Density Evolution Under Delayed Dynamics: An Open Problem (file: MYTRY-v2.tex)

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Abstract

This set of notes has arisen out of a number of attempts, over the past 40 years, to understand how one might examine the evolution of densities in systems whose dynamics are described by differential delay equations. Though we have no definitive solution to the problem, we have written these notes in an attempt to define the problem as we see it, and to sketch out several obvious attempts that have been suggested to solve the problem. We offer these in the hope that by being available to the general mathematical community they will inspire others to consider–and hopefully solve–the problem.

Contents

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1 Introduction

In examining the dynamical behavior of a system there are fundamentally two options available to the experimentalist.

1. In the first option s/he will examine the dynamical trajectories of individuals, be they fundamental particles in a cloud chamber or cells in a petri dish or animals in an ecological experiment. In this case the experimentalist may be interested in replicating the experiment many times, and building up a statistical description of the observed behavior under the assumption (among others) that the trajectory behavior will be replicated between trials given the same initial conditions.

2. In the second option this approach will be forsaken for one in which the evolving statistics of large populations are examined. This is, of course, most familiar in statistical mechanics, but is also important in many other areas. The advantage of this approach is that if one can understand the dynamics of density evolution, then many interesting statistical quantities can be computed, and the results compared with experimental results.

Which approach is taken is sometimes a matter of choice, but often dictated by the nature of the individual units being studied.

For a large class of systems in which the underlying dynamics are described by differential equations, or stochastic differential equations, or maps, there is a large *corpus* of methods that have been developed with which one can approach both of the types of data collection outlined above and the connection of that data to underlying dynamical systems theory.

However many problems in the physical, and especially the biological, sciences involve the dynamic behavior of individual entities whose dynamics involve significant delays. For problems like this, existing techniques to theoretically consider the evolution of densities are non-existent. Repeated attempts to think of ways to formulate the evolution of densities in the presence of dynamics with delays have failed in even the most elementary respects (e.g. defining the fundamental mathematical aspects of the problem) and it is because of this failure that we present this Open Problem.

To be more concrete, if we have a variable x evolving under the action of some dynamics described by a differential delay equation

$$\frac{dx}{dt} = \epsilon^{-1}\mathcal{F}(x(t), x(t-\tau)), \qquad x(t) = \varphi(t) \quad t \in [-\tau, 0], \tag{1.1}$$

then we would like to know how some density of the variable x will evolve in time, i.e. we would like to be able to write down an equation

$$\boxed{\text{UNKNOWN OPERATOR ACTING ON DENSITY} = 0.} \tag{1.2}$$

Unfortunately we don't really know how to do this, and that's the whole point of this paper. The reason that the problem is so difficult is embodied in Equation 1.1 and the infinite dimensional nature of the problem because of the necessity of specifying the *initial function* $\varphi(t)$ for $t \in [-\tau, 0]$.

However, we do have some clues about what UNKNOWN OPERATOR should look like in various limiting cases. For example, if in Equation 1.1, $\mathcal{F}(x, x(t-\tau)) = -x(t) + \mathcal{S}(x(t-\tau))$ so (1.1) becomes

$$\epsilon \frac{dx}{dt} = -x(t) + \mathcal{S}(x(t-\tau)), \qquad x(t) = \varphi(t) \quad t \in [-\tau, 0], \tag{1.3}$$

then we expect that:

1. If $\tau \to 0$ then we should recover the normal Liouville operator (see Section 2.3 below) from UNKNOWN OPERATOR;

2. If we let $\epsilon \to 0$ and restrict t to $t \in \mathbb{N}$, then UNKNOWN OPERATOR should reduce to the Frobenius Perron operator (see Section 2.5) for the map $\mathcal{S}$.

3. If $\epsilon \to 0$, then from UNKNOWN OPERATOR we should recover the operator governing the evolution of densities *in a function space* under the action of the functional map

$$x(t) = \mathcal{S}(x(t-\tau)), \tag{1.4}$$

for $t \in \mathbb{R}^+$, though we don't know what that should be (see Section 6.1)

This paper is organized as follows. In Section 2 we briefly what is known about density evolution in systems with finite dimensional dynamics, starting with a description of the connection between dynamics and densities in 2.1. Section 2.3 reviews the situation for the commonly known situation in which the dynamics are described by ordinary differential equations. Section 2.4 briefly considers dynamics described by stochastic differential equations, while 2.5 does the same for finite dimensional maps. This section concludes in 2.6 with a description of the dynamic density evolution behaviors ergodicity, mixing, exactness and asymptotic periodicity. Section 3 examines density evolution behavior in partial differential equations.

Section 4 starts to examine the situation in differential delay equations. Section 4.1 relates the formal 'density evolution' problem for differential delay equations to what is actually measured in an experimental setting. Finally, Section 4.2 gives numerical evidence for the existence of interesting ergodic properties of density evolution dynamics in the presence of delays.

Section **??** considers the real mathematical problems involved in this problem ranging from the proper nature of the underlying space to the problem of defining a density. Section 6.1 reformulates the problem as one of functional interation.

Section 5 outlines an approach that has been tried based on Hopf functional techniques, while Section 6 considers the problem reformulated as the method of steps. Finally Section 8 considers approximations to the delay problem, first looking at using a distribution of time delays in 8.1 and then in 8.2 with an approximation in which the time delay is divided into ever finer intervals. We conclude in Section 9 with a brief Discussion.

2 Density evolution in systems with finite dimensional dynamics

2.1 Dynamics and densities

In looking at ensemble behavior, the natural framework is to look at the evolution of a density as the description of the temporal behavior of the ensemble. Thus we start by looking at the operators important for describing this density evolution. For background material see Lasota and Mackey (1994).

We first start with a set X. Measure theorists often like to keep X pretty abstract, but for us X is going to be the **phase space** (more about this in the next section) on which all of our dynamics operates. Sometimes X will be a closed finite interval like $[0,1]$, sometimes it may be $\mathbb{R}^+$, or even $\mathbb{R}^d$, and sometimes X is a function space. In any event, whatever X is we are going to assume that it does not have any pathological properties.

Let X be a space, $\mathcal{A}$ a σ-algebra, μ a measure, and denote by $(X, \mathcal{A}, \mu)$ the corresponding σ-finite measure space. Let D be the subset of the space $L^1(X, \mathcal{A}, \mu)$ containing all densities, i.e. $D = \{f \in L^1 : f \geq 0, \|f\| = 1\}$. A linear map $P : L^1 \to L^1$ is a **Markov operator** if $P(D) \subset D$. If there is an $f_* \in D$ such that $Pf_* = f_*$ then f_* is called a stationary density.

In terms of dynamics, we consider $\mathcal{S}_t : X \to X$, as time t changes. Time t may be either continuous ($t \in \mathbb{R}$) as, for example, it would be for a system

whose dynamics were governed by a set of differential equations, or discrete (integer valued, $t \in \mathbb{Z}$) if the dynamics are determined by discrete time maps.

A dynamical system $\{\mathcal{S}_t\}_{t\in\mathbb{R}}$ (or, alternately, $t \in \mathbb{Z}$ for discrete time systems) on a phase space X, is simply any group of transformations $\mathcal{S}_t : X \to X$ having the properties $\mathcal{S}_0(x) = x$ and $\mathcal{S}_t(\mathcal{S}_{t'}(x)) = \mathcal{S}_{t+t'}(x)$ for $t, t' \in \mathbb{R}$ or $\mathbb{Z}$. Since, from the definition, for any $t \in \mathbb{R}$, we have $\mathcal{S}_t(\mathcal{S}_{-t}(x)) = x = \mathcal{S}_{-t}(\mathcal{S}_t(x))$, it is clear that dynamical systems are invertible, and systems of ordinary differential equations are examples of dynamical systems as are invertible maps. A semidynamical system $\{\mathcal{S}_t\}_{t\in\mathbb{R}^+}$ is any semigroup of transformations $\mathcal{S}_t : X \to X$ such that $\mathcal{S}_0(x) = x$ and $\mathcal{S}_t(\mathcal{S}_{t'}(x)) = \mathcal{S}_{t+t'}(x)$ for $t, t' \in \mathbb{R}^+$ (or $\mathbb{N}$).

As one might guess, Markov operators are extremely valuable in terms of examining the evolution of densities under the action of a variety of dynamics, and in the next few paragraphs we will give concrete examples of these operators.

2.2 Frobenius Perron operator

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $\mathcal{S}_t : X \to X$ be a measurable[1] and nonsingular[2] transformation. Then the unique operator $P^t : L^1 \to L^1$ defined by

$$\int_A P^t f(x)\, \mu(dx) = \int_{\mathcal{S}_t^{-1}(A)} f(x)\, \mu(dx) \tag{2.5}$$

is the **Frobenius-Perron operator** corresponding to $\mathcal{S}_t$.

Thus, if f is a density, then equation (2.5) defining the Frobenius-Perron operator has an intuitive interpretation. Start with an initial density f and integrate this over a set B that will evolve into the set A under the action of the transformation $\mathcal{S}_t$. However, the set B is $\mathcal{S}_t^{-1}(A)$. This integrated quantity must be equal, since $\mathcal{S}_t$ is nonsingular, to the integral over the set A of the density obtained after one application of $\mathcal{S}_t$ to f. This final density is $P^t f$. Any density f_* for which $P^t f_* = f_*$ is called a stationary density of P^t.

The fact that the Frobenius-Perron operator is unique is a straightforward consequence of the Radon-Nikodym theorem. It is clear from the definition that the Frobenius-Perron operator is a Markov operator, and so P^t is a linear contracting operator. Also, if $f \geq 0$ then $P^t f \geq 0$ and

[1] $\mathcal{S}_t^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$

[2] $\mu(\mathcal{S}_t^{-1}(A)) = 0$ for all $A \in \mathcal{A}$ such that $\mu(A) = 0$

$\| P^t f \| = \| f \|$. Finally it is easy to show that if $\mathcal{S}_{nt} = \mathcal{S}_t \circ \cdots \circ \mathcal{S}_t$, and P^{nt} and P^t are, respectively, the Frobenius-Perron operator corresponding to $\mathcal{S}_{nt}$ and $\mathcal{S}_t$, then $P^{nt} = P^t \circ \cdots \circ P^t = (P^t)^n$.

Sometimes the implicit defining Equation 2.5 for the Frobenius- Perron operator allows one to obtain an explicit formula for P^t. For example if $A = [a, x]$ then (2.5) becomes

$$\int_a^x P^t f(s)\, ds = \int_{\mathcal{S}_t^{-1}([a,x])} f(s)\, ds \tag{2.6}$$

so

$$P^t f(x) = \frac{d}{dx} \int_{\mathcal{S}_t^{-1}([a,x])} f(s)\, ds. \tag{2.7}$$

This process may be carried even further if the transformation is invertible so $\mathcal{S}_t^{-1} = \mathcal{S}_{-t}$ and $\mathcal{S}_{-t}$ has a continuous derivative with respect to x. Then, (2.7) becomes

$$P^t f(x) = f(\mathcal{S}_{-t}(x)) \left| \frac{d\mathcal{S}_{-t}(x)}{dx} \right|. \tag{2.8}$$

From this it is relatively straightforward to obtain a generalization of equation (2.8) valid for any invertible transformation $\mathcal{S}_t$ operating in $\mathbb{R}^d$. Namely

$$P^t f(x) = f(\mathcal{S}_{-t}(x)) J^{-t}(x). \tag{2.9}$$

2.3 The Liouville equation

Given a set of ordinary differential equations

$$\frac{dx_i}{dt} = \mathcal{F}_i(x), \qquad i = 1, \ldots, d \tag{2.10}$$

operating in a bounded region of $\mathbb{R}^d$, it is possible to derive an evolution equation for $P^t f(x)$ by using the invertibility of (2.10) in conjunction with (2.9). This gives the evolution equation for $f(t, x) = P^t f(x)$:

$$\frac{\partial f}{\partial t} = -\sum_{i=1}^{d} \frac{\partial (f \mathcal{F}_i)}{\partial x_i}, \tag{2.11}$$

the generalized Liouville equation.

2.4 The Fokker Planck equation

As an extension of the situation for ordinary differential equations, for stochastic differential equations of the form

$$dx = \mathcal{F}(x)dt + \sigma(x)dW(t), \tag{2.12}$$

where x is a d-dimensional vector and $W(t)$ is a standard Wiener process, then the density $f(t,x) \equiv P^t f_0(x)$ satisfies the Fokker-Planck equation

$$\frac{\partial f}{\partial t} = -\sum_{=1}^{d} i\frac{\partial(f\mathcal{F}_i)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2(a_{ij}f)}{\partial x_i \partial x_j}, \tag{2.13}$$

where $a_{ij}(x) = \sum_{k=1}^{d}\sigma_{ik}(x)\sigma_{jk}(x)$.

2.5 Density evolution in maps

For the one dimensional map

$$x_{t+1} = \mathcal{S}(x_t) \tag{2.14}$$

the Frobenius Perron operator is given by

$$P_{\mathcal{S}}f(x) = \frac{d}{dx}\int_{\mathcal{S}^{-1}([0,x])} f(u)\,du. \tag{2.15}$$

Example 1. *For the tent map*

$$\mathcal{S}(x) = \begin{cases} 2x & \text{for} \quad x \in \left[0, \frac{1}{2}\right) \\ 2(1-x) & \text{for} \quad x \in \left[\frac{1}{2}, 1\right], \end{cases} \tag{2.16}$$

$\mathcal{S} : [0,1] \to [0,1]$*, and the corresponding Frobenius Perron operator is given by*

$$P_{\mathcal{S}}f(x) = \frac{1}{2}\left[f\left(\frac{x}{2}\right) + f\left(1 - \frac{x}{2}\right)\right]. \tag{2.17}$$

It is easily verified that the stationary density is $f_*(x) = 1_{[0,1]}(x)$ *where* $1_A(x)$ *is the indicator function.*

Further, for a one dimensional map perturbed by a noise source ξ distributed with density g

$$x_{t+1} = \mathcal{S}(x_t) + \sigma\xi_t \tag{2.18}$$

then the Markov operator governing the density evolution is given by

$$P_{\mathcal{S}}f(x) = \int_X f(u)g(x - \sigma\mathcal{S}(u))\,du. \tag{2.19}$$

Example 2. *The Keener map*

$$\mathcal{S}(x) = \alpha x + \beta \qquad (\text{mod } 1), \quad \alpha, \beta \in (0,1) \tag{2.20}$$

is statistically periodic when perturbed by noise ξ distributed with density g:

$$x_{n+1} = (\alpha x_n + \beta + \sigma\xi_n) \qquad (\text{mod } 1) \tag{2.21}$$

and has a Markov operator given by

$$P_{\mathcal{S}} f(x) = \int_X f(u) g(x - \sigma(\alpha u + \beta))\, du. \tag{2.22}$$

2.6 The dynamics of density evolution

As is the case when examining the temporal evolution of single trajectories emanating from a given initial condition in a dynamical system, there can be a variety of dynamical behaviors of densities when evolving from an initial density. The first ones that we consider are ergodicity, mixing and asymptotic stability and all three of these can be characterized by the nature of the convergence of successive values of the densities.

The weakest type of convergence is contained in the property of ergodicity. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $\mathcal{S} : X \to X$ a non-singular transformation that preserves the measure μ which has density f_*. $\mathcal{S}$ is ergodic if every invariant set $A \in \mathcal{A}$ is such that either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. Ergodicity is equivalent to

Theorem 2.1. *$\mathcal{S}_t$ is ergodic with stationary density f_* operating in a finite normalized phase space X if and only if for any integrable function g the time average of g along the trajectory of $\mathcal{S}_t$ is equal to the f_* weighted average of g over the entire phase space. That is,*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} g(\mathcal{S}_k) = \int_X g(x) f_*(x)\, dx = < g > \tag{2.23}$$

in the discrete time case, or

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(\mathcal{S}_t(x))\, dt = \int_X g(x) f_*(x)\, dx = < g > \tag{2.24}$$

in the continuous time case.

