(t-\tau)$ so $x_\tau \equiv x(t+\tau)$ and equation (1.2a) can be equivalently written

$$m\frac{d^2x}{dt} = -k\left[\alpha x_{-\tau}+\beta x_\tau\right] \tag{1.2b}$$

We are curious to know if undamped harmonic motion is a possibility in this model defined by (1.2b). Since (1.2b) is linear, this is a pretty straightforward question to answer. Make the ansatz that $x(t)=\exp(\lambda t)$, substitute this into (1.2b) and obtain the eigenvalue equation

$$m\lambda^2 = -k\left[\alpha e^{-\lambda\tau}+\beta e^{\lambda\tau}\right]. \tag{1.3}$$

Assume that $\lambda=\mu+i\omega$, and substitute this into equation (1.3). After separating the real and imaginary parts of the result we have, respectively,

$$\begin{aligned} m(\mu^2-\omega^2) &= -k\cos(\omega\tau)[\alpha e^{-\mu\tau}+\beta e^{\mu\tau}] \\ 2m\mu\omega &= -k\sin(\omega\tau)\left[-\alpha e^{-\mu\tau}+\beta e^{\mu\tau}\right]. \end{aligned} \tag{1.4}$$

Since we are looking for strictly periodic motion, take $\mu=0$ so $\lambda=i\omega$ and equations (1.4) become

$$m\omega^2 = k\cos(\omega\tau) \tag{1.5a}$$

$$0 = (\beta-\alpha)\sin(\omega\tau). \tag{1.5b}$$

Note that for strictly periodic solutions of (1.2) to exist, Equations (1.5a,b) must be simultaneously satisfied.

Equation (1.5b) can be satisfied on one of two ways. Either

(1) $\sin(\omega\tau)\equiv 0$ which implies that $\omega\tau=n\pi$ where n is an integer; or
(2) $\alpha\equiv\beta$ which, in combination with the restriction that $\alpha+\beta=1$, immediately implies that $\alpha=\beta=\frac{1}{2}$.

With these two possibilities in mind, we now turn to a consideration of how (1.5a) can be satisfied.

With Possibility 1 for satisfying (1.5b), if $\omega\tau=n\pi$ with n an integer then $\cos(\omega\tau)=(-1)^n$ so, from equation (1.5a) and the fact that k, m, and ω^2 are all positive we conclude that if Possibility 1 is to be considered then n must be an even integer and $\omega\tau=2\kappa\pi$, where κ is an integer. Note that this condition places no additional constraints on α and β over and above the original one that $\alpha+\beta=1$.

To consider Possibility 2, rewrite (1.5a) as

$$\frac{m}{k\tau^2}(\omega\tau)^2 = \cos(\omega\tau). \tag{1.6}$$

For a fixed τ, solving this transcendental equation (1.6) analytically for ω is, in general, impossible. However, note that in the special case that

$$\boxed{\frac{m}{k\tau^2} << 1,} \tag{1.7}$$

then (1.6) will have a number of zeros at values given approximately by

$$\omega\tau \simeq \left(n+\frac{1}{2}\right)\pi. \tag{1.8a}$$

Since $\omega = 2\pi f$, this last relation becomes

$$\boxed{f \simeq \left(n+\frac{1}{2}\right)\frac{1}{2\tau}.} \tag{1.8b}$$

Equation (1.8b) gives, of course, the allowed frequencies of the linear harmonic oscillator at which radiation occurs–an example that is carried out in every undergraduate course in quantum mechanics.

1a. The issue of energy. In a system like (1.1), just exactly how should one go about defining the energy E? We could assume, as classically, that E is made up of a kinetic energy, which we take to be

$$E_{kin} = \frac{1}{2}mv^2, \tag{1.9}$$

but what to do about defining an analog of the potential energy? If we define an operator

$$\mathcal{D} \equiv \frac{\partial}{\partial x_{-\tau}} + \frac{\partial}{\partial x_{\tau}}, \tag{1.10}$$

so

$$F = -k\left[\alpha x_{-\tau} + \beta x_{\tau}\right] = -\mathcal{D}\phi, \tag{1.11}$$

then what should we take the potential ϕ to be? Its obvious that either of the following two choices will work equally well:

$$\phi(x_\tau, x_{-\tau}) = \frac{k}{2}\left[\alpha x_\tau^2 + \beta x_{-\tau}^2\right] \tag{1.12a}$$

$$\phi(x_\tau, x_{-\tau}) = \frac{k}{2}\left[\alpha x_\tau + \beta x_{-\tau}\right]^2 \tag{1.12b}$$

