QUANTIZED BEHAVIOUR IN THE HYDROGEN ATOM MAY BE A CONSEQUENCE OF HALF ADVANCED–HALF RETARDED ACTION AT A DISTANCE

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1. Introduction.

This paper examines the consequence of having an electron–proton pair (hydrogen atom) interacting through a force that is not instantaneous, but rather depends on both the positions of the interacting particles in the past *as well as* in the future (the so called half-advanced/half-retarded interaction case). This situation, in which causality is apparently violated, is of interest because if it is required that the laws of physics should be time reversal invariant, then the existence of retarded potentials implies the corresponding existence of symmetric advanced potentials. This notion, so contrary to common experience, has been raised a number of times over the past decades [xxx], and is an extension of the more frequently considered situation in which the particle interaction operates only with a delay [xxx]. These theoretical investigations into the potential significance of retarded and/or advanced interactions have assumed even more significance in light of the recent experimental results of Sandoghdar et al. [xxx] so elegantly summarized in a historical context by Levy [xxx].

In a very real sense, I feel that another motivation to study this problem comes from a casual examination of the Rydberg formula for the spectra of atomic hydrogen:

$$\bar{\nu}_{n,m} = R_\infty \left\{ \frac{1}{n^2} - \frac{1}{m^2} \right\} \qquad n < m \quad \text{with both being positive integers,} \tag{1.1}$$

wherein $\bar{\nu}_{n,m}$ (in units of mt^{-1}) is the wavenumber of the radiation emitted (absorbed) in passing from state $m \to n$ ($n \to m$). The wavenumber is related to the frequency $\nu_{n,m}$ (sec^{-1}) by $\nu_{n,m} = c\bar{\nu}_{n,m}$, [with c the speed of light], and R_∞ is the *Rydberg constant* appropriate for hydrogen. When $n = 1$, equation (A1.1) describes the *Lyman* series in the ultraviolet region (discovered in 1906); for $n = 2$ it describes the visible *Balmer* series first described in 1885; with $n = 3$ one recovers a good description of the infrared *Paschen* series of 1908. $n = 4$ corresponds to the 1922 *Brackett* series (also infrared); and $n = 5$ gives the far infrared *Pfund* series. The Rydberg equation (1.1) describes the wavelength of the energy emitted in going from level n to m, i.e. the wavelength of the emitted energy depends not only on the state *that it is coming from* (n), but also on the state *that it is going to* (m). One might interpret this as indicative of a dependence on what was, but also on what will be.

The plan of the paper is as follows. Section 2 gives a brief derivation of the equations of motion for a nonrelativistic hydrogen atom. Section 3 gives the conditions for a steady state characterized by constant angular momentum as well as constant energy for the pair. Constant angular momentum is a necessary and sufficient condition for the pair to perform circular uniform planar motion, and is shown to lead to a situation in which the energy of the pair is constant when the radial coordinate and angular frequency satisfy a transcendental relationship. Thus for certain allowed orbits, there is no radiation of energy as is the case in the classical situation in which instantaneous forces are

assumed. Section 4 shows how the Rydberg formula for the spectral frequencies of radiation emission or absorption can be used to provide a second relationship connecting the radial coordinate and angular frequency *if it is assumed that*, as Bohr did, *the frequency of radiation for transitions between two adjacent large orbits is equal to the frequency of rotation in that orbit* (the correspondence principle). The numerical consequences of this assumption are: (1) The calculated angular momentum in a given allowed orbit is an integral multiple of a numerically determined constant that is equal to Planck's constant divided by 2π, i.e. the constant is $\hbar$; and (2) The differences between the calculated energies of two allowed orbits are strictly proportional to the frequencies calculated from the Rydberg formula for the transitions. The constant of proportionality between these energy differences and frequencies is again numerically calculated to be equal to Planck's constant h. Section 5 is devoted to a linearization and stability analysis of the equations of motion presented in Section 2 when the pair is slightly perturbed away from the allowed orbits detailed in Sections 3 and 4. Section 6 concludes the paper with a discussion of the results in the context of previous work. Two appendices are provided. Appendix 1 briefly outlines the approach taken by Bohr [xxx] in his ground breaking work that first gave some indication of how the emission spectra of the hydrogen atom might be comprehended from a theoretical basis. This Appendix provides a concise summary of his approach that may be used for comparison with the considerations of this paper, since in a very real sense the Bohr theory stands as a zeroth order approximation to the one presented here. Appendix 2 presents some of the calculations needed for the stability analysis of Section 5.

2. The Hydrogen Atom Equations of Motion.

In this section, I give a short derivation of the equations of motion for an electron proton pair interacting through one-half advanced and one-half retarded forces. As noted in Appendix 1, the velocity in the first Bohr orbit is $\mathcal{O}(10^6\ \mathrm{ms}^{-1})$, and velocities in successively higher orbits decrease as n^{-1} where n is a positive integer. Since these velocities are much less than the speed of light, it would seem that a non-relativistic treatment is a reasonable first approximation. Thus with an interacting electron proton pair we can, in the first instance, neglect magnetic forces and view the interparticle forces as purely central.

Consider an electron of mass m_e located at the vector position $\mathbf{r_e}$ interacting with a proton of mass m_p at $\mathbf{r_p}$. The force on the electron due to the proton is $\mathbf{F_{p,e}}$ while the force on the proton due to the electron is $\mathbf{F_{e,p}}$. If the forces are central, $\mathbf{F_{p,e}} = -\mathbf{F_{e,p}}$, and we can write the equations of motion of the electron and proton as

$$m_e \ddot{\mathbf{r_e}} = \mathbf{F_{e,p}} \tag{2.1a}$$

$$m_p \ddot{\mathbf{r_p}} = -\mathbf{F_{e,p}}. \tag{2.1b}$$

Further, if the forces are central it will be easier to work in a center of mass coordinate system, so we define a new vector $\mathbf{r} = \mathbf{r_e} - \mathbf{r_p}$, and a reduced mass μ that satisfies

$$\frac{1}{\mu} = \frac{1}{m_e} + \frac{1}{m_p}. \tag{2.2}$$

Then equations (2.1a,b) take the form

$$\ddot{\mathbf{r}} = \frac{1}{\mu}\mathbf{F}, \tag{2.3}$$

where we have set $\mathbf{F_{e,p}} = \mathbf{F}$.