Next in the hierarchy is the stronger property of mixing. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $\mathcal{S} : X \to X$ a transformation that preserves the measure μ which has density f_*. $\mathcal{S}$ is mixing if

$$\lim_{t\to\infty} \mu(A \bigcap \mathcal{S}_{-t}(B)) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{A}. \tag{2.25}$$

Mixing is equivalent to

Theorem 2.2. *Let $\mathcal{S}_t$ be an ergodic transformation, with stationary density f_* of the associated Frobenius- Perron operator, operating in a phase space of finite f_* measure. Then $\mathcal{S}_t$ is f_* mixing if and only if $\{P^t f\}$ is weakly convergent to f_* for all densities f, i.e.,*

$$\lim_{t\to\infty} < P^t f, g >=< f_*, g > \tag{2.26}$$

for every bounded measurable function g.

Finally, we have the strongest property of asymptotic stability. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space, $\mathcal{S} : X \to X$ a transformation that preserves the measure μ which has density f_*, and $\mathcal{S}$ such that $\mathcal{S}(A) \in \mathcal{A}$ for each $A \in \mathcal{A}$. $\mathcal{S}$ is asymptotically stable if

$$\lim_{t\to\infty} \mu(\mathcal{S}_t(A)) = 1 \quad \text{for all } A, B \in \mathcal{A}. \tag{2.27}$$

Asymptotic stability is equivalent to

Theorem 2.3. *If $\mathcal{S}_t$ is an f_* measure preserving transformation operating on a finite normalized phase space X and P^t is the associated Frobenius-Perron operator corresponding to $\mathcal{S}_t$, then $\mathcal{S}_t$ is f_* asymptotically stable if and only if*

$$\lim_{t\to\infty} \parallel P^t f - f_* \parallel = 0,$$

i.e., $\{P^t f\}$ is strongly convergent to f_*, for all initial densities f.

Remark 1. *The three dynamic behaviors of densities we have examined are related in that asymptotic stability implies mixing which implies ergodicity. The converse is not true.*

Remark 2. *Ergodicity and mixing are properties that may be present in both dynamical and semi-dynamical systems. Asymptotic stability, however, is only possible in semi-dynamical systems.*

There is a fourth type of dynamic behavior that density evolution can display, that of asymptotic periodicity (Komorník and Lasota, 1987; Komorník, 1993). Namely, the ergodic transformation $\mathcal{S}$ is asymptotically periodic with period r if there exist a sequence of densities $g_1, \ldots, g_r$ and a sequence of bounded linear functionals $\lambda_1, \ldots, \lambda_r$ such that

$$\lim_{t\to\infty} ||P^t(f - \sum_{j=1}^{r} \lambda_j(f) g_j)|| = 0.$$

The densities g_j have disjoint supports and $Pg_j = g_{\alpha(j)}$, where α is a permutation of $(1, \ldots, r)$. The invariant density is given by

$$g_* = \frac{1}{r} \sum_{j=1}^{r} g_j \tag{2.28}$$

and $(\mathcal{S}^r, g_j)$ is exact for every $j = 1, \ldots, r$.

Remark 3. *Asymptotic periodicity is a density evolution property that may either be inherent in the dynamics (Provatas and Mackey, 1991a; Losson and Mackey, 1995) or induced by noise (Lasota and Mackey, 1987; Provatas and Mackey, 1991b) as in the next two examples.*

Example 3. *The generalized tent map on* $[0, 1]$ *is defined by:*

$$\mathcal{S}(x) = \begin{cases} ax & \text{for} \quad x \in \left[0, \frac{1}{2}\right) \\ a(1-x) & \text{for} \quad x \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{2.29}$$

Ito et al. (1979a,b) have shown that the tent map Equation 2.29 is ergodic, thus possessing a unique invariant density g_*. *The form of* g_* *has been derived in the parameter window*

$$a_{n+1} = 2^{1/2^{n+1}} < a \leq 2^{1/2^n} = a_n \qquad \text{for} \qquad n = 0, 1, 2, \cdots, \tag{2.30}$$

by Yoshida et al. (1983). Provatas and Mackey (1991a) have proved the asymptotic (statistical) periodicity of (2.29) with period $r = 2^n$, $n = 0, 1, \cdots$ *for*

$$2^{1/2^{n+1}} < a \leq 2^{1/2^n}. \tag{2.31}$$

Thus, for example, $\{P^t f\}$ *has period* 1 *for* $2^{1/2} < a \leq 2$, *period* 2 *for* $2^{1/4} < a \leq 2^{1/2}$, *period* 4 *for* $2^{1/8} < a \leq 2^{1/4}$, *etc. Equation 2.29 is exact for* $a = 2$.

The Frobenius-Perron operator corresponding to (2.29) is given by

$$Pf(x) = \frac{1}{a}\left[f\left(\frac{x}{a}\right) + f\left(1 - \frac{x}{a}\right)\right]. \tag{2.32}$$

Example 4. *Lasota and Mackey (1987); Provatas and Mackey (1991b) have studied the asymptotic periodicity induced by noise in a Keener map*

$$\mathcal{S}(x) = (ax + b) \pmod 1,\ 0 < a, b < 1 \tag{2.33}$$

by studying the dynamics of

$$x_{n+1} = (ax_n + b + \xi_n) \pmod 1,\ 0 < a, b < 1, \tag{2.34}$$

when the noise source ξ is distributed with density g.

3 Density evolution in partial differential equations

Remark 4. *Should we add a section about the ergodic theory like results in partial differential equations obtained by Rudnicki? These results are based on a Wiener measure, and I wonder if using the Wiener measure might not be a way out of part of the trouble that we have.*

4 Dynamics in ensembles of differential delay equations

4.1 What do we measure?

As pointed out in the Introduction, there are fundamentally two types of data that are taken in experimental situations, and one is related to statistical properties of large ensembles of 'units' that are typically assumed to have the same dynamics. If their dynamics are described by a differential delay equation of the form in Equation 1.1 then we must consider what is likely to be measured. Figure 4.1 will aid in this.

In Figure 4.1 we show a schematic depiction of what one would actually measure in an ensemble of units whose dynamic evolution is governed by a differential delay equation. We assume that there are N such units involved in our experiment, and that the experiment is started at time $t = 0$ with each of the N units having a history (= an initial function) on the interval $[-\tau, 0]$ preceding the start of the experiment. We let these N units evolve dynamically in time, and assume that we have a device able to record a histogram approximation ρ to the density $f(t, x)$ of the distribution of the

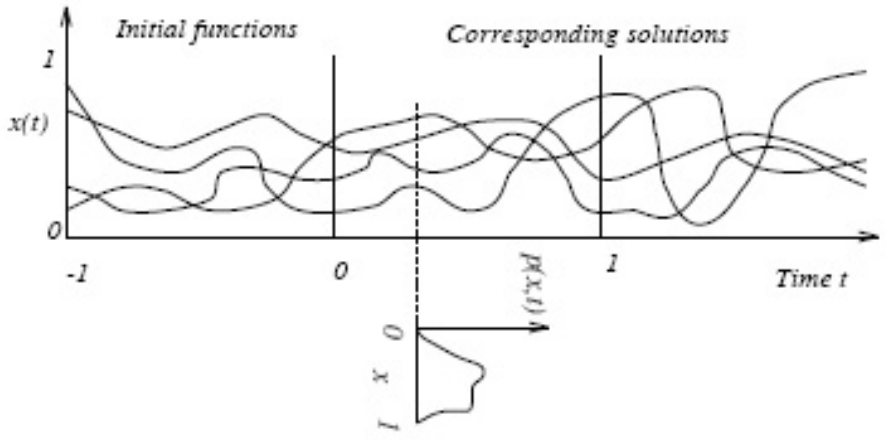

Figure 4.1: A schematic illustration of the connection between the evolution of an ensemble of initial functions and what would be measured in a laboratory. In the case that the delay has been scaled to $\tau = 1$, an ensemble of N initial functions on $[-1, 0]$ is allowed to evolve forward in time under the action of the delayed dynamics. At time t we sample the distribution of the values of x across all N trajectories and form an approximation to a density $f(t, x)$ given by ρ. Taken from Losson and Mackey (1995) with permission.

state variable x at time t.[3] Note that this measurement procedure is carried out at *successive* individual times and might be continuous.

Thus, what we *measure* is not unlike what we might measure in a system whose dynamics are evolving under the action of the system of ordinary differential equations (2.10). However, what we are able to *calculate* is far different.

4.2 Numerical evidence for interesting density dynamics in differential delay equations

Within the framework sketched in Section 4.1, what types of behavior can be observed in ensembles of differential delay equations? Numerous numerical studies of situations like this have revealed a rich and sometimes bewildering array of dynamical behaviors that become clear only within an ergodic

[3]It sometimes might be the case that we would not measure ρ, but rather might have estimates of various moments of ρ like $< x >$, $< x^2 >$, etc.

theory context. We illustrate two of those here.

Losson and Mackey (1995) numerically studied the ensemble evolution of $N = 22,500$ differential delay equations of the form

$$\frac{dx}{dt} = -\alpha x + \begin{cases} ax_\tau & \text{if } x_\tau < 1/2 \\ a(1-x_\tau) & \text{if } x_\tau \geq 1/2 \end{cases} \qquad \frac{a}{\alpha} \in (1,2], \tag{4.35}$$

formed by considering the tent map (2.29) of Example 3 as the singular perturbation limit of the differential delay equation (1.3) with $\epsilon = 10$, and $a = 1.3$ in Equation 2.29. Some of their results are shown in Figure 4.2, clearly illustrating the existence of presumptive asymptotic periodicity in a continuous time setting that depends on the choice of the ensemble of initial functions.

Losson and Mackey (1995) have also numerically examined noise induced asymptotic periodicity in a differential delay equation formed by considering the noisy Keener-map (2.34) as the singular perturbation limit of Equation 1.3 where the noise source ξ is distributed with density g:

$$\frac{dx}{dt} = -\alpha x + [(ax_\tau + b + \xi) \mod 1] \; 0 < a, b < 1 \tag{4.36}$$

The results of their simulations, shown in Figure 4.3, give circumstantial evidence for the existence of noise induced asymptotic periodicity (Figure 4.3c,d)) as well as asymptotic stability (Figure 4.3b).

4.3 Writing the Frobenius-Perron operator highlights all of the problems

I think that Equation 1.1 can be thought of as inducing a flow $\mathcal{T}_t$ on a phase space of continuous functions $C = C([-\tau, 0], \mathbb{R})$, which I guess would be written as $x_t = \mathcal{T}\varphi$. According to Hale and Verduyn Lunel (1993, page 68), $\{\mathcal{T}_t : t \geq 0\} : C \to C$ is a strongly continuous semigroup. In one sense, it would seem that the evolution of a density under the action of this semi-group would be given by an extension of Equation 2.5

$$\int_A P^t f(x)\, \mu(dx) = \int_{\mathcal{T}_t^{-1}(A)} f(x)\, \mu(dx) \quad \text{for all measurable } A \subset C. \tag{4.37}$$

This writing of the evolution of the density f under the action of the semi-group of Frobenius-Perron operators $P^t : L^1(C) \to L^1(C)$ is, however, merely formal and serves to highlight the major problems that we face. Namely the problem surfaces of what the measure μ on the space C is, what is a density f on C, what does it mean to do integration over subsets of C, and how would you actually figure out what $\mathcal{T}_t^{-1}$ is?

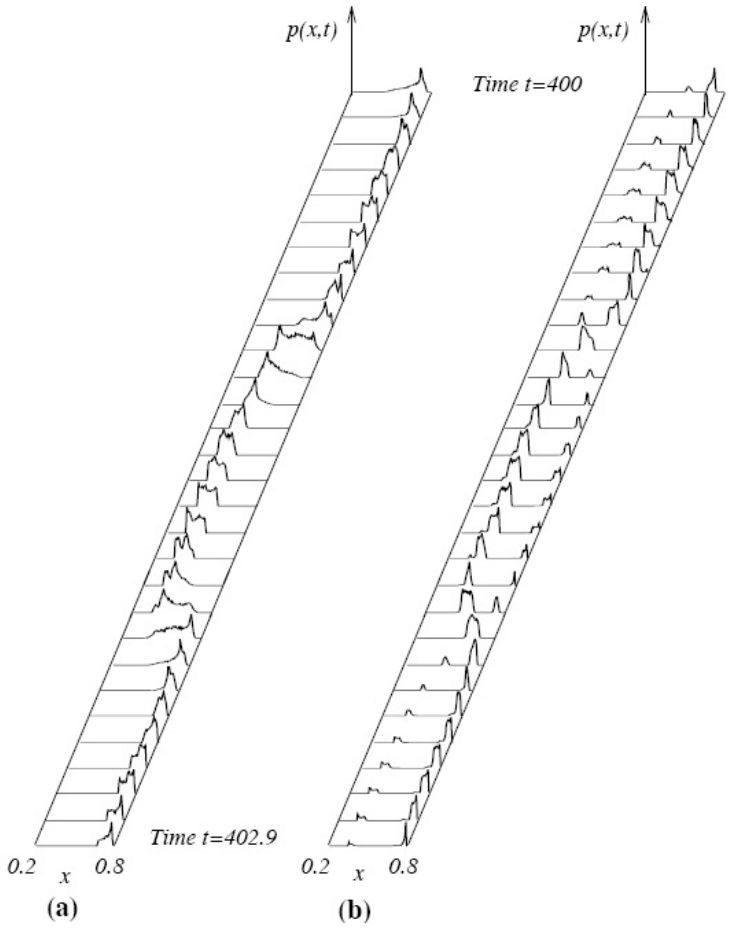

Figure 4.2: Apparent asymptotic periodicity in the differential delay equation (4.35). The parameters are $a = 13,\ \alpha = 10$. Both (a) and (b) were produced with $22,500$ initial functions. **(a)** Each of the initial functions was a random process supported uniformly on $[0.65, 0.75]$. **(b)** The initial functions were random processes supported either on $[0.65, 0.75]$ (for 17000 cases) or on $[0.35, 0.45]$ (for the remaining 5500 initial functions). The cycling is not transient, and is observed for all times. The dependence of the density cycle on the initial density reflects the dependence of the the eventual form of the statistically periodic density sequence on the initial functions. Taken from Losson and Mackey (1995) with permission.