This leaves me in a bit of a jam! What to do? Well, which ever (if either) definition of potential energy is correct the total energy should be

$$E = \frac{1}{2}mv^2 + \phi, \tag{1.13}$$

so

$$\begin{aligned} \frac{dE}{dt} &= mv\dot{v} + \frac{\partial\phi}{\partial t} \\ &= vF + \frac{\partial\phi}{\partial t} \\ &= -v\mathcal{D}\phi + \frac{\partial\phi}{\partial t}. \end{aligned} \tag{1.14}$$

1b. Nonconstant τ. Of course, this is a bit of a stupid exercise since if we are considering advances and/or delays that are a consequence of the exchange of virtual particles, then the delays and advances will not be constant. To illustrate this, it would seem that the sensible ("physically realistic") version of (1.2) should, in fact, be

$$m\frac{d^2x}{dt} = -k\left[\alpha x(t-\tau_R) + \beta x(t+\tau_A)\right], \tag{1.15}$$

where τ_R and τ_A must satisfy the functional equations

$$c\tau_R = |x(t) - x(t-\tau_R)| \tag{1.16a}$$

$$c\tau_A = |x(t) - x(t+\tau_A)| \tag{1.16b}$$

and c is the speed of light. Probably one might also want to include relativistic effects in the mass m, though that is unclear at this point. However, this becomes a total mess mathematically, and thus a different, and maybe more interesting system to study is one in which we have an electron and proton interacting through advanced and retarded potentials. This we treat in the third section, after some preliminary remarks about the old Bohr quantization.

2. Bohr Treatment of Atomic Spectra. This section outlines the assumptions and results of the old Bohr quantization rules, as these provide a touchstone for any alternative treatment of quantum mechanics.

In his work, Bohr made four assumptions that we can summarize as follows:

(1) Atomic systems have a number of *stationary states*. In a stationary state, there is no emission of radiation even though this would be expected on the basis of classical electromagnetic theory.
(2) Any emission or absorption of radiation corresponds to a *transition* between stationary states. The frequency (ν) of radiation (either emitted or absorbed) is given by
$$\nu = \frac{E_1 - E_2}{h},$$
where h is Planck's constant, and $E_{1,2}$ is the energy of the two stationary states between which the transition is occurring.
(3) When the system is is a stationary state, the dynamics are governed by classical considerations, **but this is not the case for transitions between stationary states**.
(4) Different stationary states for an electron orbiting a proton in a circular orbit are determined by
$$p = n\left(\frac{h}{2\pi}\right) = n\hbar,$$
so the angular momentum p is an integral multiple of $\hbar$.

Now, lets put flesh on these assumptions and do some calculations for an electron-proton system, where the proton is fixed (infinitely massive) and the electron has finite mass m. First, from Newton's laws $ma = F$, where a is the acceleration and F is the force, for circular motion we obtain
$$m\frac{v^2}{r} = \frac{|A|}{r^2},$$
where r is the radius of the orbit, v is the electron velocity, and the constant A is given by
$$A = -\frac{e^2}{4\pi\epsilon_0} = -\frac{e^2c^2}{10^7}.$$
(e is the electronic charge and c is the velocity of light.) Thus
$$mv^2 = \frac{|A|}{r}. \tag{2.1}$$
If E is the total energy of the electron, then
$$\begin{aligned} E &= E_{kin} + E_{pot} \\ &= \frac{1}{2}mv^2 + \frac{A}{r} = \frac{1}{2}mv^2 - \frac{|A|}{r} \end{aligned}$$
or
$$E = \frac{A}{2r} = -\frac{|A|}{2r}. \tag{2.2}$$

Since Bohr assumes that the angular momentum is quantized (assumption 4) $p_n = n\hbar$ so
$$p_n = np_1,$$
and in a circular orbit the angular momentum is
$$p = mvr,$$
which means that $mv_nr_n = n\hbar$ so $v_n = n\hbar/mr_n$ and consequently from (2.1)
$$mv_n^2 = m\left(\frac{n\hbar}{mr_n}\right)^2 = \frac{|A|}{r_n}.$$