Now we must specify the force $\mathbf{F}$. Since we assume half advanced and half retarded potentials, in the non-relativistic case

$$\mathbf{F} = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{2}\left\{\frac{\mathbf{r}_-}{|\mathbf{r}_-|^3} + \frac{\mathbf{r}_+}{|\mathbf{r}_+|^3}\right\} \tag{2.4}$$

where $\mathbf{r}_-$ and $\mathbf{r}_+$ are the vectors $\mathbf{r}$ evaluated at times $t - \tau_R$ and $t - \tau_A$ respectively. Define unit vectors $\mathbf{u_r}$, $\mathbf{u_{r+}}$, $\mathbf{u_{r-}}$ such that $\mathbf{r} = r\mathbf{u_r}$, $\mathbf{r}_+ = r_+\mathbf{u_{r+}}$, and $\mathbf{r}_- = r_-\mathbf{u_{r-}}$, and set

$$A = -\frac{e^2}{4\pi\epsilon_0}.$$

Then (2.3) and (2.4) can be combined to give

$$\mu\dot{\mathbf{v}} = \mu\ddot{\mathbf{r}} = \frac{A}{2}\left\{\frac{\mathbf{u_{r-}}}{{r_-}^2} + \frac{\mathbf{u_{r+}}}{{r_+}^2}\right\}. \tag{2.5}$$

We are interested in the total energy of the electron-proton system. We have a kinetic energy

$$E_{kin} = \frac{1}{2}\mu \mathbf{v}\cdot\mathbf{v}, \tag{2.6}$$

and define a potential by

$$\phi = \frac{A}{2}\left\{\frac{1}{r_-} + \frac{1}{r_+}\right\}, \tag{2.7}$$

so with the operator

$$\mathcal{D} \equiv \frac{\partial}{\partial r_{-\tau}}\mathbf{u}_{\mathbf{r}-} + \frac{\partial}{\partial r_{\tau}}\mathbf{u}_{\mathbf{r}+}, \tag{2.8}$$

we have $\mathbf{F} = -\mathcal{D}\phi$, and the total energy of the pair is

$$E = \frac{1}{2}\mu \mathbf{v}\cdot\mathbf{v} + \frac{A}{2}\left\{\frac{1}{r_-} + \frac{1}{r_+}\right\}, \tag{2.9}$$

while the rate of change of the energy is

$$\frac{dE}{dt} = \mathbf{v}\cdot\mathbf{F} + \frac{\partial\phi}{\partial t}. \tag{2.10}$$

Another quantity of importance will be the angular momentum

$$\mathbf{L} = \mathbf{r}\times(\mu\mathbf{v}), \tag{2.11}$$

so we have

$$\frac{d\mathbf{L}}{dt} = \mathbf{r}\times\mathbf{F}. \tag{2.12}$$

3. Consequences of Constant $|\mathbf{L}|$.

From equations (2.11) and (2.12), if the angular momentum $\mathbf{L}$ is to be a constant with respect to time, then it is necessary that the motion take place in a plane so $\mathbf{r}\times\mathbf{F} = 0$. Thus, we work in circular coordinates r, θ.

Let $\mathbf{u}_\theta$ be a unit vector orthogonal to $\mathbf{u}_\mathbf{r}$, so we can then write

$$\begin{aligned}
\mathbf{r} &= r\mathbf{u}_\mathbf{r}\\
\dot{\mathbf{r}} &= \dot{r}\mathbf{u}_\mathbf{r} + r\dot{\theta}\mathbf{u}_\theta\\
\ddot{\mathbf{r}} &= [\ddot{r} - r\dot{\theta}^2]\mathbf{u}_\mathbf{r} + [r\ddot{\theta} + 2\dot{r}\dot{\theta}]\mathbf{u}_\theta.
\end{aligned}$$

Hence, for constant angular momentum, and consequently planar motion,

$$|\mathbf{L}| = |\mathbf{r}\times(\mu\mathbf{v})| = \mu r^2|\dot{\theta}|. \tag{3.1}$$

As a consequence of equation (3.1), if $\dot{\mathbf{L}} = 0$ then it follows that

$$|\mathbf{L}| = \mu r^2|\dot{\theta}| \tag{3.2}$$

must be constant, or more explicitly

$$\mu r^2|\dot{\theta}| = \mathcal{L}. \tag{3.3}$$

Hence, if we now take the convention that variables subscripted with a "$_*$" correspond to steady states, it is clear from (3.1) for the magnitude of the angular momentum that a *sufficient condition* for $|\mathbf{L}|$ to be temporally constant is to have planar motion which simultaneously satisfies the following conditions:

(1) The planar motion is circular ($r_* = r_- = r_+$) [which implies that $\tau_R = \tau_A = r_*/c$]; *and*
(2) The planar motion is uniform so $\dot{\theta} = \omega_*$ or $\theta = \omega_* t + const.$ This, in conjunction with Condition 1, implies $\theta - \theta_- = (\theta_+ - \theta) = \omega_* r_*/c$.

With these two conditions on r_* and ω_*, it is also the case that (3.3) takes the form

$$\boxed{\mu r_*^2 \omega_* = \mathcal{L}_*}. \tag{3.4}$$

Furthermore, when these conditions are satisfied, then from (2.10) we have

$$\begin{aligned} \frac{dE}{dt} &= \frac{r_* \omega_* A}{2} \left\{ -\frac{\sin(\theta - \theta_-)}{{r_-}^2} + \frac{\sin(\theta_+ - \theta)}{{r_+}^2} \right\} \\ &= \frac{\omega_* A}{2 r_*} \left\{ -\sin\left(\frac{\omega_* r_*}{c}\right) + \sin\left(\frac{\omega_* r_*}{c}\right) \right\} \equiv 0 \end{aligned} \tag{3.5}$$

so the conditions for constant angular momentum simultaneously define a situation in which there is **absolutely no radiation of energy from the orbiting electron, in sharp contrast to the classical situation without the advanced/retarded interactions.** (What I mean by the above is that, following Nordstrom and Page, when w have half-advanced, half-retarded potentials, there is no radiation from the electron. If, however, in the "classical" situation we consider retarded potentials, then due to the radiation reaction the electron loses energy and spirals into the nucleus.) The value of the energy E in this nonradiating state is given by

$$\boxed{\mathcal{E}_* = \frac{1}{2}\mu r_*^2 \omega_*^2 + \frac{A}{r_*}}. \tag{3.6}$$