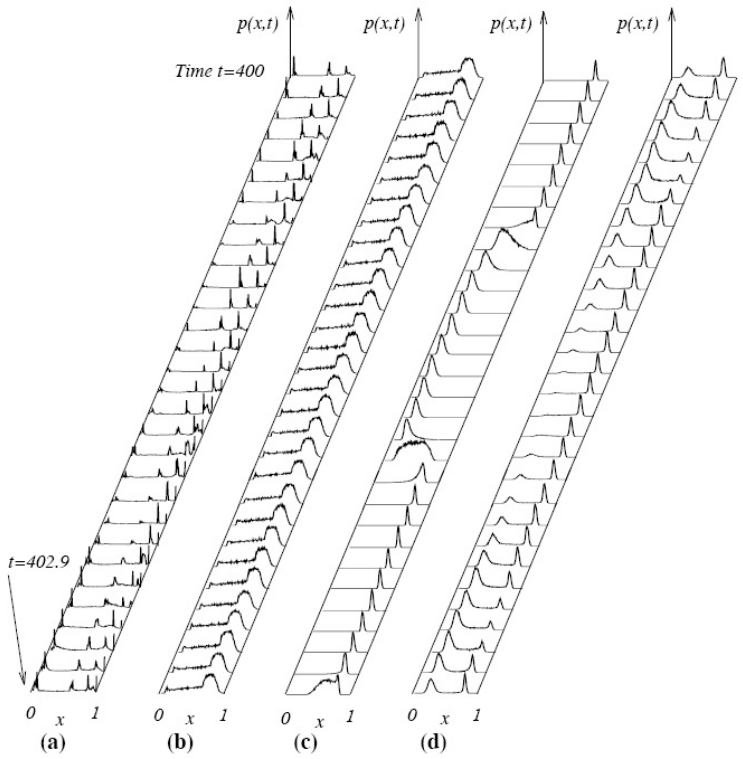

Figure 4.3: Noise induced apparent asymptotic periodicity in the stochastic differential delay equation (4.36). As in Figure 4.2, each simulation was performed with $22,500$ random initial functions. In all four panels, the parameters of the equation were $a = 0.5,\ b = 0.567,\ \alpha = 10$. For panels (a)-(c) the initial density was as in Figure 4.2(a). **(a)** No noise in the system: $p(x,t)$ is not a density, but a generalized function. **(b)** Noise supported uniformly on $[0, 0.1]$. The system is asymptotically stable, and $r = 1$ in (2.28). **(c)** Noise uniformly supported on $[0, 0.2]$, and $r = 2$ in (2.28). **(d)** Same noise as in (c), with an initial density as in Figure 4.2(b). From Losson and Mackey (1995) with permission.

5 The Hopf functional approach

5.1 Introduction to Hopf characteristic functionals and Hopf functional differential equations

This section briefly introduces a set of techniques to enable one to have a differential calculus for *functionals*, rather than just for the ordinary functions we usually consider, and how these may be used to study the behaviour of infinite dimensional systems like partial differential equations and differential delay equations.

5.1.1 Functionals and functional derivatives

A **functional** may be considered a function of infinitely many variables. Thus, one could consider a function $h : \Delta \to R$ and then a **space of functions** $C(\Delta)$, so a functional Ψ is $\Psi : C(\Delta) \to R$. The functional $\Psi(h)$ should be thought of as a natural extension and generalization of a function $F(x)$ of many variables.

In the normal calculus we are accustomed to the definition of the partial derivative of a function of several variables, e.g. with $F : R^d \to R$ with $x = (x_1, \cdots, x_i, \cdots, x_d)$ and $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_i, \cdots, \bar{x}_d)$ we have

$$\frac{\partial F(x)}{\partial x_i} = \lim_{\bar{x}_i \to x} \frac{F(\bar{x}) - F(x)}{\bar{x}_i - x_i}.$$

We want to develop a natural extension of this concept to functionals $\Psi(h)$. As in the more usual case we expect that this extension will:

1. Depend on Ψ; and
2. Depend on the choice of the "point" h.

How are we to proceed? Consider two functions h and a second one $\bar{h}$ that is just a bit different from h in some neighborhood of x_0. Then the "increase" of the functional $\Psi(h)$ in going from h to $\bar{h}$ is just

$$\Psi(\bar{h}) - \Psi(h).$$

Further, the area between h and $\bar{h}$, if $\bar{h} \geq h$, is just

$$\int_\Delta (\bar{h} - h)\, dx.$$

Thus we take the expression

$$\lim_{\bar{h}\to h,\,\mathrm{supp}(\bar{h}-h)\to x_0} \frac{\Psi(\bar{h})-\Psi(h)}{\int_\Delta(\bar{h}-h)\,dx}$$

to be the definition of the **functional** or **Volterra derivative** of $\Psi(h)$ and we denote it by

$$\frac{\delta\Psi(h)}{\delta x_0}.$$

Hence

$$\boxed{\frac{\delta\Psi(h)}{\delta x_0} = \lim_{\bar{h}\to h\ \mathrm{supp}(\bar{h}-h)\to x_0} \frac{\Psi(\bar{h})-\Psi(h)}{\int_\Delta(\bar{h}-h)\,dx}.} \tag{5.38}$$

Instead of the above we might simply write the increase of Ψ between h and $\bar{h}$ as

$$\Psi(\bar{h})-\Psi(h) = A\int_\Delta(\bar{h}-h)\,dx + O\left(\int_\Delta|\bar{h}-h|\,dx\right) \tag{5.39}$$

where

$$\lim_{\bar{h}\to h,\ \mathrm{supp}(\bar{h}-h)\to x_0} O\left(\int_\Delta|\bar{h}-h|\,dx\right) = 0,$$

and the convergence to zero is faster than that of

$$\int_\Delta(\bar{h}-h)\,dx.$$

In this case, the constant A is just taken to be the functional derivative, i.e.

$$A = \frac{\delta\Psi(h)}{\delta x_0}. \tag{5.40}$$

In the ordinary calculus the quantity

$$\frac{\partial F(x)}{\partial x_i},$$

for a fixed i, may be considered as a new function and

$$\frac{\partial^2 F(x)}{\partial x_j \partial x_i}$$

calculated. Analogously, we can fix x_0 (analogous to fixing i) in the expression

$$\frac{\delta\Psi(h)}{\delta x_0},$$

and then calculate

$$\frac{\delta^2 \Psi(h)}{\delta x_i \delta x_0}.$$

Example 5. Functional derivative of a linear functional

Having defined a functional derivative, lets try to calculate one. Specifically, lets try to calculate the functional derivative of the **linear functional**

$$\Psi(h) = \int_\Delta h(x) f(x)\, dx \tag{5.41}$$

where $f(x)$ is taken to be continuous.

Now the increase of Ψ can be simply written as

$$\begin{aligned}
\Psi(\bar{h}) - \Psi(h) &= \int_\Delta f(x)[\bar{h}(x) - h(x)]\, dx \\
&= \int_\Delta f(x_0)[\bar{h}(x) - h(x)]\, dx + \int_\Delta [f(x) - f(x_0)][\bar{h}(x) - h(x)]\, dx \\
&= f(x_0) \int_\Delta [\bar{h}(x) - h(x)]\, dx + \int_\Delta \epsilon |x - x_0| [\bar{h}(x) - h(x)]\, dx \\
&= f(x_0) \int_\Delta [\bar{h}(x) - h(x)]\, dx + O\left(\int_\Delta |\bar{h}(x) - h(x)|\, dx \right). \quad (5.42)
\end{aligned}$$

However, this is just precisely the form we had in our alternative formulation of development of the functional derivative, and we thus conclude that for Equation 5.41 the functional derivative is

$$\frac{\delta \Psi(h)}{\delta x_0} = f(x_0), \tag{5.43}$$

i.e. it is just the kernel of (5.41) evaluated at x_0, and is clearly independent of h.

Remark 5. *It is interesting to compare this result with the situation in the ordinary calculus where we differential a linear function*

$$F(x) = c_1 x_1 + \cdots + c_n x_n,$$

so

$$\frac{\partial F(x)}{\partial x_i} = c_i,$$

and again we see the independence of the result on x_i.

5.1.2 Hopf Characteristic Functionals

To introduce these we first require some notation. By the symbol $< h, f >$ we mean, as usual,

$$< h, f >= \int_{\Delta} h(x) f(x)\, dx. \tag{5.44}$$

[Note that $< h, f >\equiv \Psi(h)$.] Further, we let μ^f be a probabilistic measure on a space $C(\Delta)$ of functions f. By considering such a situation we are concentrating on the one in which a first choice gives a function, say f_1, while a second choice gives f_2. We say that the functions f form a **random field** or that $C(\Delta)$ with μ^f forms a random field.

With these notions, we define the **Hopf characteristic functional** to be

$$\boxed{\Phi(h) = \int_{C(\Delta)} e^{i<h,f>}\, d\mu^f.} \tag{5.45}$$

Note that the integration is over the space $C(\Delta)$. Furthermore, (5.45) can also be recognized as

$$\Phi(h) = \int_{C(\Delta)} e^{i\Psi(h)}\, d\mu^f.$$

Remark 6. *In normal probability theory, if we have a random vector* $(\xi_1, \cdots, \xi_n)$ *and a vector* $(\lambda_1, \cdots, \lambda_n)$*, then the characteristic function is just*

$$\begin{aligned} E\left(e^{i(\lambda_1\xi_1+\cdots+\lambda_n\xi_n)}\right) &= \int_{\Omega} e^{i(\lambda_1\xi_1+\cdots+\lambda_n\xi_n)}\, P_\xi(d\omega) \\ &= \int_{\Omega} e^{i<\lambda,\xi>}\, dP_\xi \\ &= \phi(\lambda). \end{aligned} \tag{5.46}$$

Thus replacing the discrete ξ_i *by continuous functions makes the connection with the definition of the Hopf characteristic functional more evident:*

$$E\left(e^{i<h,f>}\right) = \int_{C(\Delta)} e^{i<h,f>}\, d\mu^f \equiv \Phi(h).$$

Remark 7. *The factor* $i = \sqrt{-1}$ *in the exponential isn't too mysterious since it just serves to ensure that*

$$e^{i<h,f>}$$

is no greater than 1 since $|e^i| \leq 1$.

To calculate the functional derivative of the Hopf characteristic functional, we must first calculate the increase of Φ between h and $\bar{h}$. In carrying out this computation we will need the fact that

$$e^{ix} - e^{iy} \simeq i(x-y)e^{i\tilde{x}}, \qquad \tilde{x} \underset{x\to y}{\longrightarrow} y.$$

Thus we may write

$$\begin{aligned} \Phi(\bar{h}) - \Phi(h) &= \int \left[e^{i<\bar{h},f>} - e^{i<h,f>} \right] d\mu^f \\ &\simeq \int i < \bar{h} - h, f > e^{\tilde{z}}\, d\mu^f. \end{aligned} \tag{5.47}$$

Now the area between $\bar{h}$ and h is just

$$\Delta h = \int [\bar{h}(x) - h(x)]\, dx \tag{5.48}$$

so

$$\frac{\Delta\Phi}{\Delta h} = \int i \left[\frac{< \bar{h} - h, f >}{\Delta h} \right] e^{\tilde{z}}\, d\mu^f. \tag{5.49}$$

[N.B. Don't confuse the Δ here with the intervals introduced earlier.] Note that:

1. $\tilde{z} \to i < h, f >$ as $\Delta h \to 0$;

2. By our considerations of the linear functional (5.41),

$$\frac{< \bar{h} - h, f >}{\Delta h} \underset{\Delta h\to 0}{\longrightarrow} f(x_0).$$

Therefore, taking $\Delta h \to 0$ in (5.49) we have finally that the functional derivative of the Hopf characteristic functional (5.45) is given by

$$\boxed{\frac{\delta\Phi(h)}{\delta(x_0)} = i \int_{C(\Delta)} f(x_0) e^{i<h,f>}\, d\mu^f.} \tag{5.50}$$

Note from (5.50) that functional differential of an integral behaves just like

in the normal calculus since

$$\begin{aligned}\frac{\delta\Phi(h)}{\delta(x_0)} &= \frac{\delta}{\delta(x_0)}\int_{C(\Delta)} e^{i\Psi(h)}\,d\mu^f \\ &= \int_{C(\Delta)} \frac{\delta}{\delta(x_0)} e^{i\Psi(h)}\,d\mu^f \\ &= i\int_{C(\Delta)} \frac{\delta\Psi(h)}{\delta(x_0)} e^{i\Psi(h)}\,d\mu^f \\ &= i\int_{C(\Delta)} f(x_0) e^{i\Psi(h)}\,d\mu^f \end{aligned} \tag{5.51}$$

where we have used (5.43).

This last result, Equation 5.50, is quite interesting in the following sense. To see why, assume that $h = 0$. Then we have

$$\frac{\delta\Phi(0)}{\delta(x_0)} = i\int f(x_0)\,d\mu^f.$$

However,

$$E(f(x_0)) = \int f(x_0)\,d\mu^f,$$

and, thus, for the Hopf characteristic functional we have proved that

$$\boxed{E(f(x_0)) = \frac{1}{i}\frac{\delta\Phi(0)}{\delta(x_0)}} \tag{5.52}$$

It is interesting to pursue this procedure, for example by calculating

$$\frac{\delta^2\Phi(h)}{\delta(x_1)\delta(x_0)}.$$

By the same arguments as before, we have

$$\begin{aligned}\Delta\left(\frac{\delta\phi(h)}{\delta(x_0)}\right) &= \frac{\delta\Phi(\bar{h})}{\delta(x_0)} - \frac{\delta\Phi(h)}{\delta(x_0)} \\ &= i\int f(x_0)\left[e^{i,\bar{h},f>} - e^{i<h,f>}\right]\,d\mu^f \\ &\simeq i^2\int f(x_0) < \bar{h} - h, f > e^{\tilde{z}}\,d\mu^f. \end{aligned} \tag{5.53}$$

From this we have

$$\frac{\Delta\left(\frac{\delta\phi(h)}{\delta(x_0)}\right)}{\delta h} \simeq i^2\int f(x_0)\left[\frac{< \bar{h} - h, f >}{\delta h}\right] e^{\tilde{z}}\,d\mu^f.$$

Taking the limit of this final expression as $\delta h \to 0$ we obtain the second functional derivative of the Hopf functional which is given by

$$\frac{\delta^2 \Phi(h)}{\delta(x_1)\delta(x_0)} = i^2 \int f(x_0) f(x_1) e^{i<h,f>} \, d\mu^f.$$

Once again taking $h = 0$, we find

$$\boxed{E(f(x_0)f(x_1)) = \frac{1}{i^2} \frac{\delta^2 \Phi(0)}{\delta(x_1)\delta(x_0)}} \tag{5.54}$$

so as a special case

$$\boxed{E(f^2(x_0)) = \frac{1}{i^2} \frac{\delta^2 \Phi(0)}{\delta(x_0)^2}} \tag{5.55}$$

The consequences of equations (5.52), (5.54), and all similar versions that can be easily derived using these elementary techniques is quite surprising. Namely, given the Hopf characteristic functional (5.45):

$$\boxed{\Phi(h) = \int_{C(\Delta)} e^{i<h,f>} \, d\mu^f.}$$

with respect to the probabilistic measure μ^f of a random field of functions f, we can calculate all of the probabilistic properties (moments) of the random field by simply evaluating various functional derivatives of Φ at $h = 0$.

Remark 8. *As a simple example of these ideas, consider all of the trajectories of British Airways flight 094 from Warsaw to London over a 10 year period. Each trajectory $f(x)$ gives the altitude f at location x, and $f(x)$ can be considered as a member of a random field. We can thus make some statements about this ensemble of trajectories. For example, if x_0 is a particular location between Warsaw and London, then the expected value of the altitude at x_0 is given by*

$$E(f(x_0)) = \frac{1}{i} \frac{\delta \Phi(0)}{\delta(x_0)},$$

the variance in the altitude at x_0 is given by

$$E(f^2(x_0)) - [E(f(x_0))]^2 = \frac{1}{i^2} \frac{\delta^2 \Phi(0)}{\delta(x_0)^2} - \left[\frac{1}{i} \frac{\delta \Phi(0)}{\delta(x_0)} \right]^2,$$

and the correlation between the altitudes at points x_0 and x_1 is given by

$$E(f(x_0)f(x_1)) = \frac{1}{i^2} \frac{\delta^2 \Phi(0)}{\delta(x_1)\delta(x_0)}.$$

Remark 9. *Suppose we have a partial differential equation and an initial condition is chosen from a random field with measure μ^f. Then as time increases from zero, the measure evolves and we can consider the flow of measures. This was exactly the point of view adopted by Prodi and Foias in their study of the Navier-Stokes hydrodynamic equations. However, Hopf realized the difficulties in carrying out this approach and proposed, instead, that one one might consider the evolution of the characteristic functionals. When the Hopf point of view is adopted, we have to solve a functional differential equation involving $\Phi(h)$.*

Remark 10. *If we have a system of ordinary differential equations, then the Liouville equation describes the evolution of a density under the action of a flow generated by the system of ordinary differential equations. Alternately, we could form the Hopf functional differential equation for the evolution of $\Phi(h)$.*

5.1.3 Hopf characteristic functionals and partial differential equations

In this section we turn to a discussion of partial differential equations of the form

$$\frac{\partial u}{\partial t} = Lu \tag{5.56}$$

from the perspective of Hopf characteristic functionals. In (5.56) with initial functions $u(0,x) = f(x)$, the operator L is a linear combination of terms of the form

$$u, \quad \frac{\partial u}{\partial x_i}, \quad \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \cdots, \tag{5.57}$$

and products of these terms.