Solving this relation for r_n gives

$$r_n = \frac{n^2\hbar^2}{m|A|},$$

or

$$\boxed{r_n = n^2 r_1} \tag{2.3}$$

with

$$r_1 = \frac{\hbar^2}{m|A|}. \tag{2.3'}$$

Furthermore, we may now write v_n as

$$v_n = \frac{|A|}{n\hbar},$$

or

$$\boxed{v_n = \frac{v_1}{n}} \tag{2.4}$$

with

$$v_1 = \frac{|A|}{\hbar}. \tag{2.4'}$$

Since the orbital frequency f is given by

$$f = \frac{\omega}{2\pi} = \frac{v}{2\pi r},$$

we have

$$\begin{aligned} f_n &= \frac{v_n}{2\pi r_n} \\ &= \left(\frac{n\hbar}{2\pi m}\right)\left(\frac{m|A|}{n^2\hbar^2}\right)^2 \\ &= \left(\frac{m}{2\pi}\right)\frac{|A|^2}{(n\hbar)^3}, \end{aligned}$$

or

$$\boxed{f_n = \frac{f_1}{n^3}} \tag{2.5}$$

where

$$f_1 = \frac{m}{2\pi}\frac{|A|^2}{\hbar^3}. \tag{2.5'}$$

Since the period T_n is just $T_n = 1/f_n$ it is immediate from (2.5) that

$$\boxed{T_n = n^3 T_1,} \tag{2.6}$$

with $T_1 = 1/f_1$.

The time required for the propagation of effects from electron to proton (the *delay*) is just $\tau = r/c$, so from the above relations we find

$$\tau_n = \frac{n^2\hbar^2}{mc|A|}$$

or more simply

$$\boxed{\tau_n = n^2 \tau_1,} \tag{2.7}$$

with

$$\tau_1 = \frac{\hbar^2}{mc|A|}. \tag{2.7'}$$

Finally, the energy of the nth stationary state is

$$E_n = -\frac{|A|}{2r_n}, \tag{2.8}$$

or

$$\boxed{E_n = \frac{E_1}{n^2}} \tag{2.9}$$

wherein

$$E_1 = -\frac{m}{2}\frac{|A|^2}{\hbar^2}. \tag{2.9'}$$

From Bohr's assumption 2 the frequency of radiation in going from stationary state n_1 to n_2 is

$$\begin{aligned}
h\nu &= E_{n_1} - E_{n_2} \\
&= -\frac{|A|}{2r_{n_1}} + \frac{|A|}{2r_{n_2}} \\
&= \frac{|A|}{2}\left(\frac{1}{r_{n_2}} - \frac{1}{r_{n_1}}\right) \\
&= \frac{m|A|^2}{2\hbar^2}\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right)
\end{aligned}$$

Thus

$$\nu = \frac{m|A|^2}{4\pi\hbar^3}\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right).$$

Since, from equation (2.5) we can write

$$\frac{nf_n}{2} = \frac{m|A|^2}{4\pi n^2\hbar^3},$$

this last relation becomes

$$\nu = \frac{1}{2}\left[n_2 f_{n_2} - n_1 f_{n_1}\right].$$

Having all of these relations, its instructive to see what the numbers really look like. Taking

(1) electronic charge $e = 1.6 \times 10^{-19}$ coulombs;
(2) $\hbar = 1.05 \times 10^{-34}$ Joule-sec;
(3) speed of light $c = 3 \times 10^8$ mt/sec;
(4) electron mass $m = 9.1 \times 10^{-31}$ kg; and
(5) $K = 8.99 \times 10^9$ mt^2/sec^2

we rather easily find that

(1) Radius in the first Bohr orbit is $r_1 = 5.26 \times 10^{-11}$ mt $= 0.526A$;
(2) Velocity in the first Bohr orbit is $v_1 = 2.2 \times 10^6$ mt/sec;
(3) Delay to the first Bohr orbit is $\tau_1 = 1.75 \times 10^{-19}$ sec;
(4) Frequency in the first Bohr orbit is $f_1 = 6.66 \times 10^{15}$ sec^{-1};
(5) Period in the first Bohr orbit is $T_1 = 1.5 \times 10^{-16}$ sec;
(6) Ratio of the period to the delay in the first Bohr orbit is

$$\frac{T_1}{\tau_1} = 858 = 2\pi \times 137,$$

where 137 is the "fine structure constant". This last relation can be written in the alternate form

$$\omega_1\tau_1 = \frac{1}{137}.$$

3. The Hydrogen Atom. With the historical outline and calculations of the previous section, we now turn to a consideration of how these might be modified by the assumption that an electron and proton are interacting non-instantaneously through a combination of advanced and/or retarded forces.

As noted in the previous section, the velocity in the first Bohr orbit is $\mathcal{O}(10^6 mt/sec)$, and velocities in successively higher orbits decrease as n^{-1}. Since these velocities are much less than the speed of light, it would seem that a non-relativistic treatment is a reasonable first approximation. Thus with an interacting electron proton pair, we can in the first instance neglect magnetic forces and view the interparticle forces as purely central.