We can supplement these conditions (3.4) and (3.6) for a steady state by writing the equation of motion (2.5) in the more explicit form

$$[\ddot{r} - r\dot{\theta}^2]\mathbf{u_r} + [r\ddot{\theta} + 2\dot{r}\dot{\theta}]\mathbf{u}_\theta = \frac{A}{2\mu} \left\{ \frac{\mathbf{u_{r-}}}{{r_-}^2} + \frac{\mathbf{u_{r+}}}{{r_+}^2} \right\}. \tag{3.7}$$

Taking the dot product of (3.7) with $\mathbf{u_r}$ gives

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= \frac{A}{2\mu} \left\{ \frac{\mathbf{u_r} \cdot \mathbf{u_{r-}}}{{r_-}^2} + \frac{\mathbf{u_r} \cdot \mathbf{u_{r+}}}{{r_+}^2} \right\} \\ &= \frac{A}{2\mu} \left\{ \frac{\cos(\theta - \theta_-)}{{r_-}^2} + \frac{\cos(\theta_+ - \theta)}{{r_+}^2} \right\}, \end{aligned} \tag{3.8}$$

while the dot product of (3.7) with $\mathbf{u}_\theta$ yields

$$\begin{aligned} \frac{1}{r}\frac{d(r^2\dot{\theta})}{dt} = r\ddot{\theta} + 2\dot{r}\dot{\theta} &= \frac{A}{2\mu} \left\{ \frac{\mathbf{u}_\theta \cdot \mathbf{u_{r-}}}{{r_-}^2} + \frac{\mathbf{u}_\theta \cdot \mathbf{u_{r+}}}{{r_+}^2} \right\} \\ &= \frac{A}{2\mu} \left\{ -\frac{\sin(\theta - \theta_-)}{{r_-}^2} + \frac{\sin(\theta_+ - \theta)}{{r_+}^2} \right\}. \end{aligned} \tag{3.9}$$

Under the conditions of planar, circular, uniform motion the equation of motion (3.8) reduces to

$$\boxed{-r_* \omega_*^2 = \frac{A}{\mu r_*^2} \cos\left(\frac{\omega_* r_*}{c}\right)}, \tag{3.10}$$

while (3.9) is identically zero.

4. Numerical Characterization of the Steady States.

In the previous section we found that the equations of motion have two constants of the motion, the angular momentum $\mathcal{L}_*$ and the energy $\mathcal{E}_*$ given by (3.4) and (3.6) respectively when the steady state values (r_*, ω_*) satisfy (3.10). However, we require a second relation connecting r_* and ω_* before we may determine the explicit values of r_* and ω_* and thence $\mathcal{L}_*$ and $\mathcal{E}_*$ characteristic of a steady state. Though one could either:

(1) Assume that the angular momentum $\mathcal{L}_*$ was an integral multiple of Planck's constant divided by 2π, or
(2) Assume that the energy difference between levels n and m was proportional to the frequency $\nu_{n,m}$, with constant of proportionality h,

and then use either of the two resulting relationships as the second condition to adjoin to (3.10), neither of these alternatives seems intelectually satisfying to me since the justification for each seems to be quite a posteriori. Rather, I choose a third course of pursuit that many may feel suffers from the same criticism as the two possibilities mentioned above.

Arguably the most accurate data available concerning the behaviour of the hydrogen atom are the frequencies of radiation emission and/or absorption. As mentioned in the Introduction, these are given by the Rydberg formula

$$\nu_{n,m} = R_\infty c \left\{ \frac{1}{n^2} - \frac{1}{m^2} \right\} \qquad n < m \quad \text{with both being positive integers,} \tag{4.1}$$

where ν has dimension s^{-1}. Our task is to see how to use (4.1) to provide a second relation supplementing (3.10).

THE PART THAT FOLLOWS IS WHAT I AM MOST UNHAPPY WITH, BUT I HAVE BEEN UNABLE TO FIND ANY OTHER CONSISTENT WAY OUT OF THE PROBLEM OF FINDING A SECOND CONDITION TO USE IN ADDITION TO (3.10).

The Correspondence Principle Approach.

As with Bohr, I assume that for $m = n+1$ and large n the frequency of radiation emission in going from level n to $(n+1)$ will approximate $(\omega_*/2\pi)$. Since for large n and $m = n+1$ we have

$$\nu_{n,n+1} \simeq \frac{2R_\infty c}{n^3} \equiv \tilde{\nu}_{n,n+1}, \tag{4.2}$$

then I make the Bohr ansatz that

$$\omega_{*,n} = 2\pi\nu_{n,n+1} \simeq \frac{4\pi R_\infty c}{n^3} = 2\pi\tilde{\nu}_{n,n+1}. \tag{4.3}$$

To numerically calculate the consequences of the treatment as developed to this point, we require accurate values of the physical constants that appear in equations (3.10) and (4.3) Taking the best available modern determinations we use (unless otherwise noted, these values are taken from Cohen and Taylor (1987):

(1) speed of light $c = 2.99792458 \times 10^8$ ms^{-1};
(2) $\epsilon_0 = 10^7/4\pi c^2$ Fm^{-1};
(3) electronic charge $e = 1.60217733 \times 10^{-19}$ C
or [Cage et al., 1989]
$e = 1.60217670 \times 10^{-19}$ C ;
(4) electron mass $m_e = 9.1093897 \times 10^{-31}$ kg;
(5) proton mass $m_p = xxx \times 10^{-yy}$ kg;
(6) $h = 6.6260755 \times 10^{-34}$ Js $\implies \hbar = 1.054573 \times 10^{-34}$ Js
or [Cage et al., 1989]
$h = 6.6260704 \times 10^{-34}$ Js $\implies \hbar = 1.054572 \times 10^{-34}$ Js;
(7) $R_\infty = 1.0973731534 \times 10^7$ m^{-1}
or [Meq et al., 1989]
$R_\infty = 1.097373156830 \times 10^7$ m^{-1}
or [Andreae et al., 1992]
$R_\infty = 1.097373156841 \times 10^7$ m^{-1}