For systems whose dynamics are governed by equations like (5.56), we can always derive a functional differential equation for $\Phi(f)$. Boundary value problems are rather delicate and difficult to treat using the Hopf method, but it is rather easy to treat initial value problems

$$\begin{aligned} \frac{\partial u}{\partial t} &= Lu \\ u(0,x) &= f(x). \end{aligned} \tag{5.58}$$

Here we assume that W is a space, $f \in W$, and that we have a probabilistic measure μ^f on W. How are we to view equations (5.56) within this context? Assume that we have a set of initial functions. Then equation

(5.58) describes the evolution of functions such that a set or ensemble of initial functions evolves in time to a new ensemble. We assume that the evolution of these initial functions is such that the measure is preserved, i.e.

$$\mu^f(W_t) = \mu^f(W_0). \tag{5.59}$$

Having this probabilistic measure μ^f we couple (adjoin) it to the Hopf characteristic functional

$$\Phi_0(h) = \int e^{i<h,f>}\, d\mu^f, \tag{5.60}$$

where the integration is over the space W but the notation has been suppressed. Likewise, to μ_t^f we adjoin

$$\Phi_t(h) = \int e^{i<h,f>}\, d\mu_t^f. \tag{5.61}$$

How does this probabilistic measure evolve? As we have seen, knowing the Hopf characteristic functional gives a great deal of information concerning the random field and we will thus consider the evolution of measures *via* the evolution of $\Phi_t(h)$. Then, given $\Phi_t(h)$ we may return to a consideration of $\mu_t^f \equiv \mu^f(W_t)$.

Once again return to our basic system Equation 5.58, so it is clear that the solution $u(t,x)$ depends on f which can be indicated by writing $u(t,\cdot)$, with the "$\cdot$" indicating dependence on a whole function. The system (5.58) is equivalent to the operation of a transformation S_t operating on W (the space of initial functions f), $u(t,\cdot) = S_t(\cdot)$, and from equation (5.59) we may write

$$\mu^f(W_t) = \mu^f(W_0) = \mu^f(S_t^{-1}(W_t)). \tag{5.62}$$

Using this we have

$$\int e^{i<h,u(t,\cdot)>}\, d\mu^f = \int e^{i<h,u(t,\cdot)>}\, d\mu^f(S_t^{-1}(W_t)).$$

Changing the variables on the right hand side gives

$$\int e^{i<h,u(t,\cdot)>}\, d\mu^f = \int e^{i<h,S_t^{-1}(u(t,\cdot))>}\, d\mu^f(W_t).$$

If we fix the time and set $u(t,\cdot) = S_t(f)$, this then becomes

$$\boxed{\int e^{i<h,u(t,\cdot)>}\, d\mu^f = \int e^{i<h,f>}\, d\mu_t^f,} \tag{5.63}$$

which is a very important and fundamental relationship. Equation (5.63) is entirely analogous to the "change of variables" formula (Lasota and Mackey, 1985, page 41). The only difference is that f now plays the role of x since we are integrating over functions.

Let us now fix the following notation. Write $\Phi_t(h) \equiv \Phi(t,h)$ so

$$\Phi(0,h) = \int e^{i<h,f>}\, d\mu^f \tag{5.64}$$

and

$$\begin{aligned} \Phi(t,h) &= \int e^{i<h,f>}\, d\mu_t^f \\ &= \int e^{i<h,u(t,\cdot)>}\, d\mu^f. \end{aligned} \tag{5.65}$$

To study the evolution of the Hopf functional with respect to the system (5.58) we now differentiate $\Phi(t,h)$ with respect to t to give

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= \frac{\partial}{\partial t}\int e^{i<h,u>}\, d\mu^f \\ &= \int \frac{\partial}{\partial t} e^{i<h,u>}\, d\mu^f \\ &= i\int \frac{\partial < h,u >}{\partial t} e^{i<h,u>}\, d\mu^f \\ &= i\int \left\langle h, \frac{\partial u}{\partial t} \right\rangle e^{i<h,u>}\, d\mu^f \\ &= i\int < h, Lu > e^{i<h,u>}\, d\mu^f \\ &= i\int \left\{ \int h(x) Lu(t,x)\, dx \right\} e^{i<h,u>}\, d\mu^f \\ &= i\int h(x) \left[\int Lu e^{i<h,u>}\, d\mu^f \right] dx. \end{aligned} \tag{5.66}$$

Therefore our final formula becomes

$$\boxed{\frac{\partial \Phi}{\partial t} = i\int h(x) \left[\int Lu e^{i<h,u>}\, d\mu^f \right] dx}, \tag{5.67}$$

which we will illustrate through a series of examples.

Example 6. *Let us consider the partial differential equation*

$$\frac{\partial u}{\partial t} = u. \tag{5.68}$$

From equation (5.67), since $Lu = u$, we have

$$\frac{\partial \Phi}{\partial t} = i \int h(x) \left[\int u e^{i<h,u>} \, d\mu^f \right] dx.$$

However,

$$\frac{\delta \Phi(t,h)}{\delta(x)} = \int u e^{i<h,u>} \, d\mu^f,$$

so the functional differential equation for Φ corresponding to (5.68) is simply

$$\boxed{\frac{\partial \Phi}{\partial t} = \int h(x) \frac{\delta \Phi}{\delta(x)} \, dx.} \tag{5.69}$$

The solution of equation (5.69) is

$$\Phi(t,h) = \Phi_0(e^t h) \tag{5.70}$$

where $\Phi(0,h) = \Phi_0(h)$ is the characteristic functional of the initial measures.

Before showing you how we arrived at the solution, lets just go through the exercise of verifying that it is indeed the solution. First, recall that

$$\Phi(\bar{g}) - \Phi(g) = \int \frac{\delta \Phi(g)}{\delta(x)} [\bar{g}(x) - g(x)] \, dx + O\left(\int |\bar{g}(x) - g(x)| \, dx \right).$$

Now we have, setting $g = e^t h$ and using (25),

$$\begin{aligned}
\frac{\partial \Phi(t,h)}{\partial t} &= \lim_{\bar{t} \to t} \frac{\Phi_0(e^{\bar{t}} h) - \Phi_0(e^t h)}{\bar{t} - t} \\
&= \lim_{\bar{t} \to t} \left\{ \int [e^{\bar{t}} - e^t] h(x) \frac{\delta \Phi_0(f)}{\delta(x)} \Big|_{f = e^t h} \, dx + O\left(\int |e^{\bar{t}} h - e^t h| \, dx \right) \right\} \\
&\simeq \lim_{\bar{t} \to t} \left\{ \int [e^{\bar{t}} - e^t] h(x) \frac{\delta \Phi_0(f)}{\delta(x)} \Big|_{f = e^t h} \, dx + O\left([e^{\bar{t}} - e^t] \int h(x) \, dx \right) \right\} \\
&= e^t \int h(x) \frac{\delta \Phi_0(f)}{\delta(x)} \Big|_{f = e^t h} \, dx.
\end{aligned} \tag{5.71}$$

As an aside, note that if we have a general functional $\Phi(\lambda h)$ *then*

$$\begin{aligned}\frac{\delta[\Phi(\lambda h)]}{\delta(x)} &= \lim_{\bar{h}\to h}\frac{\Phi(\lambda\bar{h})-\Phi(\lambda h)}{\bar{h}-h}\\ &= \lambda\lim_{\bar{h}\to h}\frac{\Phi(\lambda\bar{h})-\Phi(\lambda h)}{\lambda\bar{h}-\lambda h}\\ &= \lambda\frac{\delta\Phi}{\delta(x)}(\lambda h)=\lambda\frac{\delta\Phi(f)}{\delta(x)}\Big|_{f=\lambda h}. \end{aligned} \tag{5.72}$$

Thus, taking $\lambda = e^t$ *we may write*

$$\frac{\delta[\Phi_0(e^t h)]}{\delta(x)} = e^t\frac{\delta\Phi_0(f)}{\delta(x)}\Big|_{f=e^t h},$$

and, as a consequence, equation (27) becomes

$$\begin{aligned}\frac{\partial\Phi(t,h)}{\partial t} &= \int h(x)\frac{\delta[\Phi_0(e^t h)]}{\delta(x)}\,dx\\ &= \int h(x)\frac{\delta\Phi(t,h)}{\delta(x)}\,dx,\end{aligned} \tag{5.73}$$

thereby demonstrating that (5.70) is indeed the solution of the functional differential equation (5.69).

Now that we have verified the solution (5.69), lets consider the problem of how one could have have obtained (5.69) without making a series of random guesses. This turns out to be quite straightforward since there is a general technique for solving Hopf functional differential equations corresponding to linear evolution equations.

Specifically, again consider the evolution equation

$$\frac{\partial u}{\partial t} = Lu, \tag{5.74}$$

with the initial condition $u(0,\cdot) = f$, where Lu is linear, and the corresponding Hopf functional differential equation is

$$\frac{\partial\Phi}{\partial t} = i\int h(x)\left[\int Lue^{i<h,u>}\,d\mu^f\right]dx \tag{5.75}$$

with a characteristic functional $\Phi(0,h) = \Phi_0(h)$ of the initial measure. If the solution of Equation 5.74 is written

$$u(t,\cdot) = \Gamma_t f \tag{5.76}$$

(Γ_t is now a linear operator), then the solution of the corresponding Hopf equation is simply

$$\Phi(t,h) = \Phi_0(\Gamma_t^* h), \tag{5.77}$$

where Γ_t^* is the operator adjoint to Γ_t, i.e., the operator satisfying

$$< \Gamma h, f > \;\; = \;\; < \;\; h, \; \Gamma^* f \;\; > .$$

To show that (5.77) is indeed the solution to the Hopf equation (5.75) is quite straightforward. Thus, if we start from

$$\Phi_0(h) = \int e^{i<h,f>} \, d\mu^f,$$

then

$$\Phi(t,h) = \int e^{i<h,u(t,\cdot)>} \, d\mu^f. \tag{5.78}$$

From equation (5.76), $u(t,\cdot) = \Gamma_t f$ so equation (5.78) becomes

$$\begin{aligned} \Phi(t,h) &= \int e^{i<h,\Gamma_t f>} \, d\mu^f \\ &= \int e^{i<\Gamma_t^* h, f>} \, d\mu^f \\ &= \Phi_0(\Gamma_t^* h), \end{aligned} \tag{5.79}$$

thus demonstrating equation (5.77).

With this discussion under our belt, we can now consider a second example.

Example 7. *Consider the initial value problem*

$$\frac{\partial u}{\partial t} = a(t,x) \frac{\partial u}{\partial x} \tag{5.80}$$

with $u(0,\cdot) = f(\cdot)$. From the general equation (5.67), we have directly that

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= i \int h(x) \left[\int a(t,x) \frac{\partial u}{\partial x} e^{i<h,u>} \, d\mu^f \right] dx \\ &= \int h(x) a(t,x) \frac{\partial}{\partial x} \left[i \int u e^{i<h,u>} \, d\mu^f \right] dx \end{aligned} \tag{5.81}$$

so, using equation (5.50),

$$\boxed{\frac{\partial \Phi}{\partial t} = \int h(x) a(t,x) \frac{\partial}{\partial x} \frac{\delta \Phi}{\delta(x)} dx} \tag{5.82}$$

is the functional differential equation for Φ corresponding to (5.80).

Actually, it is equally easy to show that the slightly more general initial value problem

$$\frac{\partial u}{\partial t} = a(t,x)\frac{\partial u}{\partial x} + b(t,x)u \tag{5.83}$$

has a corresponding Hopf equation

$$\boxed{\frac{\partial \Phi}{\partial t} = \int h(x)\left[a(t,x)\frac{\partial}{\partial x}\frac{\delta \Phi}{\delta(x)} + b(t,x)\frac{\delta \Phi}{\delta(x)}\right]dx.} \tag{5.84}$$

Example 8. *If we take the particular case of $a(t,x) = -x$ and $b(t,x) = \lambda$, then (5.83) becomes*

$$\frac{\partial u}{\partial t} = -x\frac{\partial u}{\partial x} + \lambda u, \tag{5.85}$$

which is a linearized version of an equation that has been applied to the problem of describing the simultaneous proliferation and maturation of a population of cells [see Lasota and Mackey (1985, Example 11.1.1, pp. 297-302), where it is shown that the solutions of (5.85) are exact with respect to the Wiener measure].

The solution of (5.85), given an initial condition $u(0,\cdot) = f(\cdot)$ is simply

$$u(t,x) = e^{\lambda t} f(e^{-t}x), \tag{5.86}$$

which may be obtained using the method of characteristics. Thus, writing (38) as

$$u(t,\cdot) = \Gamma_t f(\cdot),$$

where

$$(\Gamma_t f)(x) = e^{\lambda t} f(e^{-t}x),$$

we also have

$$< h, \Gamma_t f >= \int h(x) e^{\lambda t} f(e^{-t}x)\, dx.$$

Make the change of variables $y = e^{-t}x$ so $dx = e^t dy$ and thus

$$< h, \Gamma_t f >= \int e^{(\lambda+1)t} h(e^t y) f(y)\, dy.$$

Thus it is straightforward to see that

$$(\Gamma_t^* h)(x) = e^{(\lambda+1)t} h(e^t x)$$

and, as a consequence, the Hopf equation

$$\frac{\partial \Phi}{\partial t} = \int h(x)\left[-x\frac{\partial}{\partial x}\frac{\delta \Phi}{\delta(x)} + \lambda\frac{\delta \Phi}{\delta(x)}\right]dx \tag{5.87}$$

corresponding to (5.85) has the solution

$$\Phi(t,h) = \Phi_0(e^{(\lambda+1)t}h(e^t x)) \tag{5.88}$$

where Φ_0 is the characteristic functional of the initial measure.