Consider an electron of mass m_e located at the vector position $\mathbf{r_e}$ interacting with a proton of mass m_p at $\mathbf{r_p}$. The force on the electron due to the proton is $\mathbf{F_{p,e}}$ while the force on the proton due to the electron is $\mathbf{F_{e,p}}$. If the forces are central, $\mathbf{F_{p,e}} = -\mathbf{F_{e,p}}$, and we can write the equations of motion of the electron and proton as

$$m_e \ddot{\mathbf{r}}_\mathbf{e} = \mathbf{F_{e,p}} \tag{3.1a}$$

$$m_p \ddot{\mathbf{r}}_\mathbf{p} = -\mathbf{F_{e,p}}. \tag{3.1b}$$

Further, since the forces are central it will be easier to work in a center of mass coordinate system, so we define a new vector $\mathbf{r} = \mathbf{r_e} - \mathbf{r_p}$, and a reduced mass μ that satisfies

$$\frac{1}{\mu} = \frac{1}{m_e} + \frac{1}{m_p}.$$

Then equations (3.1a,b) take the form

$$\ddot{\mathbf{r}} = \frac{1}{\mu}\mathbf{F}, \tag{3.2}$$

where we have set $\mathbf{F_{e,p}} = \mathbf{F}$.

Now we must specify the force $\mathbf{F}$. We wish to consider the possibility of a mixture of retarded and/or advanced potentials, so in the non-relativistic case with $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$ (??does this also require that the proton be infinitely massive??),

$$\mathbf{F} = -\frac{e^2}{4\pi\epsilon_0}\left\{\alpha\frac{\mathbf{r}_-}{|\mathbf{r}_-|^3} + \beta\frac{\mathbf{r}_+}{|\mathbf{r}_+|^3}\right\} \tag{3.3}$$

where $\mathbf{r}_-$ and $\mathbf{r}_+$ are the vectors $\mathbf{r}$ evaluated at times $t - \tau_R$ and $t - \tau_A$, and τ_R and τ_A satisfy the functional equations

$$c\tau_R = |\mathbf{r}(t) - \mathbf{r}(t - \tau_R)| \tag{3.4a}$$

$$c\tau_A = |\mathbf{r}(t) - \mathbf{r}(t + \tau_A)| \tag{3.4b}$$

respectively.

Define unit vectors $\mathbf{u_r}$, $\mathbf{u_{r+}}$, $\mathbf{u_{r-}}$ such that $\mathbf{r} = r\mathbf{u_r}$, $\mathbf{r}_+ = r_+\mathbf{u_{r+}}$, and $\mathbf{r}_- = r_-\mathbf{u_{r-}}$, and set

$$A = -\frac{e^2}{4\pi\epsilon_0}.$$

Then (3.2) and (3.3) can be combined to give

$$\mu\dot{\mathbf{v}} = \mu\ddot{\mathbf{r}} = A\left\{\alpha\frac{\mathbf{u_{r-}}}{{r_-}^2} + \beta\frac{\mathbf{u_{r+}}}{{r_+}^2}\right\}. \tag{3.5}$$

We are interested in the total energy of the electron-proton system, and in contradiction to the quantized harmonic oscillator problem, there seems to be no ambiguity. As before, we have a kinetic energy

$$E_{kin} = \frac{1}{2}\mu\mathbf{v}\cdot\mathbf{v}, \tag{3.6}$$

and define a potential by

$$\phi = A\left\{\frac{\alpha}{r_-} + \frac{\beta}{r_+}\right\}, \tag{3.7}$$

so with the operator

$$\mathcal{D} \equiv \frac{\partial}{\partial x_{-\tau}}\mathbf{u_{r-}} + \frac{\partial}{\partial x_\tau}\mathbf{u_{r+}}, \tag{3.8}$$

we have $\mathbf{F} = -\mathcal{D}\phi$, and the total energy of the pair is

$$E = \frac{1}{2}\mu\mathbf{v}\cdot\mathbf{v} + A\left\{\frac{\alpha}{r_-} + \frac{\beta}{r_+}\right\}, \tag{3.9}$$

while the rate of change of the energy is

$$\frac{dE}{dt} = \mathbf{v}\cdot\mathbf{F} + \frac{\partial\phi}{\partial t}. \tag{3.10}$$