Using (4.3) in conjunction with (3.10) and the most accurate value of R_∞ as given in reference 6, I find that numerically

n	$\left(\frac{\mathcal{L}_{*,n}}{n}\right) \times 10^{34}$Js	$\mathcal{E}_{*,n} \times 10^{18}$J	$\left(\frac{\mathcal{E}_{*,n+1} - \mathcal{E}_{*,n}}{\nu_{n,n+1}}\right) \times 10^{34}$Js	$\frac{1}{n}\left(\frac{\omega_{*,n} r_{*,n}}{c}\right)^{-1}$
1	1.05436285	−2.179554718	− − −	137.01231880295
2	1.054376881	−0.5448741775	6.625163396	137.01140714295
3	1.054381429	−0.2421642123	6.624946256	137.01122731651
4	1.054380599	−0.1362175839	6.624857743	137.01117922661
5	1.054376116	−0.08717999491	6.624781451	137.01115190000
6	1.054377002	−0.06054156145	6.624963200	137.01113700079
7	1.054381176	−0.04447916232	6.625072057	137.01112808428
8	1.054381266	−0.03405435286	6.624878003	137.01112225978
9	1.054381327	−0.02690713987	6.624876099	137.01111827292
10	1.054381372	−0.02179478147	6.624874795	137.01111539982

The pattern is clear, and there seems to be no point in belaboring the issue. Rather it suffices to say that for $n = 1, \cdots, 100$ the assumption of half-advanced and half-retarded potentials in conjunction with the correspondence principle yields

$$\frac{\mathcal{L}_{*,n}}{n} \simeq \frac{h}{2\pi}, \quad \left(\frac{\mathcal{E}_{*,n+1} - \mathcal{E}_{*,n}}{\nu_{n,n+1}}\right) \simeq h \quad \text{and} \quad \frac{1}{n}\left(\frac{\omega_{*,n} r_{*,n}}{c}\right)^{-1} \simeq \text{fine structure constant.} \tag{4.4}$$

Thus this work predicts that the angular momentum (3.4) in the steady state is an integral multiple of $\hbar = h/2\pi$. However, in contrast to the Bohr treatment the steady states here characterize a situation of *no radiation.*

5. Stability Considerations.

We found, in the previous section, that the necessary and sufficient conditions for constant angular momentum translated into conditions implying uniform and planar circular motion. These, in turn, gave a condition involving r_* and ω_* embodied in equation (3.10):

$$-r_*\omega_*^2 = \frac{A}{\mu r_*^2}\cos\left(\frac{\omega_* r_*}{c}\right). \tag{3.10}$$

A reasonable question is the following. Given values of r_* and ω_* that satisfy (3.10), are they stable or unstable solutions of the equations of motion

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= \frac{A}{2\mu}\left\{\frac{\mathbf{u_r}\cdot\mathbf{u_{r-}}}{{r_-}^2} + \frac{\mathbf{u_r}\cdot\mathbf{u_{r+}}}{{r_+}^2}\right\} \\ &= \frac{A}{2\mu}\left\{\frac{\cos(\theta-\theta_-)}{{r_-}^2} + \frac{\cos(\theta_+-\theta)}{{r_+}^2}\right\}, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} r\ddot{\theta} + 2\dot{r}\dot{\theta} &= \frac{A}{2\mu}\left\{\frac{\mathbf{u}_\theta\cdot\mathbf{u_{r-}}}{{r_-}^2} + \frac{\mathbf{u}_\theta\cdot\mathbf{u_{r+}}}{{r_+}^2}\right\} \\ &= \frac{A}{2\mu}\left\{-\frac{\sin(\theta-\theta_-)}{{r_-}^2} + \frac{\sin(\theta_+-\theta)}{{r_+}^2}\right\}, \end{aligned} \tag{3.9}$$

in the face of small perturbations? In this section, we examine this question.

To deal with this problem, we assume that $R(t)$ is a small ($\mathcal{O}(\epsilon)$) perturbation of r_* satisfying (3.10) so

$$r(t) \simeq r_* + R(t),$$

while $\Theta(t)$ is a small perturbation of $\omega_* t$:

$$\theta(t) \simeq \omega_* t + \Theta(t).$$

Then, as shown in Appendix 2, to $\mathcal{O}(\epsilon)$ the linearized equation (3.8) becomes

$$\begin{aligned} &\ddot{R} - \omega_*^2 R\left[1 - \frac{\omega_* r_*}{c}\tan\left(\frac{\omega_* r_*}{c}\right)\right] - 2r_*\omega_*\dot{\Theta}(t) = \\ &\frac{\omega_*^2 r_*}{2}\tan\left(\frac{\omega_* r_*}{c}\right)\left[\Theta\left(t+\frac{r_*}{c}\right) - \Theta\left(t-\frac{r_*}{c}\right)\right] + \omega_*^2\left[R\left(t+\frac{r_*}{c}\right) + R\left(t-\frac{r_*}{c}\right)\right] \end{aligned} \tag{5.1}$$

while the linearized version of (3.9) is

$$\begin{aligned} &r_*\ddot{\Theta} - r_*\omega_*^2\Theta + 2\omega_*\dot{R} = \\ &\quad -\frac{\omega_*^2 r_*}{2}\left[\Theta\left(t+\frac{r_*}{c}\right) + \Theta\left(t-\frac{r_*}{c}\right)\right] + \omega_*^2\tan\left(\frac{\omega_* r_*}{c}\right)\left[R\left(t+\frac{r_*}{c}\right) - R\left(t-\frac{r_*}{c}\right)\right]. \end{aligned} \tag{5.2}$$

Before proceeding further with the analysis of the linearized equations (5.1) and (5.2), considerable simplification can be achieved by the judicious scaling of the variables. Thus we scale the time by ω_* and the radial perturbation by r_*, defining

$$\bar{t} = \omega_* t \qquad \text{and} \qquad \bar{R} = \frac{R}{r_*}.$$

We further set

$$x_* = \frac{\omega_* r_*}{c}, \quad \Delta = \tan x, \quad \text{and} \quad \Gamma = 1 - x\tan x.$$

Then (5.1) becomes

$$\frac{d^2\bar{R}}{d\bar{t}^2} - \Gamma\bar{R} - 2\frac{d\Theta}{d\bar{t}} = \frac{\Delta}{2}\left[\Theta(\bar{t}+x_*) - \Theta(\bar{t}-x_*)\right] + \left[\bar{R}(\bar{t}+x_*) + \bar{R}(\bar{t}-x_*)\right], \tag{5.3}$$

while (5.2) takes the dimensionless form

$$\frac{d^2\Theta}{d\bar{t}^2} - \Theta + 2\frac{d\bar{R}}{d\bar{t}} = -\frac{1}{2}\left[\Theta(\bar{t}+x_*) + \Theta(\bar{t}-x_*)\right] + \Delta\left[\bar{R}(\bar{t}+x_*) - \bar{R}(\bar{t}-x_*)\right]. \tag{5.4}$$

