Four open questions in the dynamics of maps and differential delay equations

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Outline

Background: Trajectories and densities in dynamical systems

Q1: Can dynamical systems display a 'chaotic' evolution of densities?

Q2: Asymptotic periodicity in delay equations?

- In deterministic DDE's?
- In stochastic DDE's?

Q3: Can we produce deterministic Brownian motion?

- In an ODE perturbed by a 'chaotic' map?: Yes
- In a DDE with a piecewise linear non-linearity?
- In a ODE perturbed by a 'random telegraph signal'?

Q4: How to formulate density evolution in delayed dynamics?

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- We are used to thinking about the trajectories of dynamical systems and the possible bifurcations:

stable steady state $\rightarrow$ simple limit cycle $\rightarrow$ complicated limit cycle $\rightarrow$ 'chaotic' solutions

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- This is akin to the Gibbs' notion of looking at an *ensemble* of dynamical systems, and this ensemble is described by the corresponding *density* of states.
- Which means that I'm thinking about looking at a very large number of copies of my dynamical system, under the assumption that each copy is not interacting with any others.

Density evolution in dynamical systems

- The Frobenius-Perron (FP) operator $P^t : L^1 \to L^1$

$$\int_A P^t f(x)\, \mu(dx) = \int_{\mathcal{S}_t^{-1}(A)} f(x)\, \mu(dx)$$

  maps densities to densities

See: Lasota & MCM (1994) for details

Density evolution: Continuous time systems

- ODE

$$\frac{dx_i}{dt} = \mathcal{F}_i(x), \qquad i = 1, \dots, d$$

Evolution equation for $f(t,x) = P^t f(x)$:

$$\frac{\partial f}{\partial t} = -\sum_{i=1}^{d} \frac{\partial (f\mathcal{F}_i)}{\partial x_i}$$

AKA the generalized Liouville equation

- Stochastic DE

$$dx = \mathcal{F}(x)dt + \sigma dw(t)$$

Evolution equation for density $f(t,x) \equiv P^t f_0(x)$ is

$$\frac{\partial f}{\partial t} = -\sum_{i=1}^{d} \frac{\partial (f\mathcal{F}_i)}{\partial x_i} + \frac{\sigma^2}{2}\sum_{i=1}^{d} \frac{\partial^2 (f)}{\partial x_i^2}$$

Fokker-Planck equation

Density evolution: Maps (discrete time)

- FP operator if $A = [a, x]$ becomes

$$\int_a^x P^t f(s)\, ds = \int_{\mathcal{S}_t^{-1}([a,x])} f(s)\, ds$$

so

$$P^t f(x) = \frac{d}{dx} \int_{\mathcal{S}_t^{-1}([a,x])} f(s)\, ds.$$

- Example: Tent (hat) map

$$\mathcal{S}(x) = \begin{cases} ax & \text{for} \quad x \in \left[0, \frac{1}{2}\right) \\ a(1-x) & \text{for} \quad x \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

with FP operator

$$Pf(x) = \frac{1}{a}\left[f\left(\frac{x}{a}\right) + f\left(1 - \frac{x}{a}\right)\right]$$

Important types of dynamics of density evolution

- **Ergodic**: There is a unique stationary density f_* so $Pf_* \equiv f_*$
- **Asymptotic periodicity**: There is a set of basis densities and for all initial densities $f_0(x)$

$$Pf_0(x) = \sum_{i=1}^{r} \lambda_i(f_0) g_i(x) + Qf_0(x)$$

Densities g_j have disjoint supports and $Pg_j = g_{\alpha(j)}$, where α is a permutation of $(1, \ldots, r)$. The invariant density is given by

$$g_* = \frac{1}{r} \sum_{j=1}^{r} g_j$$

- **Exact**: $f(t,x) \equiv P^t f_0(x) \to f_*(x)$ for all initial densities

Asymptotic periodicity in maps: Hat (tent) map

$$\mathcal{S}(x) = \begin{cases} ax & \text{for} \quad x \in \left[0, \frac{1}{2}\right) \\ a(1-x) & \text{for} \quad x \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

- Is **ergodic** for $a > 1$ (Ito, 1979) and we have an analytic form for f_* (Yoshida, 1983).
- Is also **asymptotically periodic** (Provatas & MCM, 1991) with period $r = 2^n$, $n = 0, 1, \cdots$ for

$$2^{1/2^{n+1}} < a \leq 2^{1/2^n}$$

  Thus, e.g., $\{P^t f\}$ has period 1 for $2^{1/2} < a \leq 2$, period 2 for $2^{1/4} < a \leq 2^{1/2}$, period 4 for $2^{1/8} < a \leq 2^{1/4}$, etc.
- Is **exact** for $a = 2$.