3a. Consequences of constant $|\mathbf{L}|$. Note from (3.2) that, with the definition of the angular momentum $\mathbf{L} = \mathbf{r}\times(\mu\mathbf{v})$, we have $\dot{\mathbf{L}} = \mathbf{r}\times\mathbf{F}$. Hence if the angular momentum $\mathbf{L}$ is to be a constant with respect to time, then it is necessary that the motion take place in a plane so $\mathbf{r}\times\mathbf{F} = 0$. Thus, we work in circular coordinates r, θ. Let $\mathbf{u}_\theta$ be a unit vector orthogonal to $\mathbf{u_r}$, so we can then write

$$
\begin{aligned}
\mathbf{r} &= r\mathbf{u_r}\\
\dot{\mathbf{r}} &= \dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta\\
\ddot{\mathbf{r}} &= [\ddot{r} - r\dot{\theta}^2]\mathbf{u_r} + [r\ddot{\theta} + 2\dot{r}\dot{\theta}]\mathbf{u}_\theta.
\end{aligned}
$$

Hence, for planar motion

$$|\mathbf{L}| = |\mathbf{r}\times(\mu\mathbf{v})| = \mu r^2\dot{\theta}. \tag{3.11}$$

Furthermore, the equation of motion (3.5) can be written in the more explicit form

$$[\ddot{r} - r\dot{\theta}^2]\mathbf{u_r} + [r\ddot{\theta} + 2\dot{r}\dot{\theta}]\mathbf{u}_\theta = \frac{A}{\mu}\left\{\alpha\frac{\mathbf{u_{r-}}}{{r_-}^2} + \beta\frac{\mathbf{u_{r+}}}{{r_+}^2}\right\}. \tag{3.12}$$

Taking the dot product of (3.12) with $\mathbf{u_r}$ gives

$$
\begin{aligned}
\ddot{r} - r\dot{\theta}^2 &= \frac{A}{\mu}\left\{\alpha\frac{\mathbf{u_r}\cdot\mathbf{u_{r-}}}{{r_-}^2} + \beta\frac{\mathbf{u_r}\cdot\mathbf{u_{r+}}}{{r_+}^2}\right\}\\
&= \frac{A}{\mu}\left\{\alpha\frac{\cos(\theta-\theta_-)}{{r_-}^2} + \beta\frac{\cos(\theta_+-\theta)}{{r_+}^2}\right\},
\end{aligned}
\tag{3.13}
$$

while the dot product of (3.12) with $\mathbf{u}_\theta$ yields

$$
\begin{aligned}
\frac{1}{r}\frac{d(r^2\dot{\theta})}{dt} = r\ddot{\theta} + 2\dot{r}\dot{\theta} &= \frac{A}{\mu}\left\{\alpha\frac{\mathbf{u}_\theta\cdot\mathbf{u_{r-}}}{{r_-}^2} + \beta\frac{\mathbf{u}_\theta\cdot\mathbf{u_{r+}}}{{r_+}^2}\right\}\\
&= \frac{A}{\mu}\left\{-\alpha\frac{\sin(\theta-\theta_-)}{{r_-}^2} + \beta\frac{\sin(\theta_+-\theta)}{{r_+}^2}\right\}.
\end{aligned}
\tag{3.14}
$$

As a consequence of equation (3.11), if $\dot{\mathbf{L}} = 0$ then it follows that

$$|\mathbf{L}| = \mu r^2\dot{\theta}$$

must be constant, or more explicitly

$$|\mathbf{L}| = \mu r^2\dot{\theta} = \mathcal{C}.$$

Since both r and r^2 must be non-negative, this then implies that with $\dot{\mathbf{L}} = 0$, we cannot have $\dot{\theta}$ changing sign, and it too must be constant. This then, in turn, implies that r^2 is constant. Hence, it is clear from (3.11) for the magnitude of the angular momentum that a necessary and sufficient condition for $|\mathbf{L}|$ to be constant is to have the planar motion satisfy the following conditions simultaneously:

(1) The planar motion must be circular ($r = r_- = r_+$) which, in turn, implies that $\tau_R = \tau_A = r/c$; and
(2) The planar motion must be uniform so $\dot{\theta} = \omega$ or $\theta = \omega t + const$. This, in conjunction with (1) implies $\theta - \theta_- = -(\theta - \theta_+) = \omega r/c$.