It is straightforward to show that the quasipolynomial for the eigenvalues λ of the linear system (5.3)–(5.4) is

$$P(\lambda)V(\lambda) - Q(\lambda)U(\lambda) = 0 \tag{5.5}$$

wherein

$$\begin{aligned} P(\lambda) &= 2\lambda - \Delta\left(e^{\lambda x_*} - e^{-\lambda x_*}\right) \\ V(\lambda) &= -\left[2\lambda + \frac{\Delta}{2}\left(e^{\lambda x_*} - e^{-\lambda x_*}\right)\right] \\ Q(\lambda) &= \lambda^2 - 1 + \frac{1}{2}\left(e^{\lambda x_*} + e^{-\lambda x_*}\right) \\ U(\lambda) &= \lambda^2 - \Gamma - \left(e^{\lambda x_*} + e^{-\lambda x_*}\right) \end{aligned} \tag{5.6}$$

It is trivial to show by substitution that $\lambda = 0$ is always a solution to (5.5). What more can be said?

As the numerics of the previous section show, $x_{*,n} \simeq x_{*,1}/n$ and $x_{*,1} \simeq 7.298613794 \times 10^{-3}$, or approximately the inverse of the fine structure constant. Consequently, to a high degree of accuracy

$$\begin{aligned} P(\lambda) &\simeq 2\lambda\Gamma \\ V(\lambda) &\simeq -\lambda\left[1+\Gamma\right] \\ Q(\lambda) &\simeq \lambda^2 \\ U(\lambda) &\simeq \lambda^2 - \Gamma - 2 \end{aligned} \tag{5.7}$$

so substituting these approximations into (5.5) we find that we have $\lambda_{1,2} = 0$ with multiplicity 2, and

$$\lambda_{3,4} \simeq \pm i\sqrt{2\Gamma^2 + \Gamma - 2} \simeq \pm i\sqrt{1 - 5x_*\tan x_*}. \tag{5.8}$$

Numerical studies of the eigenvalues given by (5.5) using the full equations (5.6) indicate that (5.8) gives a very accurate approximation to the non-zero eigenvalues. Thus we must conclude that the steady state solutions studied previously possess a type of marginal or neutral stability.

Appendix 1. The Bohr treatment of the hydrogen spectra.

This appendix summarizes the treatment followed by N. Bohr [xxx] in his explanation of the emission and absorption spectra of various elements. This work was later summarized by Bohr in yet a fifth paper [xxx]

The impetus for Bohr's work was the observation, by 1913 quite well established, that the spectra of atomic hydrogen can be very accurately described by the formula

$$\bar{\nu} = R_\infty\left\{\frac{1}{n^2} - \frac{1}{m^2}\right\} \qquad n < m \quad \text{with both being positive integers,} \tag{A1.1}$$

wherein $\bar{\nu}$ (in units of mt^{-1}) is the wavenumber [related to the frequency ν (sec^{-1}) by $\nu = c\bar{\nu}$, where c is the speed of light], and R_∞ is the *Rydberg constant* appropriate for hydrogen. When $n = 1$, equation (A1.1) describes the *Lyman* series in the ultraviolet region (discovered in 1906); for $n = 2$ it describes the visible *Balmer* series first described in 1885; with $n = 3$ one recovers a good description of the infrared *Paschen* series of 1908. $n = 4$ corresponds to the 1922 *Brackett* series (also infrared); and $n = 5$ gives the far infrared *Pfund* series.

In trying to understand the phenomenology described by equation (A1.1), Bohr used a mixture of classical physics, and what we now call the "old" quantum mechanics. Let us first examine the classical portion of his treatment.

For an electron-proton system, where the proton is fixed (infinitely massive) and the electron has finite mass m_e, from Newton's laws $m_e a = F$, where a is the acceleration and F is the force, for circular motion we obtain

$$m_e \frac{v^2}{r} = \frac{|A|}{r^2}, \tag{A1.2}$$

where r is the radius of the orbit, v is the electron velocity, and the constant A is given by

$$A = -\frac{e^2}{4\pi\epsilon_0} = -\frac{e^2 c^2}{10^7}. \tag{A1.3}$$

(e is the electronic charge and c is the velocity of light.)

Since, in a circular orbit, $v = r\omega$, where ω is the angular frequency, equation (A1.2) can also be written in form

$$m_e r\omega^2 = \frac{|A|}{r^2}, \tag{A1.4}$$

If E is the total energy of the electron, then it is made up of the sum of the kinetic and potential energy:

$$E = E_{kin} + E_{pot} \tag{A1.5}$$

where the kinetic energy is defined in a usual fashion

$$E_{kin} = \frac{1}{2} m_e v^2. \tag{A1.6}$$

From the customary definition of the potential energy,

$$E_{pot} = -\int_r^\infty \frac{|A|}{r'^2} dr' = -\frac{|A|}{r}. \tag{A1.7}$$

This potential energy corresponds to the work required to remove the electron from position r to ∞, and the point at which the potential energy goes to zero must correspond to the ionization potential for the electron-proton pair. Thus, we can write the total energy of equation (A1.5) as

$$E = \frac{1}{2} m_e v^2 + \frac{A}{r} = \frac{1}{2} m_e v^2 - \frac{|A|}{r} = -\frac{|A|}{2r}, \tag{A1.8}$$

where we have used (A1.2) to arrive at the final result. Alternately, we can solve (A1.4) for r:

$$r = \left(\frac{|A|}{m_e \omega^2} \right)^{\frac{1}{3}}, \tag{A1.9}$$

to give

$$E = -\frac{|A|^{\frac{2}{3}}}{2} (m_e \omega^2)^{\frac{1}{3}}. \tag{A1.10}$$

So far, the treatment is entirely classical, and equations (A1.9) and (A1.10) point out the inadequacy of any classical treatment to explain the spectral phenomena described by (A1.1). Namely, from equation (A1.9) in a classical situation all energies are allowed as well as all radii. However, since the electron is moving in a circular orbit, it is under constant acceleration, and consequently should be radiating electromagnetic waves at a frequency equivalent to its orbital frequency, i.e.

$$\nu = \frac{\omega}{2\pi}. \tag{A1.11}$$

Consequently the electron should lose energy in a continuous fashion and eventually spiral into the nucleus. All of this is quite elementary, and simply a summary of the well known classical paradox with respect to the observed discrete radiation spectrum described by equation (A1.1).

