NOTES SUMMARIZING MY CONFUSION CONCERNING TRAJECTORY AND ENSEMBLE AVERAGES PREPARED FOR THE NOISY AND RETARDED WORKING GROUP FILE: CONFUSED.TEX 29 NOVEMBER, 1995

MICHAEL C. MACKEY
CENTRE FOR NONLINEAR DYNAMICS IN PHYSIOLOGY AND MEDICINE
DEPARTMENTS OF PHYSIOLOGY, PHYSICS AND MATHEMATICS
MCGILL UNIVERSITY
MONTREAL, QUEBEC, CANADA

1. The Origin of the Problem

On my recent visit to Chicago, John Hunter [Hun95] showed me his results obtained from a study of the **trajectory** auto-correlation of the solution of the stochastic differential delay equation

$$\frac{dx(t)}{dt} = ax(t) + bx(t-\tau) + \sigma\eta(t), \tag{1.1}$$

where the additive "noise" η is filtered white noise–i.e., coloured noise–given by the solution of the equation

$$d\eta(t) = -\alpha\eta(t)dt + \alpha dw(t) \tag{1.2}$$

and $w(t)$ is the white noise term.

Briefly, as I understand the results, at fixed values of a, b, σ and α he finds that the limiting autocorrelation, defined as

$$\bar{x^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt, \tag{1.3}$$

as a function of increasing τ is initially increasing, reaching a maximum at some value of τ, and then decreasing thereafter to finally approach an apparently constant and positive value at large values of τ.

This I find quite interesting, since [MN95] found that the **ensemble** second moment of precisely the same system had a steady state value given by [cf. their equation (4.22)]

$$\lim_{t\to\infty}\langle x^2(t)\rangle = \langle x^2\rangle^* = \frac{\sigma^2\alpha}{2}\frac{1}{(a+b)(a+b-\alpha)} \tag{1.4}$$

that is manifestly *independent* of the delay in the system. My "hand waving" explanation of the differences in the results between the two approaches is that the delay acts like a phase shift in a linear filter, and that in examining trajectories these "phase shifts" are preserved while in the ensemble approach the phase shift information is wiped out. BUT IS THIS REALLY TRUE?

However, **if this is true then the important implication is that the trajectory and ensemble statistical quantities are not necessarily the same.** Now for systems working in finite dimensional phase spaces ergodicity is a necessary and sufficient condition for the equality of trajectory and ensemble averages, but this necessary and sufficient condition may well fail in the current circumstance since differential delay equations (even stochastic ones) are definitely not operating in finite dimensional phase space.

2. An Approximation to Look at the Ensemble Approach with Additive White Noise

Given the interesting differences between what [Hun95] and [MN95] found for the trajectory and ensemble second moments with additive colored noise, I thought that it might be interesting to go back to an examination of the additive white noise case and utilize an approximation to see what kind of dependences on τ, if any, might surface. The rather surprising, and disturbing, result is that the result one obtains depends on the order in which the computations are carried out. These are detailed below in the following two subsections. Since the two computations are apparently the same but involve doing procedures in a different order, I have tried to be as precise as possible. In both cases, I am considering the stochastic differential delay equation

$$dx(t) = [ax(t) + bx(t-\tau)]dt + \sigma dw(t) \tag{2.1}$$

so it is clear that this is a case in which we are looking at additive white noise.

2.1. Ensemble Approximation Approach 1.

1. Our first step is to approximate the delayed term in (2.1) through the expansion

$$x(t-\tau) \simeq x(t) + [(t-\tau) - t]\dot{x}(t) = x(t) - \tau\dot{x}. \tag{2.2}$$

   Substituting this approximation (2.2) into equation (2.1) and collecting terms, we end up with the approximate stochastic ordinary differential equation

$$dx(t) = -Bx(t)dt + \bar{\sigma}dw(t), \tag{2.3}$$

   wherein the constants B and $\bar{\sigma}$ are related to the original parameters of the problem by

$$B = \frac{a+b}{1+b\tau} \qquad \text{and} \qquad \bar{\sigma} = \frac{\sigma}{1+b\tau}. \tag{2.4}$$

2. Having obtained the ordinary stochastic differential equation (2.3) we next write down the Ito formula for x^2 using standard techniques [Gar94] [see, in particular, his (4.3.14)]:
$$dx^2(t) = -2Bx^2(t)dt + \bar{\sigma}^2 dt + 2\bar{\sigma}x(t)dw(t). \tag{2.5}$$
3. We next integrate equation (2.5) from 0 to t to obtain
$$x^2(t) - x^2(0) = -2B\int_0^t x^2(s)ds + \bar{\sigma}^2 t + 2\bar{\sigma}\int_0^t x(s)dw(s). \tag{2.6}$$
4. The next step in our procedure is to take the expectation of equation (2.6) to obtain
$$Ex^2(t) - Ex^2(0) = -2B\int_0^t Ex^2(s)ds + \bar{\sigma}^2 t + 2\bar{\sigma}E\int_0^t x(s)dw(s). \tag{2.7}$$
Realizing, however, that the expectation in the last integral of equation (2.7) is identically zero [LM94], equation (11.4.2), this equation becomes
$$Ex^2(t) - Ex^2(0) = -2B\int_0^t Ex^2(s)ds + \bar{\sigma}^2 t. \tag{2.8}$$
5. Next, we take the time derivative of (2.8) to obtain an ordinary differential equation for the ensemble second moment Ex^2:
$$\frac{dEx^2(t)}{dt} = -2BEx^2(t) + \bar{\sigma}^2. \tag{2.9}$$
6. This equation (2.9) is easily solved, but the important result for us is that it predicts that there should be a steady state second moment given by
$$(Ex^2(t))^* = -\frac{\bar{\sigma}^2}{2B} = -\frac{\sigma^2}{2(a+b)}\frac{1}{(1+b\tau)} \simeq -\frac{\sigma^2}{2(a+b)}(1-b\tau). \tag{2.10}$$