Example 9. *Consider the system of ordinary differential equations*

$$\frac{dx_k}{dt} = F_k(x), \qquad k = 1, \cdots, d \tag{5.89}$$

and the corresponding Liouville equation for the evolution of the density $u(t,x)$ under the action of the flow generated by (5.89):

$$\frac{\partial u}{\partial t} = -\sum_{k=1}^{d} \frac{\partial [F_k u]}{\partial x_k}. \tag{5.90}$$

Rewriting (5.90) as

$$\frac{\partial u}{\partial t} = -u\sum_{k=1}^{d} \frac{\partial F_k}{\partial x_k} - \sum_{k=1}^{d} F_k \frac{\partial u}{\partial x_k}, \tag{5.91}$$

and identifying L in an obvious manner, we have from (5.67) that

$$\frac{\partial \Phi}{\partial t} = -i\int h(x)\left[\int \sum_{k=1}^{d} \frac{\partial F_k}{\partial x_k} u e^{i<h,u>} d\mu^f - \sum_{k=1}^{d} F_k \frac{\partial}{\partial x_k}\int u e^{i<h,u>} d\mu^f\right] dx. \tag{5.92}$$

Then, using (5.50) and some simple manipulations it is almost immediate that the differential equation for the Hopf functional Φ is given by

$$\frac{\partial \Phi}{\partial t} = -i\int h(x)\left[\sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(F_k \frac{\delta \Phi}{\delta x}\right)\right] dx. \tag{5.93}$$

5.2 Characteristic functionals for delay equations

We consider differential delay equations of the form

$$\frac{du}{ds} = -\alpha u(s) + F(u(s-1)) \text{ for } 1 < s, \tag{5.94}$$

in which the delay τ is taken to be 1 without loss of generality, with the initial function

$$u(s) = v(s) \text{ if } s \in [0,1]. \tag{5.95}$$

From now on we write Equations 5.94-5.95 as the combined system

$$\begin{aligned} u(s) &= v(s) \text{for } s \in [0,1], \\ \frac{du(s)}{ds} &= -\alpha u(s) + F(v(s-1)) \text{for } 1 < s < 2 \end{aligned} \tag{5.96}$$

and denote by $\mathcal{S}_t$ the corresponding semidynamical system $\mathcal{S}_t : \mathcal{C}([0,1]) \longmapsto \mathcal{C}([0,1])$ given by

$$\mathcal{S}_t v(x) = u_v(x+t), \tag{5.97}$$

where u_v denotes the solution of Equation 5.96 corresponding to the initial function v. Equation 5.97 defines a *semidynamical system* because a differential delay equation is **noninvertible**, *i.e.* it cannot be run unambiguously forward and backwards in time.

From Equation 5.97, the system (5.96) is equivalent to considering

$$\frac{\partial}{\partial t}\mathcal{S}_t v(x) = \begin{cases} \dfrac{\partial}{\partial t} v(x+t) & \text{for } x \in [0, 1-t], \\ -\alpha u(x+t) + F(v(x+t-1)) & \text{for } x \in (1-t, 1]. \end{cases} \tag{5.98}$$

Thus, we consider a segment of a solution of (5.96) defined on an interval $I_t = [t, t+1]$, as t increases (continuously) [*i.e.* the differential delay equation (5.96) operates on a *buffer* of length 1, "sliding" it along the time axis]. Equation (5.98) states that the contents of this buffer are the initial condition v when the argument $(x+t)$ is less than 1, and the solution u of the equation otherwise.

We next introduce the characteristic functional $\mathcal{Z}_t$ of a family of probability measures evolving from an initial measure. We define the characteristic functional $\mathcal{Z}_t$ for (5.98) by

$$\mathcal{Z}_t[J_1, J_2] = \int_{\mathcal{C}} \exp\left[i \int_0^1 J_1(x) u_v(x+t)\, dx + i \int_0^1 J_2(x) v(x)\, dx \right] d\mu_0(\mathcal{T}_t^{-1}(v, u_v)). \tag{5.99}$$

The source functions J_1 and J_2 are elements of $\mathcal{C}([0,1])$ and the measure of integration is the initial measure μ_0 (describing the initial distribution of functions) composed with $\mathcal{T}_t^{-1}(v, u_v)$ where $\mathcal{T}_t(v) : \mathcal{C}([0,1]) \longmapsto \mathcal{C} \times \mathcal{C}$ is defined by

$$\mathcal{T}_t(v) = (v, u_v). \tag{5.100}$$

For simplicity, we will use the notation $\mu_0(\mathcal{T}_t^{-1}(v, u_v)) \equiv \mathcal{W}[v, \mathcal{S}_t(v)]$, so (5.99) becomes

$$\mathcal{Z}_t[J_1, J_2] = \int_{\mathcal{C}} \exp\left[i \int_0^1 J_1(x) u_v(x+t)\, dx + i \int_0^1 J_2(x) v(x)\, dx \right] d\mathcal{W}[v, \mathcal{S}_t(v)]. \tag{5.101}$$

When no confusion is possible, we write $\mathcal{W}_t$ for $\mathcal{W}[v, \mathcal{S}_t(v)]$.

If f and g are two functions defined on an interval I, we denote their scalar product by

$$< f, g > \equiv \int_I f(x)g(x)\, dx.$$

To simplify the notation we also write

$$\Upsilon[J_1, J_2; v] = \exp\left[i < J_1(x), u_v(x+t)\right\} + i\{J_2(x), v(x) >\right]. \tag{5.102}$$

Υ is used from now on to denote the function of J_1, J_2 and v defined in (5.102). We begin by noting the following relations

$$\frac{\delta^n \mathcal{Z}_t}{\delta J_1^n(\xi)} = i^n \left\langle \Upsilon u_v^n(\xi + t) \right\rangle, \tag{5.103}$$

$$\frac{\delta^n \mathcal{Z}_t}{\delta J_2^n(\xi)} = i^n \left\langle \Upsilon v^n(\xi) \right\rangle, \tag{5.104}$$

where it is understood that

$$\left\langle \begin{pmatrix} \vdots \end{pmatrix} \right\rangle = \int_{\mathcal{C}} \begin{pmatrix} \vdots \end{pmatrix}\, d\mathcal{W}[v, \mathcal{S}_t(v)].$$

Note that if μ_0 is the probability measure on the space of all initial functions v, and A is any subset of $\mathcal{C}([0,1])$, then

$$\mu_t(A) \equiv \mu_0(\mathcal{S}_t^{-1}(A)). \tag{5.105}$$

In other words, the probability that a randomly chosen function belongs to A at time t equals the probability that the counterimage of that function (under the action of $\mathcal{S}_t$) belonged to the counterimage of the set A. This defines the family of measures characterized by the solutions $\mathcal{Z}_t$ of a functional differential equation which is the Fourier transform of the infinite dimensional version of the Kramers-Moyal expansion Risken (1984). The derivation of such an equation for a differential delay equations was first considered by Capiński (1991). If the semiflow $\mathcal{S}_t$ is measure-preserving with respect to μ_0, then $\mu_0(A) \equiv \mu_0(\mathcal{S}_t^{-1}(A))$. In this case, we alternately say that the measure μ_0 is invariant with respect to $\mathcal{S}_t$.

We are now in a position to derive an evolution equation for the characteristic functional.

5.2.1 A functional differential equation for $\mathcal{Z}_t$.

Time differentiation of the characteristic functional $\mathcal{Z}_t$ defined in (5.101) yields, in conjunction with (5.98),

$$\begin{aligned}
\frac{\partial \mathcal{Z}_t}{\partial t} &= i\left\langle \Upsilon \int_0^1 J_1(x) \frac{\partial u_v(x+t)}{\partial t}\, dx \right\rangle \\
&= i\left\langle \Upsilon \int_0^1 J_1(x) \frac{\partial u_v(x+t)}{\partial x}\, dx \right\rangle \\
&= i\left\langle \Upsilon \int_0^{1-t} J_1(x) \frac{\partial v(x+t)}{\partial x}\, dx - \alpha \Upsilon \int_{1-t}^1 J_1(x) u(x+t)\, dx \right\rangle \\
&\quad + i\left\langle \Upsilon \int_{1-t}^1 J_1(x) F(v(x+t-1))\, dx \right\rangle . \qquad (5.106)
\end{aligned}$$

Therefore, from Equation 5.103 and the definition (5.98), we obtain

$$\begin{aligned}
\frac{\partial \mathcal{Z}_t}{\partial t} &= \int_0^{1-t} J_1(x) \frac{\partial}{\partial x}\left(\frac{\delta \mathcal{Z}_t}{\delta J_1(x)} \right) dx - \alpha \int_{1-t}^1 J_1(x) \frac{\delta \mathcal{Z}_t}{\delta J_1(x)}\, dx \\
&\quad + i\left\langle \Upsilon \int_{1-t}^1 J_1(x) F(v(x+t-1))\, dx \right\rangle . \qquad (5.107)
\end{aligned}$$

Equation 5.107 is related to the Hopf functional differential equation for the evolution of the characteristic functional $\mathcal{Z}_t$, and contains all the statistical information describing the evolution of a density of initial functions under the action of the differential delay system (5.94,5.95). An equation similar to (5.107) was first obtained by Capiński (1991) for a differential delay equation with a quadratic nonlinearity (see Example 10 below).

In order to derive the Hopf equation *per se*, we restrict our attention to situations where the feedback function F in the differential delay equation (5.94) is a polynomial

$$F(v) = \sum_{k=1}^{n} a_k v^k . \qquad (5.108)$$

With nonlinearity (5.108), Equation 5.107 becomes, with identity (5.104),

$$\begin{aligned}
\frac{\partial \mathcal{Z}_t}{\partial t} &= \int_0^{1-t} J_1(x) \frac{\partial}{\partial x}\left(\frac{\delta \mathcal{Z}_t}{\delta J_1(x)} \right) dx - \alpha \int_{1-t}^1 J_1(x) \frac{\delta \mathcal{Z}_t}{\delta J_1(x)}\, dx \\
&\quad + \sum_{k=1}^{n} i^{(1-k)}\, a_k \int_{1-t}^1 J_1(x) \frac{\delta^k \mathcal{Z}_t}{\delta J_2^k(x+t-1)}\, dx. \qquad (5.109)
\end{aligned}$$

Analytically solving the Hopf equation (5.109) is not possible at present, though a correct method of solution should make use of integration with respect to measures defined on the space $\mathcal{C}$. Presently, the theory of such integrals only allows their consistent utilization in solving functional differential equations when the measure of integration is the *Wiener measure* Sobczyk (1984).

Before proceeding to treat the Hopf equation in a perturbative manner, we illustrate its specific form for a simple nonlinear delay equations.

Example 10. *The differential delay equation*

$$\frac{du}{ds} = -\alpha u(s) + ru(s-1)[1-u(s-1)], \tag{5.110}$$

can be considered as a continuous analogue of the discrete time quadratic map

$$u_{n+1} = ru_n(1-u_n) \tag{5.111}$$

because Equation 5.110 is the singular perturbation of the quadratic map (5.111) as defined in Ivanov and Šarkovskiĭ (1991). The characteristic functional is defined by (5.101), and the functional differential equation corresponding to (5.109) was shown by Capiński (1991) to be

$$\begin{aligned}\frac{\partial \mathcal{Z}_t}{\partial t} &= \int_0^{1-t} J_1(x)\frac{\partial}{\partial x}\left(\frac{\delta \mathcal{Z}_t}{\delta J_1(x)}\right)\,dx - \alpha\int_{1-t}^{1} J_1(x)\frac{\delta \mathcal{Z}_t}{\delta J_1(x)}\,dx \\ &\quad + r\int_{1-t}^{1} J_1(x)\frac{\delta \mathcal{Z}_t}{\delta J_2(x+t-1)}\,dx \qquad (5.112)\\ &\quad - ri^{-1}\int_{1-t}^{1} J_1(x)\frac{\delta^2 \mathcal{Z}_t}{\delta J_2^2(x+t-1)}\,dx. \qquad (5.113)\end{aligned}$$

In spite of the fact that we cannot solve the Hopf equation analytically, relatively mild assumptions allow one to gain significant insight into the dynamics of $\mathcal{Z}_t$. More precisely, if $\mathcal{Z}_t$ is analytic we can expand it in a power series and treat the Hopf equation in a perturbative manner. We follow this approach in the next section.

5.2.2 The moments of the measure $\mathcal{W}_t$

The statistical properties of the random field of functions v and u are described by an infinite hierarchy of moments of the measure $\mathcal{W}_t$. For fixed t, the average value of the contents of the buffer defined on $I_t = [t, t+1]$ (*i.e.*

v on $[t,1]$ and u_v on $(1,1+t]$), which is just the first order moment of the measure $\mathcal{W}_t$, is

$$M_v^1(t,x) \equiv \int_{\mathcal{C}} v(x+t)\, d\mu_0(v) \qquad \text{for } x \in [0,1-t], \tag{5.114}$$

$$M_u^1(t,x) \equiv \int_{\mathcal{C}} u_v(x+t)\, d\mu_0(v) \qquad \text{for } x \in (1-t,1]. \tag{5.115}$$

These two equations can be written as one relation:

$$M^1(t,x) \equiv \int_{\mathcal{C}} u_v(x+t)\, d\mathcal{W}_t \qquad \text{for } x \in [0,1]. \tag{5.116}$$

The definition of the second order moment (or covariance function) $M^2(t,x,y)$ is, with the same notation,

$$\begin{aligned}
M^2(t,x,y) &= \int_{\mathcal{C}} v(x+t)v(y+t)\, d\mu_0(v) \equiv M_{vv}^2(t,x,y) \quad \text{for } x,y \in [0,1-t]\times[0,1-t],\\
M^2(t,x,y) &= \int_{\mathcal{C}} u_v(x+t)v(y+t)\, d\mu_0(v) \equiv M_{uv}^2(t,x,y) \quad \text{for } x,y \in (1-t,1]\times[0,1-t],\\
M^2(t,x,y) &= \int_{\mathcal{C}} v(x+t)u_v(y+t)\, d\mu_0(v) \equiv M_{vu}^2(t,x,y) \quad \text{for } x,y \in [0,1-t]\times(1-t,1],\\
M^2(t,x,y) &= \int_{\mathcal{C}} u_v(x+t)u_v(y+t)\, d\mu_0(v) \equiv M_{uu}^2(t,x,y) \quad \text{for } x,y \in (1-t,1]\times(1-t,1].
\end{aligned}$$

The subscripts of the various components of M^2 refer to the segments of the solution whose correlation is given by the particular component. For example, M_{uv}^2 describes the correlation between u and v segments of the solution as is illustrated in Figure 1. Remember that the initial function is defined on an interval $[0,1]$, so to complete the description of the statistical dependence of the solution u_v on the initial function it is necessary to introduce the functions M_{ou}^2. M_o^1 is the first order moment of measure μ_0, M_{oo}^2 is the second order moment of μ_0 *etc.*.

The moments of the measure $\mathcal{W}_t$ are also given by the power series expansion of the characteristic functional $\mathcal{Z}_t$ as we next discuss.