Faced with this impasse from classical theory, Bohr made the following postulates:

(1) Although the classical equations of motion (as sketched out above for circular orbits) are valid for an electron in an atom, only certain orbits (denoted by the index $n = 1, 2, \cdots$) are allowed, and in these orbits the electron has a fixed energy E_n.

(2) In these allowed orbits (which can be thought of as corresponding to *stationary states*) there is no emission of radiation even though this would be expected on the basis of classical electromagnetic theory.

(3) An electron going from a stationary state of energy E_m (remember that on a classical basis that the binding energy is negative) at radius r_m to a second stationary state with a binding energy $E_n < E_m$ at a smaller radius r_n will radiate electromagnetic waves at a frequency ν proportional to the difference in the energies in the two orbits.

$$E_m - E_n \propto \nu.$$

Bohr assumed the constant of proportionality is Planck's constant h:

$$E_m - E_n = h\nu = hc\bar{\nu}. \tag{A1.12}$$

The adsorption of energy would be described by the reverse process. Through a comparison of equations (A1.1) and (A1.12), Bohr then concluded that the energies in the stationary states must have the form

$$E_n = -\frac{R_\infty hc}{n^2}. \tag{A1.13}$$

(4) Things in the Bohr papers now become a bit clouded, but the fourth postulate essentially utilizes a primitive form of what he later enunciated as the *correspondence principle*. Bohr argued that in the situation that we are observing a transition from a state n to $n+\delta$, and n is large relative to δ, we should identify ν with the classical orbital frequency $\omega/2\pi$. In such a circumstance, from equation (A1.1) and the relation between wavenumber and frequency we have

$$\begin{aligned}\nu_{n,n+\delta} &= R_\infty c\left[\frac{1}{(n)^2} - \frac{1}{(n+\delta)^2}\right] \\ &= \frac{R_\infty c}{n^2}\left[1 - \frac{1}{[1+\delta/n]^2}\right] \\ &\simeq \frac{2R_\infty c\delta}{n^3}.\end{aligned} \tag{A1.14}$$

Thus, in going from the n^{th} to $(n+1)^{st}$ level with n large, we have that

$$\nu_{n,n+1} \simeq \frac{2R_\infty c}{n^3}. \tag{A1.15}$$

Now setting $\omega = 2\pi\nu$, with ν given by (A1.15), in equation (A1.10), along with equation (A1.13), we immediately obtain an expression for the Rydberg constant, namely

$$R_\infty = \frac{m_e e^4}{8\epsilon_0^2 h^3 c}. \tag{A1.16}$$

Equation (A1.16) was Bohr's main result, and with the value of h estimated from black body radiation and experiments on the photoelectric effect, and the other constants, he was able to obtain a numerical value for the Rydberg constant consistent with spectroscopic measurements.

With these relations, we can put some more flesh on the character of these stationary orbits assumed by Bohr. From equation (A1.13) we can write the energy of the nth stationary state as

$$E_n = -\frac{|A|}{2r_n}, \tag{A1.17}$$

or

$$\boxed{E_n = \frac{E_1}{n^2}} \tag{A1.18}$$

wherein

$$E_1 = -\frac{m_e}{2}\frac{|A|^2}{\hbar^2}, \tag{A1.19}$$

and we use the now customary notation $\hbar = h/2\pi$. Using the expression for the energy in equation (A1.8) we can solve for r_n to give

$$r_n = \frac{n^2\hbar^2}{m_e|A|}, \tag{A1.20}$$

or

$$\boxed{r_n = n^2 r_1} \tag{A1.21}$$

with

$$r_1 = \frac{\hbar^2}{m_e|A|}. \tag{A1.22}$$

Equation (A1.4) now yields

$$\begin{aligned}\omega_n &= \left(\frac{n\hbar}{m_e}\right)\left(\frac{m_e|A|}{n^2\hbar^2}\right)^2 \\ &= m_e\frac{|A|^2}{(n\hbar)^3},\end{aligned} \tag{A1.23}$$

or

$$\boxed{\omega_n = \frac{\omega_1}{n^3}} \tag{A1.24}$$

where

$$\omega_1 = m_e\frac{|A|^2}{\hbar^3}. \tag{A1.25}$$

Since the period T_n is just $T_n = 1/f_n$ and $f_n = 2\pi\omega_n$ it is immediate from (A1.24) that

$$\boxed{T_n = n^3 T_1,} \tag{A1.26}$$

with $T_1 = /2\pi/\omega_1$. The time required for the propagation of effects from electron to proton (the *delay*) is just $\tau = r/c$, so from the above relations we find

$$\tau_n = \frac{n^2\hbar^2}{m_e c|A|} \tag{A1.27}$$

or more simply

$$\boxed{\tau_n = n^2\tau_1,} \tag{A1.28}$$

with

$$\tau_1 = \frac{\hbar^2}{m_e c|A|}. \tag{A1.29}$$

From equation (A1.2) in conjunction with (A1.21), we may write v_n as

$$v_n = \frac{|A|}{n\hbar}, \tag{A1.30}$$

or

$$\boxed{v_n = \frac{v_1}{n}} \tag{A1.31}$$

with

$$v_1 = \frac{|A|}{\hbar}. \tag{A1.32}$$

In a circular orbit, the angular momentum is

$$p = m_e v r,$$

which means that

$$m_e v_n r_n = n\hbar \tag{A1.33}$$

so as a consequence of this entire treatment the angular momentum also ends up being quantized. Many authors incorrectly present the Bohr treatment as if quantized angular momentum was one of his postulates, but this is not the case.