Q1: Chaotic density evolution: Is it possible?

- Nobody knows–it's an open problem!
- The trajectory sequence of potential solution behaviors through bifurcations in dynamical or semi-dynamical systems is:

  stable steady state → simple limit cycle → complicated limit cycle → 'chaotic' solutions
- The bifurcation structure in the evolution of sequences of densities under the action of a FP (or Markov) operator is:

  stable stationary density → simple asymptotic periodicity → complicated asymptotic periodicity
- Question: "How could (can) one construct an evolution operator for densities that would display a 'chaotic' evolution of densities?" Is it even possible?
- Markov and Frobenius-Perron operators are linear, so (my) suspicion is that in order to have a chaotic density evolution it would be necessary to have a non-linear evolution operator.
- Density dependent?

A toy example: Density dependent hat map

- Consider a density dependent hat map

$$x_{n+1} = \begin{cases} a[f_n]x_n & x_n \in [0, \frac{1}{2}] \\ a[f_n](1 - x_n) & x_n \in (\frac{1}{2},], \end{cases}$$

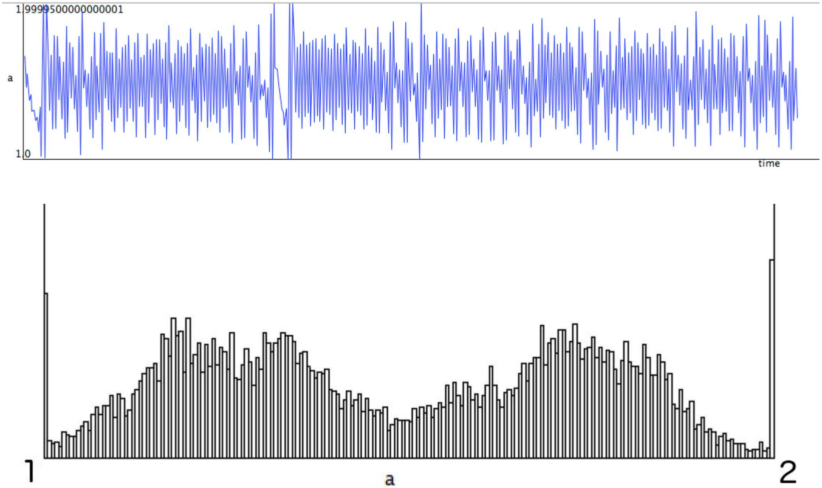
- Functional $a[f]$ is defined by

$$a[f] = 1 + \int_{A}^{A+\delta} f(x)dx \qquad a \in [1, 2]$$

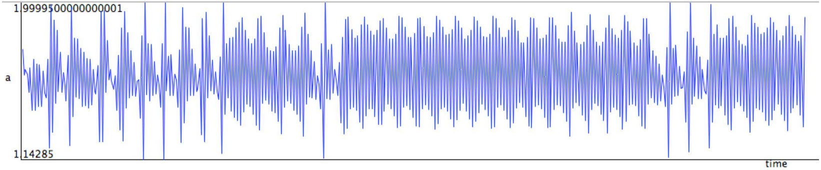
- Nonlinear evolution (pseudo-Frobenius-Perron) operator is