5.2.3 Taylor series expansion of the functional $\mathcal{Z}_t$.

The expression for the series expansion of a functional can be understood with the following argument. Let

$$F(y_1,\cdots,y_k) = F(\mathbf{y})$$

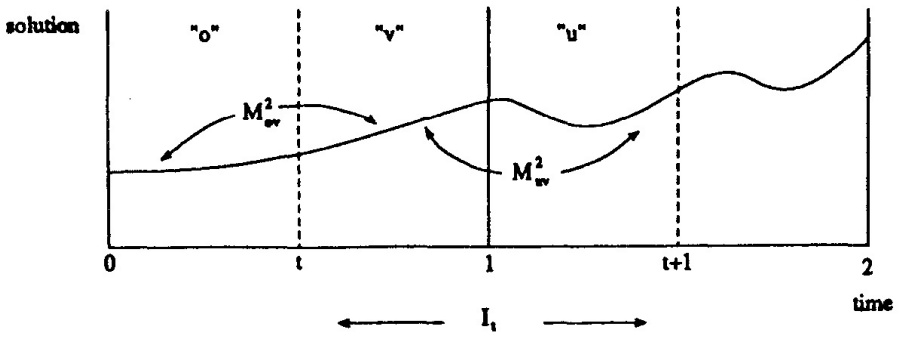

Figure 5.4: A D.D.E transforms a function defined on $[0,1]$ into a function defined on I_t. Illustration of the "o", "v" and "u" segments of the solution. Reproduced from Losson and Mackey (1992) with permission.

be a function of k variables. The power series expansion of F is

$$F(\mathbf{y}) = \sum_{n=0}^{\infty} \sum_{i_1=0}^{k} \cdots \sum_{i_n=0}^{k} \frac{1}{n!} \mathcal{E}_n(i_1, \cdots, i_n)(y_1, \cdots, y_n), \tag{5.117}$$

where

$$\mathcal{E}_n(i_1, \cdots, i_n) = \left. \frac{\partial^n F(\mathbf{y})}{\partial y_1 \cdots \partial y_n} \right|_{\mathbf{y}=0}.$$

Passing to a continuum in the following sense

$$\begin{aligned} i &\longrightarrow x_i, \\ y_i (i = 1, \cdots, k) &\longrightarrow y(x), \\ -\infty < \quad x \quad &< \infty, \\ \sum_i &\longrightarrow \int_{\mathbb{R}} dx, \end{aligned} \tag{5.118}$$

we obtain the corresponding series expansion of a *functional* $\mathcal{F}$:

$$\mathcal{F}[y] = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} dx_1 \cdots dx_n \mathcal{E}_n(x_1, \cdots, x_n) y(x_1) \cdots y(x_n), \tag{5.119}$$

where

$$\mathcal{E}_n(x_1, \cdots, x_n) = \frac{1}{n!} \left. \frac{\delta^n \mathcal{F}[y]}{\delta y(x_1) \cdots \delta y(x_n)} \right|_{y=0}. \tag{5.120}$$

$\mathcal{F}[y]$ is called the *characteristic functional* of the functions $\mathcal{E}_n$.

With these conventions, the expansion of the characteristic functional (5.101) is

$$\mathcal{Z}_t[J_1, J_2] = \sum_{p=0}^{\infty} \sum_{q=0}^{p} \int_0^1 \cdots \int_0^1 \mathcal{E}_{pq}(t, x_1, \cdots, x_p) \left(\prod_{j=1}^{q} J_1(x_j)\, dx_j \right) \left(\prod_{j=q+1}^{p} J_2(x_j)\, dx_j \right). \tag{5.121}$$

The kernels $\mathcal{E}_{pq}$ in the expansion are proportional to the moment functions of the measure $\mathcal{W}[v, \mathcal{S}_t v]$. From equations (5.103) and (5.104) they are given by

$$\begin{aligned} \mathcal{E}_{pq}(t, x_1, \cdots, x_p) &= \frac{1}{p!} \frac{\delta^p \mathcal{Z}_t}{\delta J_1^q\, \delta J_2^{p-q}} \\ &= \frac{i^p}{p!} \langle u_v(x_1) \cdots u_v(x_q) v(x_{q+1}) \cdots v(x_p) \rangle \quad (5.122) \\ &= \frac{i^p}{p!} M^p_{u^q v^{(p-q)}}(t, x_1, \cdots, x_p), \quad (5.123) \end{aligned}$$

where from now on we use the notation $M^p_{u^q v^{(p-q)}}(t, x_1, \cdots, x_p) = M^p_{u^q v^{(p-q)}}(t, \mathbf{x})$. Equation (5.121) is the infinite dimensional generalization of the well known expansion of a characteristic function in terms of the corresponding probability moments (or their Legendre transforms, the cumulants).

5.2.4 P.D.E's for the moments.

The evolution equation of the k^{th} moment is given by substituting the moment in question into (5.109) and then using formula (5.121) to the appropriate order.

Consider the first order moments of the measure $\mathcal{W}_t$. If we substitute the definitions (5.122)-(5.123) and the expansion (5.121) into equation (5.109), we obtain a P.D.E for the moment $M^1(t, x)$:

$$\begin{aligned} \frac{\partial}{\partial t} M^1_v(t, x) &= \frac{\partial}{\partial x} M^1_v(t, x) \quad \text{for } x \in [0, 1-t], \\ \frac{\partial}{\partial t} M^1_u(t, x) &= -\alpha M^1_u(t, x) + \sum_{k=1}^{n} a_k M^k_{o^k}(x+t-1, .\overset{k}{.}., x+t-1) \\ &\qquad \text{for } x \in (1-t, 1]. \quad (5.124) \end{aligned}$$

Equation 5.124 is simply the Hopf equation (5.109) for the first order moments. In (5.124) the k arguments of $M^k_{o^k}$ indicate that it is the k-point

autocorrelation function of the initial function distribution described by μ_0. Moments whose label does not contain u are moments of the initial measure.

The second order moment functions $M^2(t,x)$ are given by the solutions of the four equations:

$$\begin{aligned}
\frac{\partial}{\partial t}M^2_{vv}(t,x,y) &= \frac{\partial}{\partial x}M^2_{vv}(t,x,y)+\frac{\partial}{\partial y}M^2_{vv}(t,x,y) \\
&\qquad \text{for } (x,y)\in[0,1-t)\times[0,1-t), \qquad (5.125)\\
\frac{\partial}{\partial t}M^2_{uv}(t,x,y) &= \frac{\partial}{\partial y}M^2_{uv}(t,x,y)-\alpha M^2_{uv}(t,x,y) \\
&\quad +\sum_{k=2}^{n} a_k M^k_{o^{(k-1)}v}(t,x+t-1,\overset{(k-1)}{\dots},x+t-1,y) \\
&\qquad \text{for } (x,y)\in(1-t]\times[0,1-t], \qquad (5.126)\\
\frac{\partial}{\partial t}M^2_{vu}(t,x,y) &= \frac{\partial}{\partial x}M^2_{vu}(t,x,y)-\alpha M^2_{vu}(t,x,y) \\
&\quad +\sum_{k=2}^{n} a_k M^k_{vo^{(k-1)}}(t,x,y+t-1,\overset{(k-1)}{\dots},y+t-1) \\
&\qquad \text{for } (x,y)\in[0,1-t]\times(1-t,1], \qquad (5.127)\\
\frac{\partial}{\partial t}M^2_{uu}(t,x,y) &= -2\alpha M^2_{uu}(t,x,y) \\
&\quad +\sum_{k=1}^{n} a_k \{\, M^k_{o^{(k-1)}u}(t,x+t-1,\overset{(k-1)}{\dots},x+t-1,y)+ \\
&\quad +M^k_{uo^{(k-1)}}(t,x,y+t-1,\overset{(k-1)}{\dots},y+t-1)\,\}, \\
&\qquad \text{for } (x,y)\in(1-t,1]\times(1-t,1]. \qquad (5.128)
\end{aligned}$$

The functions M^2_{ou} and M^2_{oou} are given by

$$\begin{aligned}
\frac{\partial}{\partial t}M^2_{ou}(t,x,y) &= -\alpha M^2_{ou} \\
&\quad + \sum_{k=2}^{n} a_k M^k_{o^k}(x,y+t-1,\overset{k}{\dots},y+t-1), \qquad (5.129)\\
\frac{\partial}{\partial t}M^3_{oou}(t,x,y,z) &= -\alpha M^3_{oou}(t,x,y,z)+ \\
&\quad + \sum_{k=3}^{n} a_k M^k_{o^k}(x,y,z+t-1,\overset{k}{\dots},z+t-1) \qquad (5.130)
\end{aligned}$$

and similar equations give the moments $M^k_{o^{(k-1)}u}$.

A pattern clearly emerges from the preceding analysis: The moment $M^p(t,\mathbf{x})$ is given by 2^p partial differential equations of the same form as (5.125) through (5.128) since $M^p(t,\mathbf{x})$ is a function of p variables, each of which can belong to one of two possible intervals ($[0,1-t]$ or $(1-t,1]$). The first of these equations (when all the x_k's belong to $[0,1-t]$) is

$$\frac{\partial}{\partial t}M^p_{v^p}(t,\mathbf{x}) = \sum_{j=1}^{p}\frac{\partial}{\partial x_j}M^p_{v^p}(t,\mathbf{x}). \tag{5.131}$$

We call the equations which give the moments of the form $M^p_{v^l u^{(p-l)}}$ **mixed** equations because they yield functions which correlate mixed u and v segments of the solution. For the moment of order p, there are (2^p-2) mixed equations and 2 **pure** equations. The pure equations give $M^p_{v^p}$ and $M^p_{u^p}$, the p-point autocorrelation functions of the v and u segments of the solution.

If $x_j \in [0,1-t]$ for $j=1,\cdots,l$ and $x_j \in (1-t,1]$ for $j=l+1,\cdots,p$, then when the forcing term F of equation (5.94) is the polynomial (5.108), the generic form of the mixed equation for $M_{v^l u^{(p-l)}}$ is

$$\begin{aligned}\frac{\partial}{\partial t}M^p_{v^l\,u^{(p-l)}}\qquad (t,\mathbf{x}) &= \sum_{i=1}^{l}\frac{\partial}{\partial x_i}M^p_{v^l\,u^{(p-l)}}(t,\mathbf{x}) - \alpha(p-l)\,M^p_{v^l\,u^{(p-l)}}(t,\mathbf{x}) \\ &+ \sum_{j=0}^{n-1} a_j\left\{M^{(p+j)}_{v^l\,o^j\,u^{(p-l)}}(t,\mathbf{x}) + M^{(p+j)}_{v^l\,u^{(p-l)}\,o^j}(t,\mathbf{x})\right\}. \end{aligned}\tag{5.132}$$

Once again, this equation is one representative of the (2^p-2) mixed equations to be solved to obtain the moment of order p. Deriving these equations is tedious, but the task is greatly simplified by the similarity existing between the systems of equations for moments of different orders.

Before proceeding, we illustrate the ideas presented above and derive the partial differential equations analogous to (5.124) and (5.125) through (5.128) for the nonlinear D.D.E (5.110) considered in Example 10.

Example 11. *When the D.D.E is*

$$\frac{du}{ds} = -\alpha u(s) + ru(s-1) - ru^2(s-1), \tag{5.133}$$

the first order moment equations are given by

$$\frac{\partial M^1_v(t,x)}{\partial t} = \frac{\partial M^1_v(t,x)}{\partial x}, \tag{5.134}$$

$$\begin{aligned}\frac{\partial M^1_u(t,x)}{\partial t} &= -\alpha M^1_u(t,x) + rM^1_o(x+t-1) \\ &- rM^1_{oo}(x+t-1,x+t-1).\end{aligned}\tag{5.135}$$

The four evolution equations for the second order moments are

$$\begin{aligned}
\frac{\partial M^2_{vv}(t,x,y)}{\partial t} &= \frac{\partial M^2_{vv}(t,x,y)}{\partial x} + \frac{\partial M^2_{vv}(t,x,y)}{\partial y}, \\
&\quad \textit{for} \quad x,y \in [0,1-t] && (5.136)\\
\frac{\partial M^2_{vu}(t,x,y)}{\partial t} &= \frac{\partial M^2_{vu}(t,x,y)}{\partial x} - \alpha M^2_{vu}(t,x,y) + rM^2_{vo}(t,x,y+t-1) \\
&\quad -rM^3_{voo}(t,x,y+t-1,y+t-1), \\
&\quad \textit{for} \quad x \in [0,1-t],\ y \in (1-t,1] && (5.137)\\
\frac{\partial M^2_{uv}(t,x,y)}{\partial t} &= \frac{\partial M^2_{uv}(t,x,y)}{\partial y} - \alpha M^2_{uv}(t,x,y) + rM^2_{ov}(t,x+t-1,y) \\
&\quad -rM^3_{oov}(t,x+t-1,x+t-1,y), \\
&\quad \textit{for} \quad x \in (1-t,1],\ y \in [0,1-t] && (5.138)\\
\frac{\partial M^2_{uu}(t,x,y)}{\partial t} &= -2\alpha M^2_{uu}(t,x,y) + r\left[M^2_{ou}(t,x+t-1,y) + M^2_{uo}(t,x,y+t-1)\right] - \\
&\quad r\left[M^3_{oou}(t,x+t-1,x+t-1,y) + M^3_{uoo}(t,x,y+t-1,y+t-1)\right], \\
&\quad \textit{for} \quad x,y \in (1-t,1]. && (5.139)
\end{aligned}$$

To solve these equations, one needs to solve first for the moments M^2_{ou}, M^2_{uo}, and M^3_{oou} which satisfy equations of the following form

$$\begin{aligned}
\frac{\partial M^2_{ou}(t,x,y)}{\partial t} &= -\alpha M^2_{ou}(t,x,y) + \beta M^2_{oo}(t,x,y), && (5.140)\\
\frac{\partial M^3_{oou}(t,x,y,z)}{\partial t} &= -\alpha M^3_{oou}(t,x,y,z) + rM^3_{ooo}(x,y,z+t-1) \\
&\quad -rM^4_{oooo}(x,y,z+t-1,z+t-1). && (5.141)
\end{aligned}$$

Hence, the moments can be obtained by successively solving ordinary or hyperbolic partial differential equations. Suppose for illustration that first order moments of the initial measure are real positive constants:

$$\begin{aligned}
M^1_o &= m_1 && (5.142)\\
M^2_{oo} &= m_2 && (5.143)\\
M^3_{ooo} &= m_3 && (5.144)\\
M^4_{oooo} &= m_4. && (5.145)
\end{aligned}$$

First Moment:
For $M^1_u(t,x)$, the evolution equation (5.135) reduces to

$$\frac{\partial M^1_u(t,x)}{\partial t} = -\alpha M^1_u(t,x) + r(m_1 - m_2), \qquad (5.146)$$

whose solution is

$$M_u^1(t,x) = \gamma_1 + \left[M_u^1(0,x) - \gamma_1\right] e^{-\alpha t} \quad \textit{where } \gamma_1 \equiv \frac{r(m_1 - m_2)}{\alpha}. \quad (5.147)$$

At $t = 0$, from (5.97) and (5.98) we know that $v(1) = u_v(1)$. In addition, $M_o^1(t,x) \equiv M_v^1(t,x)$. Therefore, from (5.114)-(5.115),

$$M_o^1(t=0, x=1) = \int_{\mathcal{C}} v(1) d\mu_0 = \int_{\mathcal{C}} u_v(1) d\mu_0(v) = M_u^1(t=0, x=1)$$

and from the initial condition (45) we conclude $M_u^1(t=0,x) = m_1$. Hence

$$M_u^1(t,x) = \gamma_1 + [m_1 - \gamma_1] e^{-\alpha t}. \quad (5.148)$$

Second Moments:

To obtain expressions for M_{uv}^2, M_{vu}^2, M_{uu}^2 we have to solve their respective equations of motion (remember that M_{vv}^2 is given). We first tackle (5.138) (this choice is arbitrary; (5.137) can be dealt with in the same manner):

$$\frac{\partial M_{uv}^2(t,x,y)}{\partial t} = \frac{\partial M_{uv}^2(t,x,y)}{\partial y} - \alpha M_{uv}^2(t,x,y) + r(m_2 - m_3) \quad (5.149)$$

with initial condition $M_{uv}^2(0,x,y) = M_{vv}^2(0,x,y) \equiv m_2$ for all x, y in the domains defined in (41). This initial condition is, as for the first moment, obtained from Equations 5.114-5.115. Equation 5.149 is solved using the method of characteristics, and the solution is

$$M_{uv}^2(t,x,y) = \gamma_2 + [m_2 - \gamma_2] e^{-\alpha t} \quad \textit{where } \gamma_2 \equiv \frac{r(m_2 - m_3)}{\alpha}. \quad (5.150)$$

The moment $M_{vu}^2(t,x,y)$ can be obtained in a similar way and the result is

$$M_{vu}^2(t,x,y) = M_{uv}^2(t,x,y). \quad (5.151)$$

This equality is due to the fact that the moments of the initial measure are constant. Finally, it is necessary to solve (5.140) and (5.141) before obtaining M_{uu}^2. Using (5.143)-(5.145),

$$\begin{aligned}
M_{ou}^2 &= \gamma_2 + [m_2 + \gamma_2] e^{-\alpha t} && (5.152)\\
M_{uo}^2 &= \gamma_2 + [m_2 + \gamma_2] e^{-\alpha t} && (5.153)\\
M_{oou}^2 &= \gamma_3 + [m_3 + \gamma_3] e^{-\alpha t} \quad \textit{where } \gamma_3 \equiv \frac{r(m_3 - m_4)}{\alpha} && (5.154)\\
M_{uoo}^2(t) &= \gamma_3 + [m_3 + \gamma_3] e^{-\alpha t} && (5.155)
\end{aligned}$$

so that the evolution equation for M^2_{uu} becomes

$$\frac{\partial M^2_{uu}(t,x,y)}{\partial t} = -2\alpha M^2_{uu}(t,x,y) + 2r\gamma_2 + 2r\gamma_3 + 2re^{\alpha t}\left[m_2 - m_3 + \gamma_2 - \gamma_3\right]. \tag{5.156}$$

The above is a linear first order ODE which can be solved to give

$$M^2_{uu}(t) = \frac{-2re^{-\alpha t}}{3\alpha}\left[m_2 - m_3 + \gamma_2 - \gamma_3\right] - \frac{r}{\alpha}\left[\gamma_2 - \gamma_3\right] + \mathcal{K}e^{2\alpha t}, \tag{5.157}$$

where

$$\mathcal{K} \equiv \frac{2r}{3\alpha}\left(m_2 - m_3 + \frac{1}{2}(\gamma_3 - \gamma_2)\right) + m_2 \quad \bullet$$

This analysis can be carried out in a similar way when the moments are not constants, but such that the equations derived above remain solvable analytically.