Having all of these relations, its instructive to see what the numbers really look like. Taking the best available modern determinations we use (unless otherwise noted, these values are taken from Cohen and Taylor (1987):

(1) speed of light $c = 2.99792458 \times 10^8$ ms^{-1};
(2) $\epsilon_0 = 10^7/4\pi c^2$ Fm^{-1};
(3) electronic charge $e = 1.60217733 \times 10^{-19}$ C
or [Cage et al., 1989]
$e = 1.60217670 \times 10^{-19}$ C ;
(4) electron mass $m_e = 9.1093897 \times 10^{-31}$ kg;
(5) $h = 6.6260755 \times 10^{-34}$ Js $\implies \hbar = 1.054573 \times 10^{-34}$ Js
or [Cage et al., 1989]
$h = 6.6260704 \times 10^{-34}$ Js $\implies \hbar = 1.054572 \times 10^{-34}$ Js;
(6) $R_\infty = 1.0973731534 \times 10^7$ m^{-1}
or [Meq et al., 1989]
$R_\infty = 1.097373156830 \times 10^7$ m^{-1}
or [Andreae et al., 1992]
$R_\infty = 1.097373156841 \times 10^7$ m^{-1}

we first find that **calculating** h from equation (A1.16) using the CODATA [Cohen and Taylor, 1987] values, especially for R_∞, that the precise value quoted in the CODATA figures results. Namely,

$$h = 6.6260755 \times 10^{-34} \quad \text{Js},$$

which is a bit suspicious since in the CODATA report they make no mention of how their best figure for h was arrived at. Going further we rather easily find, using the 1986 CODATA values, that

(1) Energy in the first Bohr orbit is $E_1 = -2.1798741 \times 10^{-18}$J;
(2) Radius in the first Bohr orbit is $r_1 = 5.29177249 \times 10^{-11}$ m;
(3) Angular frequency in the first Bohr orbit is $\omega_1 = 4.13413732 \times 10^{16}$ s^{-1};
(4) Period in the first Bohr orbit is $T_1 = 1.51982989 \times 10^{-16}$ s;
(5) Delay to the first Bohr orbit is $\tau_1 = 1.7651453 \times 10^{-19}$ s;
(6) Velocity in the first Bohr orbit is $v_1 = 2.18769141 \times 10^6$ ms^{-1};
(7) Frequency in the first Bohr orbit is $f_1 = 6.57968374 \times 10^{15}$ sec^{-1};
(8) Ratio of the period to the delay in the first Bohr orbit is

$$\frac{T_1}{\tau_1} = 2\pi \times 137.03590744504,$$

where $137.\cdots$ is the "fine structure constant". This last relation can be written in the alternate form

$$\frac{1}{\omega_1 \tau_1} = 137.03590744504.$$

Interestingly, this is not the same value as given in the CODATA report [Cohen and Taylor, 1986], namely

$$\alpha^{-1} = 137.0359895,$$

nor does it match the recent value [Cage et al., 1989] determined from measurements on the quantized Hall effect and the Josephson junction carried out at the NBS:

$$\alpha^{-1} = 137.0359940.$$

A simpler route to the Bohr results.

It is interesting that a much simpler and more transparent way is available to derive the essence of the Bohr results. It is the following.

First, we have equation (A1.4), which I repeat for ease of exposition:

$$m_e r\omega^2 = \frac{|A|}{r^2}. \tag{A1.4}$$

Furthermore, for circular planar motion the angular momentum of the electron is constant, so

$$m_e \omega r^2 = \mathcal{L}, \tag{A1.34}$$

where $\mathcal{L}$ is an unknown quantity to be determined. Solving (A1.4) and (A1.34) for the pair (r, ω) gives

$$r = \frac{\mathcal{L}^2}{m_e |A|} \tag{A1.35}$$

and

$$\omega = \frac{m_e |A|^2}{\mathcal{L}^3} \tag{A1.36}$$

respectively. If we also adopt the Bohr approximation derived from the Rydberg equation:

$$\nu \simeq \frac{2R_\infty c\delta}{n^3}, \tag{A1.14}$$

and assume

$$\omega = 2\pi\nu \simeq \frac{4\pi R_\infty c\delta}{n^3}, \tag{A1.37}$$

then in conjunction with (A1.36) we have after a bit of algebra the result

$$\left(\frac{2\pi\mathcal{L}}{n}\right)^3 = \frac{m_e e^4}{8R_\infty \epsilon_0^2}. \tag{A1.38}$$

The right hand side of equation (A1.38) is numerically equal to the value of the Planck constant raised to the third power:

$$\left(\frac{2\pi\mathcal{L}}{n}\right)^3 = \frac{m_e e^4}{8R_\infty \epsilon_0^2} = h^3. \tag{A1.39}$$

Consequently, we must have

$$\mathcal{L}_n = n\frac{h}{2\pi} = n\hbar, \tag{A1.40}$$

so

$$r_n = \frac{n^2\hbar^2}{m_e |A|} \tag{A1.41}$$

and

$$\omega_n = \frac{m_e |A|^2}{n^3\hbar^3} \tag{A1.10}$$

respectively.

Finally, we note that from equation (A1.8), we can write

$$E_n = -\frac{m_e A^2}{2n^2\hbar^2}, \tag{A1.11}$$

so

$$E_n - E_m = \frac{m_e A^2}{2\hbar^2}\left[\frac{1}{m^2} - \frac{1}{n^2}\right] = hR_\infty\left[\frac{1}{m^2} - \frac{1}{n^2}\right] = h\bar{\nu}, \tag{A1.44}$$

thereby demonstrating that Bohr's third assumption is unnecessary.

Appendix 2. Derivation of the perturbation equations.

To examine the stability of equations of motion (3.8) and (3.9) written in the slightly modified form

$$\begin{aligned}\ddot{r} - r\dot{\theta}^2 &= \frac{A}{2\mu}\left\{\frac{\mathbf{u_r}\cdot\mathbf{u_{r-}}}{{r_-}^2} + \frac{\mathbf{u_r}\cdot\mathbf{u_{r+}}}{{r_+}^2}\right\} \\ &= \frac{A}{2\mu}\left\{\frac{\cos(\theta-\theta_-)}{{r_-}^2} + \frac{\cos(\theta-\theta_+)}{{r_+}^2}\right\},\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}r\ddot{\theta} + 2\dot{r}\dot{\theta} &= \frac{A}{2\mu}\left\{\frac{\mathbf{u}_\theta\cdot\mathbf{u_{r-}}}{{r_-}^2} + \frac{\mathbf{u}_\theta\cdot\mathbf{u_{r+}}}{{r_+}^2}\right\} \\ &= -\frac{A}{2\mu}\left\{\frac{\sin(\theta-\theta_-)}{{r_-}^2} + \frac{\sin(\theta-\theta_+)}{{r_+}^2}\right\},\end{aligned} \tag{3.9}$$

in the face of small perturbations, we assume that $R(t)$ is a small ($\mathcal{O}(\epsilon)$) perturbation of r_* satisfying (3.10) so

$$r(t) \simeq r_* + R(t),$$

while $\Theta(t)$ is a small perturbation of $\omega_* t$:

$$\theta(t) \simeq \omega_* t + \Theta(t).$$

Then to $\mathcal{O}(\epsilon)$ the left hand sides of (3.8) and (3.9) become

$$\ddot{r}(t) - r(t)\dot{\theta}^2(t) \simeq \ddot{R}(t) - \omega_*^2 R(t) - 2r_*\omega_*\dot{\Theta}(t) - r_*\omega_*^2 \tag{A2.1}$$

and

$$r(t)\ddot{\theta}(t) + 2\dot{r}(t)\dot{\theta}(t) \simeq r_*\ddot{\Theta}(t) + 2\omega_*\dot{R}(t) \tag{A2.2}$$

respectively.