Note in particular that when these conditions are satisfied, then from (3.10) we have

$$\frac{dE}{dt} \equiv 0$$

so these conditions define a situation in which there is **absolutely no radiation of energy from the orbiting electron, in sharp contrast to the classical situation without the advanced/retarded interactions.**

Under these conditions we have from the two component equations of motion (3.13) and (3.14)

$$-r\omega^2 = \frac{A}{\mu r^2}\cos\left(\frac{\omega r}{c}\right) \tag{3.15}$$

(remember that $\alpha + \beta = 1$) and

$$0 = (\beta - \alpha)\frac{A}{\mu r^2}\sin\left(\frac{\omega r}{c}\right) \iff \frac{d\mathbf{L}}{dt} \equiv 0. \tag{3.16}$$

There are two ways in which the latter equation (3.16) can be satisfied:

(1) If

$$\frac{\omega r}{c} = n\pi, \tag{3.17}$$

where n is an integer; or

(2) With an assumption of half-advanced and half-retarded interactions, $\alpha = \beta = \frac{1}{2}$.

3a.i. Satisfying (3.16): Consequences of n an integer. Condition (3.17) implies that $\cos(n\pi) = (-1)^n$ and this in conjunction with (3.15) implies first of all that

$$\frac{\mu}{|A|}r^3\omega^2 = (-1)^n. \tag{3.18}$$

Since the left hand side is nonnegative, this places the further requirement on n that it be an even integer, $n = 2\kappa$ and thus

$$\frac{\omega r}{c} = 2\kappa\pi, \tag{3.19}$$

Consequently (3.15) takes the form

$$r^3\omega^2 = \frac{|A|}{\mu}, \tag{3.20}$$

or, with $\omega r = 2\kappa\pi c$, we have more specifically

$$r = \frac{1}{\kappa^2}\frac{|A|}{\mu(2\pi c)^2}. \tag{3.21}$$

We can write this in the form equivalent to (2.3) as

$$\boxed{r_\kappa = \frac{r_1}{\kappa^2}} \tag{3.22}$$

wherein

$$r_1 = \frac{|A|}{\mu(2\pi c)^2}. \tag{3.23}$$

Furthermore, since $\omega = 2\kappa\pi c/r$, we can write

$$\boxed{\omega_\kappa = \kappa^3\omega_1} \tag{3.24}$$

where

$$\omega_1 = \frac{\mu}{|A|}(2\pi c)^3. \tag{3.25}$$

With the above results, it is straightforward that the angular momentum is quantized according to

$$|\mathbf{L}|_\kappa = \frac{|\mathbf{L}|_1}{\kappa},$$

with

$$|\mathbf{L}|_1 = \frac{|A|}{2\pi c}.$$

What about the energy under these conditions? First note that with circular orbits the potential ϕ will be independent of time so with the additional assumption of uniform motion we have

$$\frac{dE}{dt} = \mathbf{v}\cdot\mathbf{F} = (\alpha - \beta)\frac{A\omega}{r}\sin(\frac{\omega r}{c}), \tag{3.26}$$

which is obviously zero. Thus in all of the quantized orbits defined by (3.13) the energy is independent of time. It has a value given by

$$\begin{aligned} E &= \frac{1}{2}\mu\omega^2 r^2 + \frac{A}{r} \\ &= \frac{1}{2}\mu(2\kappa\pi c)^2 + \frac{A}{r_1}\kappa^2 \\ &= -\frac{1}{2}(2\pi)^2\mu c^2\kappa^2, \end{aligned}$$

or

$$\boxed{E_\kappa = -\frac{1}{2}(2\pi)^2\mu c^2\kappa^2.} \tag{3.27}$$

Notice that this quantized behaviour is not only highly relativistic, but also non-relativistic, since the smallest energy is $|E_1| = \mu c^2(2\pi)^2/2$, $2\pi^2 \simeq 18$ times **larger** than μc^2.

Strangely enough, it would appear that these conditions specifying the orbital properties for which the angular momentum is quantized are satisfied without any conditions on α and β. However, this is not the case as the following computation shows. Using (3.5) we may write a more explicit equation for $\dot{\mathbf{L}}$, namely

$$\dot{\mathbf{L}} = Ar\left\{\alpha\frac{\mathbf{u_r}\times\mathbf{u_{r-}}}{{r_-}^2} + \beta\frac{\mathbf{u_r}\times\mathbf{u_{r+}}}{{r_+}^2}\right\}. \tag{3.28}$$

Since we are searching for conditions for $\dot{\mathbf{L}} = 0$, an examination of (3.11) makes it obvious that if r is finite this requires that the bracketed term must be identically zero for all time. Since retarded and/or advanced potentials have been assumed, this condition certainly requires at the minimum that $(\mathbf{u_r}\times\mathbf{u_{r-}})$ and $(\mathbf{u_r}\times\mathbf{u_{r+}})$ have opposite signs, which in turn means that *both* advanced and retarded interactions must be operative. Thus neither α nor β can be identically zero.