5.2.5 Invariant measures.

It is of physical interest to investigate the constraint to be satisfied by a measure μ_*, invariant under the action of a differential delay equation. For the nonlinear DDE (5.133), the characteristic functional $\mathcal{Y}$ of such a measure is defined as

$$\mathcal{Y}[J_1] = \int_{\mathcal{C}} \exp\left[i\int_0^1 J_1(x)u_v(x+t)\,dx\right]\,d\mu_* \tag{5.158}$$

and so we have

$$\mathcal{Y}[J_1] = \mathcal{Z}_t[J_1, 0] \qquad \text{for all } t.$$

where $\mathcal{Z}_t[J_1, J_2]$ is given by (5c). The Hopf equation (16) becomes

$$\begin{aligned} 0 &= \int_0^{1-t} J_1(x)\frac{\partial}{\partial x}\left(\frac{\delta\mathcal{Y}}{\delta J_1(x)}\right)\,dx - \alpha\int_{1-t}^1 J_1(x)\frac{\delta\mathcal{Y}}{\delta J_1(x)}\,dx \\ &+ r\int_{1-t}^1 J_1(x)\frac{\delta\mathcal{Y}}{\delta J_1(x+t-1)}\,dx \\ &- ri^{-1}\int_{1-t}^1 J_1(x)\frac{\delta^2\mathcal{Y}}{\delta J_1^2(x+t-1)}\,dx, \quad \forall t. \end{aligned} \tag{5.159}$$

By choosing $t = 0$, the first integral in the Hopf equation must vanish so that we have,

$$\frac{\partial}{\partial x}\left(\frac{\delta\mathcal{Y}}{\delta J_1(x)}\right) = 0 \quad \text{a.e.} \tag{5.160}$$

From this relation a **necessary** condition for the invariant measure follows: using (5.123), the moments must satisfy

$$\sum_{k=1}^{n} \frac{\partial}{\partial x_k} M^n_{u^q v^{(n-q)}}(x_1, \cdots, x_k) = 0. \tag{5.161}$$

6 The method of steps

In treating the problem using the "method of steps" we restrict our attention to systems of the form

$$\frac{dx}{dt} = G\left(x(t)\right) + F\left((x(t-\tau)\right) \tag{6.162}$$

where F and G are real functions and τ is a positive real constant. The initial function for the system is denoted

$$x(t) \equiv \varphi(t) \qquad \text{for} \quad t \in [-\tau, 0). \tag{6.163}$$

To simplify matters, we start with linear differential delay equations of the form

$$\frac{dx}{dt} = ax(t) + bx(t-\tau), \tag{6.164}$$

with a and b real constants. The first avenue we explore is the rewriting of a differential delay equation as a set of ordinary differential equations.

Before proceeding it is useful to define some notation. The interval $\mathcal{I}_i$ $(i = 0, 1, \cdots)$ is defined to be

$$\mathcal{I}_i \equiv [(i-1)\tau, i\tau),$$

while the restriction of the solution $x(t)$ of equation (6.164) to the interval $t \in \mathcal{I}_i$ will be denoted $x_i(t)$:

$$x_i(t) \equiv x(t) \qquad \text{when } (i-1)\tau \le t < i\tau \qquad i = 0, 1, \cdots$$

with $x_0(t) = \varphi(t)$. Finally the real function $\mathcal{F}_0(x_0)$ is defined to be such that:

$$\frac{dx_0}{dt} = \mathcal{F}_0(x_0) \Longleftrightarrow x_0(t) = \varphi(t).$$

6.1 Functional iteration

Consider the differential delay equation

$$\epsilon \frac{dx}{dt} = -ax(t) + \mathcal{S}(x(t-\tau)), \tag{6.165}$$

write out the formal solution using integrating factors, and make the substitution $t - \tau \to t$. Then then we can rewrite the resulting equation as a mapping $\mathcal{L} : C \to C$

$$x_{m+1}(t) = e^{-at/\epsilon} \left\{ x_m(\tau) + \frac{1}{\epsilon} \int_0^t e^{az/\epsilon} \mathcal{S}(x_m(z)) dz \right\} = \mathcal{L}(x_m(t)) \tag{6.166}$$

for iterating functions supported on $[0, \tau]$ to functions supported on $[0, \tau]$. In (6.166), $m = 0, 1, \ldots$ indexes the number of iterations that have taken place on an initial function $x_0(u)$ with $u \in [0, \tau]$.

Remark 11. *What, if anything, can be accomplished with the operator Equation 6.166? Would this be of any help, or should the whole section be tossed out?*

Remark 12. *15 November, 1993. (Sharkovsky et al., 1993, page 134) point out that equation (6.166) "may have continuous (including arbitrarily smooth) and bounded solutions which are not uniformly continuous when $t \to \infty$. The existence of these solutions distinguishes difference equations with a continuous argument from ordinary differential equations with lag qualitatively". Thus it is highly likely that it will not be of material benefit to study the evolution of densities under the action of (6.166) in order to understand the evolution of densities under the action of a differential delay equation.*

Remark 13. *8 January, 2006. I now do not know what to make of this note that I added some years ago. I must go back to Sharkovsky et al. (1993) and try to understand their point again.*

Remark 14. *We can use the result contained in (6.166) to generate a numerically useful algorithm by dividing up the interval $[0, \tau]$ into n subintervals of length Δ so $\tau = n\Delta$. If we set $x_m(t = j\Delta) = x_m^j$ with $j = 1, \ldots, n$, then the integral on the right hand side of (6.166) can be written as*

$$\int_0^{t=j\Delta} e^{az/\epsilon} \mathcal{S}(x_m(z)) dz \simeq \sum_{k=1}^{j} \Delta \exp\left[k \frac{a\Delta}{\epsilon} \right] \mathcal{S}(x_m^k). \tag{6.167}$$

Consequently, a finite difference approximation to (6.166) is

$$x_{m+1}^j = x_m^n e^{-aj\Delta/\epsilon} + \frac{\Delta}{\epsilon}\sum_{k=1}^{j} \exp\left[-\left((j-k)\frac{a\Delta}{\epsilon}\right)\right]\mathcal{S}(x_m^k). \tag{6.168}$$

If we set $\kappa = \exp[-a\Delta/\epsilon]$*, then (6.168) becomes*

$$x_{m+1}^j = x_m^n \kappa^j + \frac{\Delta}{\epsilon}\sum_{k=1}^{j} \kappa^{j-k}\mathcal{S}(x_m^k). \tag{6.169}$$

If we define matrices

$$A = \frac{\Delta}{\epsilon}\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \kappa & 1 & 0 & 0 & \cdots & 0 \\ \kappa^2 & \kappa & 1 & 0 & \cdots & 0 \\ \kappa^3 & \kappa^2 & \kappa & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \kappa^{n-1} & \kappa^{n-2} & \kappa^{n-3} & \kappa^{n-4} & \cdots & 1 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & \kappa \\ 0 & 0 & 0 & 0 & \cdots & \kappa^2 \\ 0 & 0 & 0 & 0 & \cdots & \kappa^3 \\ 0 & 0 & 0 & 0 & \cdots & \kappa^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \kappa^n \end{bmatrix},$$

and vectors

$$\mathcal{S} = \begin{bmatrix} \mathcal{S} & \cdots & \mathcal{S} \end{bmatrix},$$

and

$$\mathcal{X}^T = \begin{bmatrix} x^1 & \cdots & x^n \end{bmatrix},$$

then we can write (6.169) in the form

$$\mathcal{X}_{m+1} = A\mathcal{S}\circ\mathcal{X}_m + B\mathcal{X}_m. \tag{6.170}$$

For large m *this becomes approximately*

$$\mathcal{X}_{m+1} \simeq A\mathcal{S}\circ\mathcal{X}_m. \tag{6.171}$$

6.2 The idea

Consider the linear equation (6.164). Suppose that one wishes to solve the differential delay equation for $t \in \mathcal{I}_1$, given the initial function $x_0(t)$ defined for $t \in \mathcal{I}_0$.

$$\begin{aligned} \frac{dx_1}{dt} &= ax_1(t) + bx_1(t-\tau) \\ &= ax_1(t) + bx_0(t) \qquad (6.172) \\ \frac{dx_0}{dt} &= \mathcal{F}_0(x_0). \qquad (6.173) \end{aligned}$$

Similarly, the restriction of $x(t)$ to the interval $\mathcal{I}_2$ can be obtained by rewriting the original differential delay equation as the following set of ordinary equations:

$$\begin{aligned} \frac{dx_2}{dt} &= ax_2(t) + bx_1(t) \qquad (6.174) \\ \frac{dx_1}{dt} &= ax_1(t) + bx_0(t) \qquad (6.175) \\ \frac{dx_0}{dt} &= \mathcal{F}_0(x_0). \qquad (6.176) \end{aligned}$$

This can be easily generalized to the restriction of the solution of the differential delay equation to the interval $\mathcal{I}_k$ with k an arbitrary positive integer: In that case, the function $x_k(t)$ can be computed by solving the following system of equations:

$$\begin{aligned} \frac{dx_0}{dt} &= \mathcal{F}_0(x_0). \qquad (6.177) \\ &\vdots \\ \frac{dx_{k-1}}{dt} &= ax_{k-1}(t) + bx_{k-2}(t) \qquad (6.178) \\ \frac{dx_k}{dt} &= ax_k(t) + bx_{k-1}(t) \qquad (6.179) \end{aligned}$$

This procedure is trivially extended to nonlinear differential delay equations of the form (6.162), since the solution of (6.162) restricted to the

interval $\mathcal{I}_k$ is also the solution of the system

$$\begin{aligned} \frac{dx_0}{dt} &= \mathcal{F}_0(x_0) && (6.180)\\ &\vdots \\ \frac{dx_{k-1}}{dt} &= G\left(x_{k-1}(t)\right) + F\left(x_{k-2}(t)\right) && (6.181)\\ \frac{dx_k}{dt} &= G\left(x_k(t)\right) + F\left(x_{k-1}(t)\right) && (6.182) \end{aligned}$$

It is possible, when describing the density of trajectories $f(t,x)$ to write this density *for the whole trajectory* as a sum of densities for the restrictions of $x(t)$ on the various $\mathcal{I}_k$'s. We will use the following notation in the next section

$$f_i(t,x_0,\cdots,x_i) \equiv f(t,x_0,\cdots,x_i) \qquad \text{for} \quad t \in \mathcal{I}_i. \tag{6.183}$$

i.e. f_i is the restriction of the phase space density f to the interval $\mathcal{I}_i$. It will simplify the notation to define

$$\tilde{f}_{i+1}(t,x_{i+1}) \equiv \int_{\alpha_i}^{\beta_i} \cdots \int_{\alpha_0}^{\beta_0} f(t,x_0,\cdots,x_i)\, dx_0 \cdots dx_i. \tag{6.184}$$

6.3 The Liouville equation

Recall that for the n dimensional system of ordinary differential equations

$$\frac{dx_i}{dt} = \mathcal{F}_i(x_0,\cdots,x_n), \qquad i = 0,1,\cdots,n \tag{6.185}$$

the Liouville equation (describing the evolution of phase-space densities $f(t,x_0,\cdots,x_n)$ is

$$\frac{\partial f}{\partial t} = -\sum_{i=0}^{n} \frac{\partial f \mathcal{F}_i}{\partial x_i}. \tag{6.186}$$

6.3.1 The linear case

In this section we write the evolution equation for the density of trajectories generated by a density of initial conditions to equation (6.164) rewritten as system (6.173). The Liouville equation for system (6.173) is

$$\frac{\partial f_1(t,x_0,x_1)}{\partial t} = -\frac{\partial\{\mathcal{F}_0(x_0) f_1(t,x_0,x_1)\}}{\partial x_0} - \frac{\partial}{\partial x_1}\left\{[ax_1 + bx_0] f_1(t,x_0,x_1)\right\} \tag{6.187}$$

The evolution equation we are interested in governs the evolution of a density

$$\tilde{f}_1(t, x_1) \equiv \int_{\alpha_0}^{\beta_0} f_1(t, x_0, x_1)\, dx_0$$

where $x_0 \in [\alpha_0, \beta_0]$. By integrating (6.187) over x_0, it is reduced to an evolution equation for the density of the variable $x_1(t)$;

$$\begin{aligned} \frac{\partial \tilde{f}_1(t, x_1)}{\partial t} = & - \int_{\alpha_0}^{\beta_0} \frac{\partial \{\mathcal{F}_0(x_0) f_1(t, x_0, x_1)\}}{\partial x_0}\, dx_0 \\ & - \int_{\alpha_0}^{\beta_0} \frac{\partial}{\partial x_1} \{[a x_1 + b x_0] f_1(t, x_0, x_1)\}\, dx_0. \end{aligned} \quad (6.188)$$

Explicitly, (6.188) becomes

$$\begin{aligned} \frac{\partial \tilde{f}_1(t, x_1)}{\partial t} &= -\{\mathcal{F}_0(\beta_0) f_1(t, \beta_0, x_1) - \mathcal{F}_0(\alpha_0) f_1(t, \alpha_0, x_1)\} - a \tilde{f}_1(t, x_1) \\ &- a x_1 \frac{\partial \tilde{f}_1(t, x_1)}{\partial x_1} \\ &- b \frac{\partial}{\partial x_1} \int_{\alpha_0}^{\beta_0} x_0 f_1(t, x_0, x_1)\, dx_0. \end{aligned} \quad (6.189)$$

Note that $\int_{\alpha_0}^{\beta_0} x_0 f_1(t, x_0, x_1)\, dx_0$ is the average value of $x_0(t)$ and it is given by the initial conditions to the original problem.