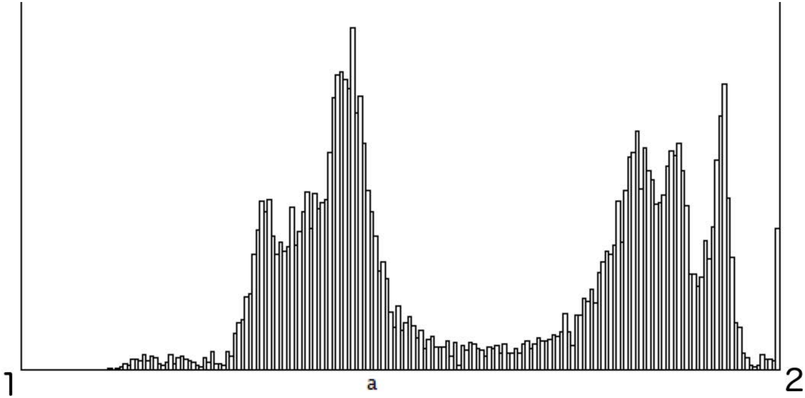
$$Pf(x) = \frac{1_{[0,a[f]/2]}(x)}{a[f]} \left\{ f\left(\frac{x}{a[f]}\right) + f\left(1 - \frac{x}{a[f]}\right) \right\}$$

Numerical: 10^4 iterations, $f_0(x) = 1_{[0,1]}(x)$, $A = 0$, $\delta = 0.5$

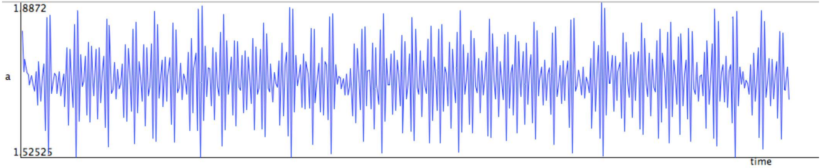


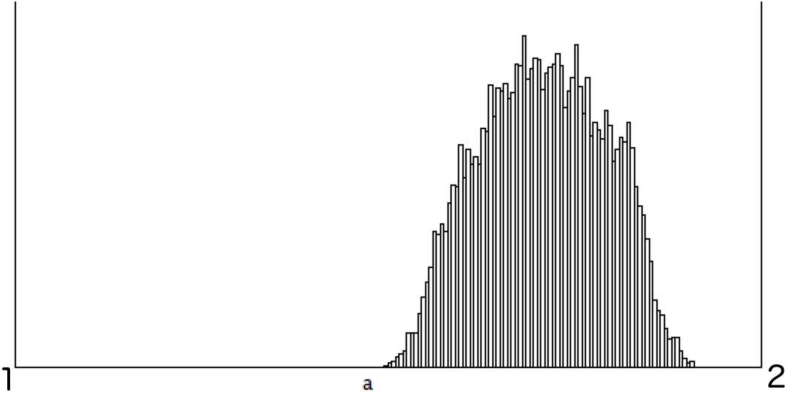
Numerical: 10^4 iterations, $f_0(x) = 1_{[0,1]}(x)$, $A = 0$, $\delta = 0.6$



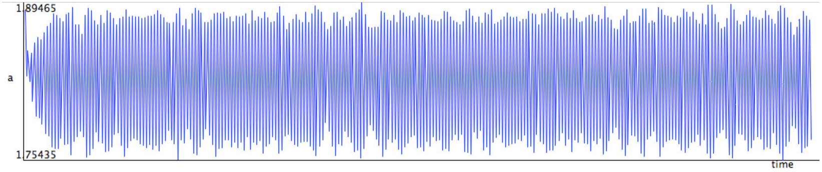


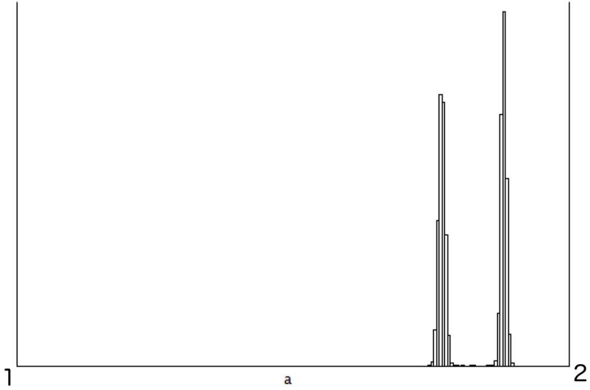
Numerical: 10^4 iterations, $f_0(x) = 1_{[0,1]}(x)$, $A = 0$, $\delta = 0.7$