2.2. **Ensemble Approximation Approach 2.**

1. Our first step in this approach is to write down the Ito formula for $x^2(t)$ when the dynamics are described by equation (2.1) using standard techniques [Gar94]:
$$dx^2(t) = 2x(t)[ax(t) + bx(t-\tau)]dt + \sigma^2 dt + 2\sigma x(t)dw(t). \tag{2.11}$$
2. We next integrate equation (2.11) from 0 to t to obtain
$$x^2(t) - x^2(0) = 2a\int_0^t x^2(s)ds + 2b\int_0^t x(s)x(s-\tau)ds + \sigma^2 t + 2\sigma\int_0^t x(s)dw(s). \tag{2.12}$$
3. The next step in our procedure is to take the expectation of equation (2.12) to obtain
$$Ex^2(t) - Ex^2(0) = 2a\int_0^t Ex^2(s)ds + 2b\int_0^t E[x(s)x(s-\tau)]ds + \sigma^2 t + 2\sigma E\int_0^t x(s)dw(s). \tag{2.13}$$

Realizing, however, that again the expectation in the last integral of equation (2.13) is identically zero [LM94], this equation becomes

$$Ex^2(t) - Ex^2(0) = 2a \int_0^t Ex^2(s)ds + 2b \int_0^t E[x(s)x(s-\tau)]ds + \sigma^2 t. \tag{2.14}$$

4. Next, we take the time derivative of (2.14) to obtain a differential delay equation for the ensemble second moment Ex^2:

$$\frac{dEx^2(t)}{dt} = 2aEx^2(t) + 2bE[x(t)x(t-\tau)] + \sigma^2. \tag{2.15}$$

5. Our next step is to once again approximate the delayed term in (2.15) through the expansion

$$x(t-\tau) \simeq x(t) + [(t-\tau) - t]\dot{x}(t) = x(t) - \tau\dot{x}, \tag{2.16}$$

so we have

$$x(t)x(t-\tau) \simeq x^2(t) - \tau x(t)\dot{x}(t) = x^2(t) - \frac{\tau}{2}\frac{dx^2(t)}{dt}. \tag{2.17}$$

Substituting this result back into equation (2.15) and collecting terms, we end up with the approximating ordinary differential equation

$$(1 + b\tau)\frac{dEx^2(t)}{dt} = 2(a+b)Ex^2(t) + \sigma^2, \tag{2.18}$$

6. This equation (2.18) is also easily solved, but once again the important result that we want is that it predicts that there should be a steady state second moment given by

$$(Ex^2(t))^* = -\frac{\sigma^2}{2(a+b)} \tag{2.19}$$

which is not only absolutely independent of the delay τ, but which is also identically equal to the result that [MN95] obtained for the additive white noise case [cf. their equation (3.6)].

2.3. **What is the Problem?** Well, now the problem becomes obvious. In Approach 1, we conclude that the second moment (ensemble) is dependent on the delay τ while Approach 2 (using exactly the same set of steps, but in different order) yields a result that has no dependence on the delay τ.

3. More Confusion on the Trajectory Scene

If the above discrepancies between:

1. the computed second moments from a trajectory point of view (Section 1) and the ensemble point of view [MN95];
2. The Section 2, Approach 1 approximation to the ensemble point of view and [MN95]; and
3. The two different approximation approaches of Section 2

were not enough, there seems to be yet another from the trajectory standpoint.

Namely, I found a paper [KM92] that claims to have done exactly the same trajectory calculation as done by [Hun95], but for the case of additive white noise. Though for obvious personal reasons the past few days I haven't been able to check in detail, it seems that the essence of what we are interested in is contained in equation (2.28) of [KM92], though I freely confess that I may have misread their results.

Nevertheless, it seems that using the same notation as has been used throughout this note they have

$$\bar{x}^2 = \begin{cases} \dfrac{b\sinh(q\tau) - q}{2q(a + b\cosh(q\tau))} & \text{if } b < -|a| \\ \dfrac{b\tau - 1}{4b} & \text{if } b = a \\ \dfrac{b\sin(q\tau) - q}{2q\{a + b\cos(q\tau)\}} & \text{if } |b| < -a[1ex] \end{cases} \tag{3.1}$$

where

$$q = \sqrt{|a^2 - b^2|}. \tag{3.2}$$

Now this is supposed to be the exact analytic result corresponding to what was carried out in [Hun95], and when I used the results that John was using, namely $a = -1$ and $b = -\dfrac{1}{2}$, and graphed the result of [KM92] versus the delay τ I got a curve that was a monotone increasing function of increasing τ—but not a single humped function of τ as John had showed me.

SO—YET ANOTHER MYSTERY THAT I DON'T UNDERSTAND. HOWEVER, IT IS LATE ON MONDAY EVENING AND I'M GOING TO SEND THIS OFF TO ALL OF YOU SO THAT YOU HAVE SOMETHING TO CHEW ON. LETS TRY TO GET THIS CRAZY BUSINESS RESOLVED!

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