Remembering further that uniform circular motion of the electron is a consequence of $\dot{\mathbf{L}} = 0$ which, in turn, implies that $\tau_R = \tau_A$ and further that $\mathbf{u_r}\times\mathbf{u_{r-}} = -(\mathbf{u_r}\times\mathbf{u_{r+}})$, (3.11) takes the simpler form

$$\dot{\mathbf{L}} = \frac{A}{r}(\mathbf{u_r}\times\mathbf{u_{r-}})(\alpha-\beta). \tag{3.29}$$

It is clear that considering the half-advanced, half-retarded situation with $\alpha = \beta = \frac{1}{2}$ will be the only way in which $\dot{\mathbf{L}} = 0$ given the other assumptions about the uniform circular motion of the electron. Thus condition (2) for equation (3.16) to hold follows as an immediate consequence of condition (1) when uniform circular motion is assumed.

3a.ii. Satisfying (3.15).

If we go back to equation (3.15)

$$-r\omega^2 = \frac{A}{\mu r^2}\cos\left(\frac{\omega r}{c}\right), \tag{3.15}$$

how can we deal with this? First, note from 3.11 and its consequences that

$$|\mathbf{L}| = \mu r^2\dot{\theta} = \mu r^2\omega = \Gamma(r,\omega) \tag{3.30}$$

is a constant independent of time. This, in conjunction with (3.15), gives

$$-\frac{\Gamma(r,\omega)c}{A}\cdot\left(\frac{\omega r}{c}\right) = \cos\left(\frac{\omega r}{c}\right). \tag{3.31}$$

For $[\Gamma(r,\omega)c/A << 1]$, this implies that

$$\left(\frac{\omega r}{c}\right) \simeq \left(n+\frac{1}{2}\right)\pi \qquad n \quad \text{an integer.} \tag{3.32}$$

The immediate consequence of this in (3.16), since

$$\sin\left(\frac{\omega r}{c}\right) \simeq \sin[(n+\frac{1}{2})\pi] = (-1)^n, \tag{3.33}$$

is to require half advanced and half retarded interations: $\alpha = \beta = \dfrac{1}{2}$.

4. Stability Considerations.

We found, in the previous section, that the necessary and sufficient conditions for constant angular momentum translated into conditions implying uniform and planar circular motion. These, in turn, gave conditions involving r and ω that are embodied in equations (3.15) and (3.16):

$$-r\omega^2 = \frac{A}{\mu r^2}\cos\left(\frac{\omega r}{c}\right) \tag{3.15}$$

(remember that $\alpha+\beta=1$) and

$$0 = (\alpha-\beta)\frac{A}{\mu r^2}\sin\left(\frac{\omega r}{c}\right). \tag{3.16}$$

A reasonable questions is the following. Given values of r and ω that satisfy (3.15)-(3.16), are they stable or unstable solutions of the equations of motion

$$\begin{aligned}\ddot r - r\dot\theta^2 &= \frac{A}{\mu}\left\{\alpha\frac{\mathbf{u_r}\cdot\mathbf{u_{r-}}}{{r_-}^2} + \beta\frac{\mathbf{u_r}\cdot\mathbf{u_{r+}}}{{r_+}^2}\right\}\\ &= \frac{A}{\mu}\left\{\alpha\frac{\cos(\theta-\theta_-)}{{r_-}^2} + \beta\frac{\cos(\theta-\theta_+)}{{r_+}^2}\right\},\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}r\ddot\theta + 2\dot r\dot\theta &= \frac{A}{\mu}\left\{\alpha\frac{\mathbf{u_\theta}\cdot\mathbf{u_{r-}}}{{r_-}^2} + \beta\frac{\mathbf{u_\theta}\cdot\mathbf{u_{r+}}}{{r_+}^2}\right\}\\ &= \frac{A}{\mu}\left\{\alpha\frac{\sin(\theta-\theta_-)}{{r_-}^2} + \beta\frac{\sin(\theta-\theta_+)}{{r_+}^2}\right\},\end{aligned} \tag{3.14}$$

in the face of small perturbations? In this section, we examine this question.