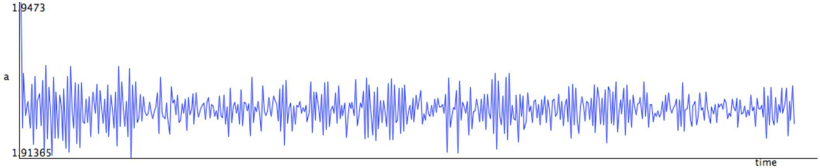


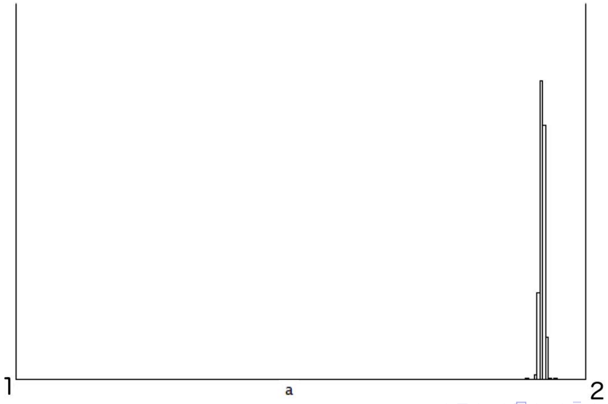
Numerical: 10^4 iterations, $f_0(x) = 1_{[0,1]}(x)$, $A = 0$, $\delta = 0.8$





Numerical: 10^4 iterations, $f_0(x) = 1_{[0,1]}(x)$, $A = 0$, $\delta = 0.9$





Q2: Asymptotic periodicity in delay equations?

Since asymptotic periodicity is a known property of discrete time maps it raises the question of whether or not it might also arise in differential delay equations.

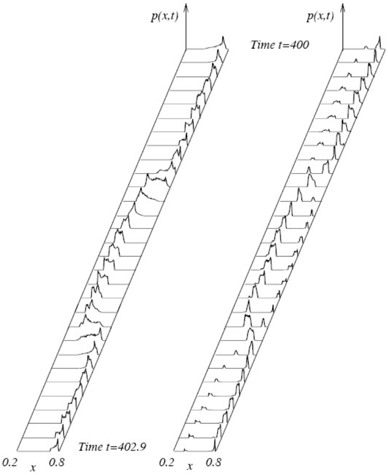
The two sub-questions are:

- In deterministic DDE's?
- In stochastic DDE's?

Here all I have is numerical evidence.

Analytic & numerical results for the 'hat' DDE

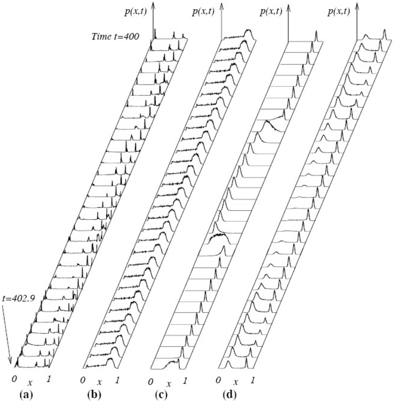
$$\frac{dx}{dt} = -\alpha x + \begin{cases} a x_\tau & \text{if } x_\tau < 1/2 \\ a(1 - x_\tau) & \text{if } x_\tau \geq 1/2 \end{cases} \qquad \frac{a}{\alpha} \in (1, 2],$$



(a) (b)

'Keener' map DDE with noise ξ (Losson & MCM, 1995): Analytic & numerical results

$$\frac{dx}{dt} = -\alpha x + [(ax_\tau + b + \xi) \mod 1] \; 0 < a, b < 1$$



Q3: A deterministic Brownian motion?

- We are accustomed to ‘noise’ in our world
- Mathematicians have abstracted this into elaborate and beautifully developed mathematics dealing with ‘random’ events
- While is clear that the assumption of ‘random’ events is sufficient to explain aspects of data
- It is by no means clear that it is necessary
- Question: “Can one produce completely deterministic theories that have the character of randomness that we see in the real world?”

