CORRELATION FUNCTIONS AND DENSITIES

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We first express correlations for ergodic transformations in terms of phase space averages. Let $S : [0,1] \to [0,1]$ be an ergodic transformation with unique stationary density f_*, and $F, G : [0,1] \to R$ be integrable functions. Then the correlation of F with G is given by

$$\begin{aligned} C_{F,G}(\tau) &\equiv \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} F(x_{t+\tau})G(x_t) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} F(S^{t+\tau}(x)G(S^t(x)) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} F(S^\tau \circ S^t(x))G(S^t(x)) \\ &= \int_0^1 F(S^\tau(x))G(x)f_*(x)dx, \end{aligned}$$

where the last line follows by the assumed ergodicity and the Birkhoff individual ergodic theorem.

If we identify $F = G = x$ so $C_{xx}(\tau)$ is the autocorrelation of x, and assume $S^\tau(x)$ and $f_*(x)$ are even functions about the point $x = \frac{1}{2}$ for $\tau \geq 1$, then

$$\begin{aligned} C_{xx}(\tau) &= \int_0^1 xS^\tau(x)f_*(x)dx \\ &= \int_0^1 \left(x - \frac{1}{2}\right) S^\tau(x)f_*(x)dx + \frac{1}{2}\int_0^1 S^\tau(x)f_*(x)dx \\ &= \begin{cases} < x^2 > & \tau = 0 \\ \dfrac{1}{2} < x > & \tau \geq 1. \end{cases} \end{aligned}$$

Since f_* was assumed to be even about $x = \frac{1}{2}$, we have $< x >= \frac{1}{2}$ so $\frac{1}{2} < x > \equiv < x >^2$. Now let $\rho_{xx}(\tau)$ be the covariance of x: $\rho_{xx}(\tau) \equiv C_{xx}(\tau) - < x >^2$. Then

$$\rho_{xx}(\tau) = \begin{cases} < x^2 > - < x >^2 & \tau = 0 \\ 0 & \tau \geq 1. \end{cases}$$

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Independence of η. The auto-correlation of η_t^i defined by (19) is

$$C_{\eta^2}^i(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \eta_t^i \eta_{t+\tau}^i \qquad \tau = 0, 1, \cdots . \tag{20}$$

If it is the case that the covariance $\rho_{\eta^2}^i \equiv 0$ for all $\tau \geq 1$, then the η_t^i can be treated as independent variables. If further for t sufficiently large the η_t^i have achieved a stationary density g, then we can consider the η_t^i to be not only independent but also identically distributed (i.i.d.) random variables.

Inserting η from (19) we may write (20) explicitly as

$$C_{\eta^2}^i(\tau) = \frac{\epsilon^2}{4} \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \{S(x_t^{i-1})S(x_{t+\tau}^{i-1}) + S(x_t^{i+1})S(x_{t+\tau}^{i+1}) + S(x_t^{i-1})S(x_{t+\tau}^{i+1}) + S(x_t^{i+1})S(x_{t+\tau}^{i-1})\} \tag{21}$$

Note that the first two terms of $C_{\eta^2}^i$ are the auto-correlation functions of the map S at lattice sites $(i-1)$ and $(i+1)$ respectively, $C_{S^2}^{i-1}$ and $C_{S^2}^{i+1}$. This and the symmetry in the lattice allows us to write (21) in the form

$$\begin{aligned} C_{\eta^2}^i(\tau) &= \frac{\epsilon^2}{2} \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} S(x_t^{i-1})S(x_{t+\tau}^{i+1}) + \frac{\epsilon^2}{2}\begin{cases} < x^2 > & \tau = 0 \\ \frac{1}{2} < x > & \tau \geq 1. \end{cases} \\ &= \frac{\epsilon^2}{2}\int_0^1\int_0^1 x^{i-1}S^\tau(x^{i+1})f_*(x^{i-1},x^{i+1})dx^{i-1}dx^{i+1} + \frac{\epsilon^2}{2}\begin{cases} < x^2 > & \tau = 0 \\ \frac{1}{2} < x > & \tau \geq 1. \end{cases} \end{aligned}$$

For ϵ small we expect that x^{i-1} and x^{i+1} should be approximately independent so the integral may be evaluated to give

$$C_{\eta^2}^i = \frac{\epsilon^2}{2}\begin{cases} < x^2 > + < x >^2 & \tau = 0 \\ \frac{1}{2} < x > + < x >^2 & \tau \geq 1, \end{cases}$$

and thus $\rho_{\eta^2}(\tau)$ takes the form

$$\rho_{\eta^2}(\tau) \simeq \frac{\epsilon^2}{2}\begin{cases} < x^2 > - < x >^2 & \tau = 0 \\ 0 & \tau \geq 1. \end{cases}$$

Thus for small ϵ the covariance of η is of $\mathcal{O}(\epsilon^2)$ for $\tau = 0$ and approximately 0 for $\tau \geq 1$, and we can take the η_t to be independent of one another. Numerical calculations indicate that for $\epsilon \leq 10^{-1}$ our estimate of $\rho_{\eta^2}(\tau)$ is accurate.

Density of the Distribution of η. For small ϵ we expect that g should be given by the convolution of the stationary density of each of the terms making up η:

$$g(z) = \int_0^\epsilon f_*(z-y)f_*(y)dy.$$

If $\operatorname{supp} f_* = [0,1]$ then $\operatorname{supp} g = [0,\epsilon]$.

Cross Correlation of T with η. The nearest neighbor spatial-averaged cross-correlation of T with η is defined by

$$C_{T\eta}(t) = \frac{1}{L}\sum_{i=1}^{L} T(x_t^i)\ \eta_t^i \tag{22}$$

If it can be shown that $C_{T\eta}(t)$ is negligible for large t, then T and η can be taken as approximately independent as was assumed in the derivation of (18).

To estimate the cross-correlation $C_{T\eta}(t)$ between T and η, we insert the expressions for T and η, so equation (22) can be written in the explicit form

$$C_{T\eta}(t) = \epsilon(1-\epsilon)\left\{\frac{1}{L}\sum_{i=1}^{L} S(x_t^i)S(x_t^{i+1})\right\} \tag{23}$$

because of the cyclicity of the lattice. If the lattice is large so $1 \ll L$, then the bracketed term in (23) should approximate $< x^2 >$. Therefore

$$C_{T\eta}(t) \simeq < x^2 > \epsilon(1-\epsilon) \tag{24}$$

Numerical computations of $C_{T\eta}(t)$ indicate that (24) accurately predicts its value.