To make our solution of the Liouville equation clearer, define the following functions

$$\mathcal{Q}_1^{\alpha_0, \beta_0}(t, x_1) \equiv \mathcal{F}_0(\beta_0) f_1(t, \beta_0, x_1) - \mathcal{F}_0(\alpha_0) f_1(t, \alpha_0, x_1) \quad (6.190)$$

$$\mathcal{R}_1^{\alpha_0, \beta_0}(t, x_1) \equiv \frac{\partial}{\partial x_1} \int_{\alpha_0}^{\beta_0} f_1(t, x_0, x_1)\, dx_0 \quad (6.191)$$

so that (6.189) becomes

$$\frac{\partial \tilde{f}_1(t, x_1)}{\partial t} + a x_1 \frac{\partial \tilde{f}_1(t, x_1)}{\partial x_1} = -\mathcal{Q}_1^{\alpha, \beta}(t, x_1) - \mathcal{R}_1^{\alpha_0 \beta_0}(t, x_1) - a \tilde{f}_1(t, x_1). \quad (6.192)$$

This is a hyperbolic partial differential equation which can be solved using the method of characteristics. Note that α_0, β_0, $f_1(t, \alpha_0, x_1)$ and $f_1(t, \beta_0, x_1)$ are specified initially. Equation (6.192) therefore describes the evolution of an ensemble of trajectories under the equation of the system (6.173), and we know that the solutions of this system are also solutions of the linear

differential delay equation (6.164). Therefore, we have obtained the Liouville equation for the differential delay equation for times restricted to the interval $\mathcal{I}_1$.

The solution of the differential delay equation on the interval $\mathcal{I}_2$ is given by the system (6.176). For this system, the Liouville equation is

$$\begin{aligned}\frac{\partial f_2(t,x_0,x_1,x_2)}{\partial t} &= -\frac{\partial}{\partial x_2}\{[ax_2(t)+bx_1(t)]f_2(t,x_0,x_1,x_2)\} \\ &- \frac{\partial}{\partial x_1}\{[ax_1(t)+bx_0(t)]f_2(t,x_0,x_1,x_2)\} \\ &- \frac{\partial}{\partial x_0}\{\mathcal{F}_0(x_0)f_2(t,x_0,x_1,x_2)\}. \end{aligned} \tag{6.193}$$

Again, integrating with respect to x_0 and x_1 yields the density

$$\tilde{f}_2(t,x_2) \equiv \int_{\alpha_1}^{\beta_1}\int_{\alpha_0}^{\beta_0} f_2(t,x_0,x_1,x_2)\,dx_0dx_1,$$

where $\min(x_1(t)) = \alpha_1$ and $\sup(x_1(t)) = \beta_1$. The evolution equation for the density $\tilde{f}_2(t,x_2)$ is obtained after double integration of (6.193):

$$\begin{aligned}\frac{\partial \tilde{f}_2(t,x_2)}{\partial t} &= -\left\{a\tilde{f}_2(t,x_2) + ax_2\frac{\partial \tilde{f}_2(t,x_2)}{\partial x_2} + b\frac{\partial}{\partial x_2}\int_{\alpha_1}^{\beta_1} x_1\int_{\alpha_0}^{\beta_0} f_2(t,x_0,x_1,x_2)dx_0dx_1\right\} \\ &- a\tilde{f}_2(t,x_2) - a\int_{\alpha_1}^{\beta_1}\frac{\partial}{\partial x_1}x_1\int_{\alpha_0}^{\beta_0} f_2(t,x_0,x_1,x_2)dx_0dx_1 \\ &- b\int_{\alpha_1}^{\beta_1}\frac{\partial}{\partial x_1}\int_{\alpha_0}^{\beta_0} x_0 f_2(t,x_0,x_1,x_2)dx_0dx_1 \\ &+ \int_{\alpha_1}^{\beta_1}\{\mathcal{F}_0(\beta_1)f_2(t,\beta_1,x_1,x_2) - \mathcal{F}_0(\alpha_1)f_1(t,\alpha_1,x_1,x_2)\}\,dx_1. \end{aligned} \tag{6.194}$$

Again, this is a hyperbolic PDE which can be solved using the method of characteristics, given appropriate initial conditions. Before solving this equation for specific examples, we will write the Liouville equation for the interval $\mathcal{I}_k$, integrate that equation $(k-1)$ times and obtain an evolution equation for the density $\tilde{f}_k(t,x_k)$. The restriction of the Liouville equation to the interval $\mathcal{I}_k$ is

$$\begin{aligned}\frac{\partial f_k(t,x_0,\cdots,x_k)}{\partial t} = &- \sum_{i=1}^{k}\frac{\partial}{\partial x_i}\{[ax_i+bx_{i-1}]f_k(t,x_0,\cdots,x_k)\} \\ &- \frac{\partial}{\partial x_0}\{\mathcal{F}_0(x_0)f_k(t,x_0,\cdots,x_k)\}. \end{aligned} \tag{6.195}$$

Integrating (6.195) with respect to $x_0, \cdots, x_{k-1}$ yields

$$\begin{aligned}
\frac{\partial \tilde{f}_k(t,x_k)}{\partial t} &+ ax_k \frac{\partial \tilde{f}_k(t,x_k)}{\partial x_k} = \\
&- \int_{\alpha_{k-1}}^{\beta_{k-1}} \cdots \int_{\alpha_1}^{\beta_1} x_{k-1} \frac{\partial f_k(t,x_0,\cdots,x_k)}{\partial x_{k-1}} dx_0 \cdots dx_{k-1} \\
&- \sum_{i=1}^{k-1} \int_{\alpha_{k-1}}^{\beta_{k-1}} \cdots \int_{\alpha_1}^{\beta_1} [ax_i + bx_{i-1}] \frac{\partial f_k(t,x_0,\cdots,x_k)}{\partial x_i} dx_0 \cdots dx_{k-1} \\
&- \int_{\alpha_{k-1}}^{\beta_{k-1}} \cdots \int_{\alpha_1}^{\beta_1} \{\mathcal{F}_0(\beta_0)(f_k(t,\beta_0,\cdots,x_k) \\
&\qquad -\mathcal{F}_0(\alpha_0) f_k(t,\alpha_0,\cdots,x_k)\} \, dx_0 \cdots dx_{k-1} \qquad (6.196)
\end{aligned}$$

6.3.2 The problem of initial conditions

In the previous Section, we did not explicitly show how to specify the initial conditions for the various sets of ordinary differential equations mimicking the solution of the differential delay equation on the intervals $\mathcal{I}_k$. It should be clear that the only quantity that is specified initially is the function $f_0(0,x_0)$, from which all the $f_k(0,x_0,\cdots,x_k)$'s can be obtained.

The initial density $f_1(0,x_0,x_1)$ cannot be postulated *a priori* since the density of x_1's at time $t=0$ is not independent of the initial density $f_0(0,x_0)$. In fact, $\tilde{f}_1(0,x_1) \equiv f_0(\tau,x_0)$ since the distribution of the x_0's at time $t=\tau$ becomes the distribution of x_1's at time $t=0$ (remember that from Definition 4.2, $x_{i-1}(\tau) = x_i(0)$). This observation is a direct consequence of our definition of the various x_k's. From (6.195), with $\mathbf{x} = (x_0,\cdots,x_k)$, the Liouville equation on $\mathcal{I}_k$ is

$$\begin{aligned}
\frac{\partial f_k(t,\mathbf{x})}{\partial t} + \mathcal{F}_0(x_0)\frac{\partial f_k(t,\mathbf{x})}{\partial x_0} &+ \sum_{i=1}^{k} [ax_i(t) + bx_{i-1}(t)] \frac{\partial f_k(t,\mathbf{x})}{\partial x_i} \\
&= -\left(\frac{\partial \mathcal{F}_0(x_0)}{\partial x_0} + ka\right) f_k(t,\mathbf{x}) \qquad (6.197)
\end{aligned}$$

with initial condition $f_k(0,\mathbf{x})$. The problem is to construct this density from $f_0(0,x_0)$. It is solved by noting that from our definitions

$$\tilde{f}_k(0,x_k) \equiv \tilde{f}_{k-1}(\tau,x_k).$$

Therefore,

$$f_k(0,\mathbf{x}) = f_0(0,x_0) \prod_{i=1}^{k} \tilde{f}_i(0,x_i). \qquad (6.198)$$

7 Stochastic differential delay equations

Here we should discuss the work of Frank (2002), Guillouzic et al. (1999) as well as Mohammed (1984).

8 Approximating the differential delay equation

8.1 Distributed delay

Remark 15. *If we keep this section then references to Fargue (1973, 1974) and Blythe et al. (1985) need to be added.*

Rather than just considering a single delay one can more generally consider systems whose memory extends over the whole past. The extent to which values in the past affect their present evolution is determined by a kernel $K(t)$:

$$\frac{dx}{dt} = F_0(x(t), z(t)) \qquad \text{where} \qquad z(t) = \int_{-\infty}^{t} K(t-u)x(u)\,du \tag{8.199}$$

The fixed delay case corresponds to choosing a Dirac delta function for the kernel, i.e. $K(t-u) = \delta(t-u-\tau)$.

Under certain conditions, a differential delay equation is equivalent (but in what sense??) to an infinite set of ordinary differential equations. This can be shown using the following approach. Assume the kernel in Equation 8.199 is normalized and has the form of a gamma distribution

$$G_a^m(q) = \frac{a^m}{m!} q^m e^{-aq}\,, \qquad a, m \geq 0 \tag{8.200}$$

where m is an integer. This kernel has a maximum at $q = \dfrac{m}{a}$ and the average delay is given by

$$\overline{\tau} = \frac{\int_0^\infty q G_a^m(q)\,dq}{\int_0^\infty G_a^m(q)\,dq} = \frac{m+1}{a}\,. \tag{8.201}$$

The important property of this kernel is

$$\lim_{m,a\to\infty,\overline{\tau}\ \text{const}} G_a^m(q) = \delta(q-\overline{\tau}) \tag{8.202}$$

so that in this limit

$$z(t) = x(t-\overline{\tau})\,. \tag{8.203}$$

We now define

$$y_0(t) \equiv x(t), \qquad y_i(t) \equiv \int_{-\infty}^{t} x(u) G_a^{i-1}(t-u)\, du \qquad i = 1, ..., m+1 \tag{8.204}$$

The equations satisfied by the $y_i(t)$'s are obtained using the recursive relation

$$\frac{dG_a^p(t-u)}{dt} = a\big\{G_a^{p-1}(t-u) - G_a^p(t-u)\big\}, \tag{8.205}$$

to give

$$\begin{aligned} \frac{dy_0}{dt} &= F_0(y_0, y_{m+1}) \\ \frac{dy_i}{dt} &= a(y_{i-1} - y_i) \equiv F_i(y_{i-1}, y_i) \quad i = 1, 2, ..., m+1. \end{aligned} \tag{8.206}$$

Hence (8.199) is strictly equivalent to the system of equations (8.206), which is an $(m+2)$-dimensional system of ordinary differential equations, all of which are linear except for the first one which contains F. If the limit in (8.202) is taken, the original system (8.199) becomes a differential delay equation, and it is equivalent to an infinite set of linear ordinary differential equations plus one nonlinear ordinary differential equation.

The Liouville equation corresponding to (8.206) is given by

$$\frac{\partial f}{\partial t} = -\frac{\partial (f F_0)}{\partial y_0} - a \sum_{i=1}^{m+1} \frac{\partial [f(y_{i-1} - y_i)]}{\partial y_i} \equiv -\sum_{i=0}^{m+1} \frac{\partial (f F_i)}{\partial y_i}, \tag{8.207}$$

or, using (8.201),

$$\frac{\partial f}{\partial t} + \sum_{i=0}^{m+1} F_i \frac{\partial f}{\partial y_i} = -f \left\{ \frac{\partial F_0}{\partial y_0} - \frac{(m+1)^2}{\bar{\tau}} \right\}. \tag{8.208}$$

In the limit $m \to \infty$, the initial condition for both the integro-differential equation (8.199) (a function on $(-\infty, 0]$) and the system (8.206) is infinite-dimensional. However, for finite m, the initial condition for (8.199) is still infinite dimensional, while that for (8.206) is a point in an $(m+2)$-dimensional phase space. This "dimension reduction" which occurs in the transformation from the distributed delay system to the finite-dimensional set of ordinary differential equations is, however, only apparent.

Remark 16. *What the heck does this term*

$$\frac{(m+1)^2}{\bar{\tau}}$$

in Equation 8.208 mean as $m \to \infty$? Is there a mistake, or is it nonsense?

Let $I(t)$, $t \in (-\infty, 0]$ be the initial condition of (8.199). Then the initial conditions $y_i(0)$ of (8.206) are constants given by

$$y_i(0) = \int_{-\infty}^{0} I(u) G_a^{i-1}(-u)\, du\,. \tag{8.209}$$

Further, if $I(u)$ equals a constant C, then $y_i(0) = C$ for all i because the kernels are normalized. What this means is that although the initial condition is infinite dimensional, the solution depends only on a finite number of constants given by the integrals in (8.208). This becomes more apparent when the problem is formulated in terms of the $(m+2)$ coupled ordinary differential equations.

Critique

- First note that though the differential delay equation (1.1) is non-invertible, the system of ordinary equations (8.206) is invertible. This is an unsatisfactory and apparently contradictory situation which I discussed with AL some years ago–without resolution.
- It is completely unclear what the equation resulting from Equation 8.208 the limit $m \to \infty$ should be.

8.2 Dividing up the delay interval

Consider the differential delay equation

$$\frac{dx}{dt} = F(x(t), x(t-\tau)), \tag{8.210}$$

and following Banks (1977, 1979a,b) define

$$y_i(t) = x\left(t - \frac{i\tau}{n}\right) \qquad i = 0, \cdots, n. \tag{8.211}$$

Note that

$$z(t) - z(t-\theta) = \int_{\theta}^{t} \dot{z}(s) ds \simeq \theta \dot{z}(t-\theta), \tag{8.212}$$

so

$$z(t) \simeq z(t-\theta) + \theta \dot{z}(t-\theta). \tag{8.213}$$

Using the approximation (8.213) with equations like (8.211) we have

$$y_k(t) \simeq y_k\left(t - \frac{\tau}{n}\right) + \frac{\tau}{n} \dot{y}_k\left(t - \frac{\tau}{n}\right), \tag{8.214}$$

so we can rewrite (8.210) as the system ($i = 1, \ldots, n$)

$$\begin{aligned} \frac{dy_0(t)}{dt} &= F_0(y_0(t), y_n(t)) \\ \frac{dy_i(t)}{dt} &= \frac{n}{\tau}[-y_i(t) + y_{i-1}(t)] \equiv F_i(y_i(t), y_{i-1}(t)). \end{aligned} \tag{8.215}$$

Thus, from the system of ordinary differential equations (8.215) and the considerations of the previous section, we know that a density $f(t, y_0, \cdots, y_n)$ will evolve according to

$$\frac{\partial f}{\partial t} + f \sum_{i=0}^{n} \frac{\partial F_i}{\partial y_i} + \sum_{i=0}^{n} F_i \frac{\partial f}{\partial y_i} = 0, \tag{8.216}$$

or

$$\frac{\partial f}{\partial t} + \sum_{i=0}^{n} F_i \frac{\partial f}{\partial y_i} = -f \left\{ \frac{\partial F_0}{\partial y_0} - \frac{n^2}{\tau} \right\}. \tag{8.217}$$

Critique

- The Bank's approach gives precisely the same result as the distribution of delays approach, compare (8.208) with (8.217), though for myself I have a better feeling for the Banks approach.
- Again we seem to have this pesky divergence in the evolution Equation 8.217 as $n \to \infty$.
- What, if any, is the relation of this approach to the approximation we have in Remark 14?

9 Discussion and conclusions

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