To deal with this problem, we assume that $R(t)$ is a small ($\mathcal{O}(\epsilon)$) perturbation of r satisfying (3.15)-(3.16) so

$$r(t) \simeq r + R(t),$$

while $\Theta(t)$ is a small perturbation of ωt:

$$\theta(t) \simeq \omega t + \Theta(t).$$

Then, to $\mathcal{O}(\epsilon)$ the left hand sides of (3.13) and (3.14) become

$$\ddot r(t) - r(t)\dot\theta^2(t) \simeq \ddot R(t) - \omega^2 R(t) - 2r\omega\dot\Theta(t) - r\omega^2 \tag{3.34}$$

and

$$r(t)\ddot\theta(t) + 2\dot r(t)\dot\theta(t) \simeq r\ddot\Theta(t) + 2\omega\dot R(t) \tag{3.35}$$

respectively.

In examining the angular components,

$$\theta(t)-\theta_\pm = \mp\frac{\omega r}{c} + \Theta(t) - \Theta\left(t \pm \frac{r}{c} \pm \frac{R}{c}\right) \simeq \mp\frac{\omega r}{c} \pm \frac{r}{c}\dot\Theta(t)$$

so, to the same accuracy,

$$\cos(\theta(t)-\theta_\pm) \simeq \cos\left(\frac{\omega r}{c}\right) + \dot\Theta(t)\frac{r}{c}\sin\left(\frac{\omega r}{c}\right) \tag{3.36}$$

and

$$\sin(\theta(t)-\theta_\pm) \simeq \mp\sin\left(\frac{\omega r}{c}\right) \pm \dot\Theta(t)\frac{r}{c}\cos\left(\frac{\omega r}{c}\right). \tag{3.37}$$

Furthermore, in the radial motion we have

$$r_\pm(t) = r + R\left(t \pm \frac{r}{c} \pm \frac{R}{c}\right) \simeq r + R(t) \mp \frac{r}{c}\dot R(t), \tag{3.38}$$

so

$$\frac{1}{r_{\pm}^2(t)} \simeq \frac{1}{r^2}\left[1 - \frac{2R(t)}{r} \pm \frac{2\dot{R}(t)}{c}\right]. \tag{3.39}$$

Consequently,

$$\frac{\cos(\theta(t)-\theta_{\pm})}{r_{\pm}^2(t)} \simeq \frac{1}{r^2}\cos\left(\frac{\omega r}{c}\right)\left[1 - \frac{2R(t)}{r} \pm \frac{2\dot{R}(t)}{c}\right] + \dot{\Theta}(t)\frac{1}{rc}\sin\left(\frac{\omega}{rc}\right), \tag{3.40}$$

and

$$\frac{\sin(\theta(t)-\theta_{\pm})}{r_{\pm}^2(t)} \simeq \mp\frac{1}{r^2}\sin\left(\frac{\omega r}{c}\right)\left[1 - \frac{2R(t)}{r} \pm \frac{2\dot{R}(t)}{c}\right] \pm \dot{\Theta}(t)\frac{1}{rc}\cos\left(\frac{\omega r}{c}\right). \tag{3.41}$$

Combining the estimations of (3.34) and (3.40), equation (3.13) to $\mathcal{O}(\epsilon)$ becomes

$$\ddot{R} - \omega^2 R - 2r\omega\dot{\Theta} - r\omega^2 = \frac{A}{\mu}\left\{\frac{1}{r^2}\cos\left(\frac{\omega r}{c}\right)\left[1 - \frac{2R}{r}\right] + \dot{\Theta}\frac{1}{rc}\sin\left(\frac{\omega r}{c}\right)\right\}. \tag{3.42}$$

In a similar fashion, from (3.35) and (3.41), equation (3.14) becomes (remember that $\alpha + \beta = 1$)

$$r\ddot{\Theta} + 2\omega\dot{R} = \frac{A}{\mu}\left\{(\alpha-\beta)\frac{1}{r^2}\sin\left(\frac{\omega r}{c}\right)\left[1 - \frac{2R}{r}\right] - \frac{2\dot{R}}{r^2 c}\sin\left(\frac{\omega r}{c}\right) + \dot{\Theta}\frac{1}{rc}\cos\left(\frac{\omega r}{c}\right)\right\}. \tag{3.43}$$

Using the equilibrium condition (3.15) in (3.42) we obtain

$$\ddot{R} - 3\omega^2 R = \dot{\Theta}\left[\frac{A}{\mu rc}\sin\left(\frac{\omega r}{c}\right) + 2\omega r\right]. \tag{3.44}$$

In a similar fashion, (3.15) and (3.16) simplify (3.43) to

$$r\ddot{\Theta} = -2\dot{R}\left[\frac{A}{\mu r^2 c}\sin\left(\frac{\omega r}{c}\right) + \omega\right] \tag{3.45}$$