In examining the angular components,

$$\theta(t) - \theta_\pm \simeq \mp\frac{\omega_* r_*}{c} \mp \frac{\omega_* R}{c} + \Theta(t) - \Theta\left(t \pm \frac{r_*}{c}\right) \tag{A2.3}$$

so, to the same accuracy,

$$\cos(\theta-\theta_\pm) \simeq \cos\left(\frac{\omega_* r_*}{c}\right) - \left[\frac{\omega_* R}{c} \mp \Theta(t) \pm \Theta\left(t \pm \frac{r_*}{c}\right)\right]\sin\left(\frac{\omega_* r_*}{c}\right) \tag{A2.4}$$

and

$$\sin(\theta-\theta_\pm) \simeq \mp\sin\left(\frac{\omega_* r_*}{c}\right) \mp \left[\frac{\omega_* R}{c} \mp \Theta(t) \pm \Theta\left(t \pm \frac{r_*}{c}\right)\right]\cos\left(\frac{\omega_* r_*}{c}\right) \tag{A2.5}$$

Furthermore, in the radial coordinate we have

$$r_\pm(t) \simeq r_* + R\left(t \pm \frac{r_*}{c}\right), \tag{A2.6}$$

so

$$\frac{1}{r_\pm^2(t)} \simeq \frac{1}{r_*^2}\left[1 - \frac{2}{r_*}R\left(t \pm \frac{r_*}{c}\right)\right]. \tag{A2.7}$$

Consequently,

$$\begin{aligned}\frac{\cos(\theta(t)-\theta_\pm)}{r_\pm^2(t)} \simeq{}& \frac{1}{r_*^2}\cos\left(\frac{\omega_* r_*}{c}\right) \\ &- \frac{1}{r_*^2}\left[\frac{\omega_* R}{c} \mp \Theta(t) \pm \Theta\left(t \pm \frac{r_*}{c}\right)\right]\sin\left(\frac{\omega_* r_*}{c}\right) \\ &- \frac{2}{r_*^3}R\left(t \pm \frac{r_*}{c}\right)\cos\left(\frac{\omega_* r_*}{c}\right)\end{aligned} \tag{A2.8}$$

and

$$\frac{\sin(\theta(t)-\theta_{\pm})}{r_{\pm}^2(t)} \simeq \mp\frac{1}{r_*^2}\sin\left(\frac{\omega_* r_*}{c}\right) \mp \frac{1}{r_*^2}\left[\frac{\omega_* R}{c} \mp \Theta(t) \pm \Theta\left(t \pm \frac{r_*}{c}\right)\right]\cos\left(\frac{\omega_* r_*}{c}\right) \pm \frac{2}{r_*^3}R\left(t \pm \frac{r_*}{c}\right)\sin\left(\frac{\omega_* r_*}{c}\right) \tag{A2.9}$$

Combining the estimations of (A2.34) and (A2.8), equation (A2.8) to $\mathcal{O}(\epsilon)$ becomes

$$\ddot{R} - \omega_*^2 R - 2r_*\omega_*\dot{\Theta} - r_*\omega_*^2 = \frac{A}{\mu r_*^2}\cos\left(\frac{\omega_* r_*}{c}\right) - \frac{A}{2\mu r_*^2}\left[\frac{2\omega_* R}{c} + \Theta\left(t+\frac{r_*}{c}\right) - \Theta\left(t-\frac{r_*}{c}\right)\right]\sin\left(\frac{\omega_* r_*}{c}\right) - \frac{A}{\mu r_*^3}\left[R\left(t+\frac{r_*}{c}\right) + \left[R\left(t-\frac{r_*}{c}\right)\right]\cos\left(\frac{\omega_* r_*}{c}\right)\right. \tag{A2.10}$$

In a similar fashion, from (A2.35) and (A2.9), equation (A2.9) becomes

$$r_*\ddot{\Theta} + 2\omega_*\dot{R} = \frac{A}{2\mu r_*^2}\left[\Theta\left(t+\frac{r_*}{c}\right) - 2\Theta(t) + \Theta\left(t-\frac{r_*}{c}\right)\right]\cos\left(\frac{\omega_* r_*}{c}\right) - \frac{A}{\mu r_*^3}\left[R\left(t+\frac{r_*}{c}\right) - R\left(t-\frac{r_*}{c}\right)\right]\sin\left(\frac{\omega_* r_*}{c}\right) \tag{A2.11}$$

Using the equilibrium condition (A2.10) in (A2.10) we obtain

$$\ddot{R} - \omega_*^2 R\left[1 - \frac{\omega_* r_*}{c}\tan\left(\frac{\omega_* r_*}{c}\right)\right] - 2r_*\omega_*\dot{\Theta}(t) = \frac{\omega_*^2 r_*}{2}\tan\left(\frac{\omega_* r_*}{c}\right)\left[\Theta\left(t+\frac{r_*}{c}\right) - \Theta\left(t-\frac{r_*}{c}\right)\right] + \omega_*^2\left[R\left(t+\frac{r_*}{c}\right) + R\left(t-\frac{r_*}{c}\right)\right] \tag{A2.12}$$

while (A2.11) can be written as

$$r_*\ddot{\Theta} - r_*\omega_*^2\Theta + 2\omega_*\dot{R} = -\frac{\omega_*^2 r_*}{2}\left[\Theta\left(t+\frac{r_*}{c}\right) + \Theta\left(t-\frac{r_*}{c}\right)\right] + \omega_*^2\tan\left(\frac{\omega_* r_*}{c}\right)\left[R\left(t+\frac{r_*}{c}\right) - R\left(t-\frac{r_*}{c}\right)\right] \tag{A2.13}$$