Usual Brownian motion (BM)

$$
\begin{aligned}
\frac{dx}{dt} &= v \\
m\frac{dv}{dt} &= -\gamma v + f(t)
\end{aligned}
$$

- f is a fluctuating "force" given by

$$f(t) = \sigma\xi(t)$$

- $\xi = \dfrac{dw}{dt}$: a 'white noise' (delta correlated) which is the 'derivative' of a Wiener process $w(t)$
- $\xi(t)$: normally distributed with $\mu = 0$, $\sigma = 1$

BM: ODE perturbed by a chaotic map

$$\begin{aligned}\frac{dx}{dt} &= v \\ m\frac{dv}{dt} &= -\gamma v + f(t)\end{aligned}$$

- $f(t) = m\kappa \sum_{n=0}^{\infty} \xi(t)\delta(t - n\tau)$,
- ξ: a "highly chaotic" deterministic variable generated by
- $\xi(t) = T(\xi(t - \tau))$,
- where T is an *exact* map or semi-dynamical system, e.g. the tent map on $[-1, 1]$
- c.f. MCM & Tyran-Kamińska (2006): Here we have rigorous results, and it really does produce a Brownian motion

BM: From a piecewise linear differential delay equation

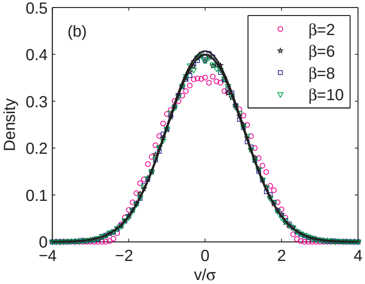
$$
\begin{aligned}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= -\gamma v + \sin(2\pi\beta v(t-1)) \\
v(t) &= \phi(t), \ -1 \leq t \leq 0
\end{aligned}
$$

or

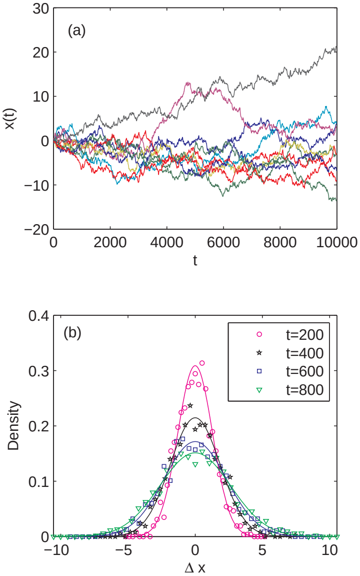
$$
\frac{dv}{dt} = -\gamma v + 2\left[H(\sin(2\pi\beta v(t-1)) - \frac{1}{2})\right]
$$

- H is the Heavyside step function
- The 'random' force is discontinuous
- Solutions are piecewise exponentials, increasing and decreasing
- Lei & MCM (2011): Mostly numerical results (and conjectures)

BM: Piecewise linear delay equation simulations, velocity



BM: Piecewise linear delay equation simulations, position



BM: ODE perturbed by a random telegraph signal (RTS)

$$\frac{dx}{dt} = v$$
$$m\frac{dv}{dt} = -\gamma v + \xi$$

- Consider a signal $\xi(t)$ that switches between $+1$ and -1 'randomly'
- $+1 \longrightarrow -1$ with transition probability $k_d \Delta t + o(\Delta t)$
- $-1 \longrightarrow +1$ with transition probability $k_u \Delta t + o(\Delta t)$
- This is the random telegraph signal (RTS)
- Fully characterized analytically

BM: ODE perturbed by a random telegraph signal cont.

$$\frac{dv}{dt} = -\gamma v + \xi$$

- The 'random' force ξ is the random telegraph signal
- Pick $k_d = k_u \equiv \alpha$
- Solutions are continuous and consist of segments that are piecewise exponentials, increasing and decreasing
- State space is $V\left(-\frac{1}{\gamma}, \frac{1}{\gamma}\right)$
- Stationary density is

$$p_*(x) = \frac{\gamma}{B\left(\frac{1}{2}, \frac{\alpha}{\gamma}\right)}\left(1 - \gamma^2 x^2\right)^{\alpha/\gamma - 1}$$

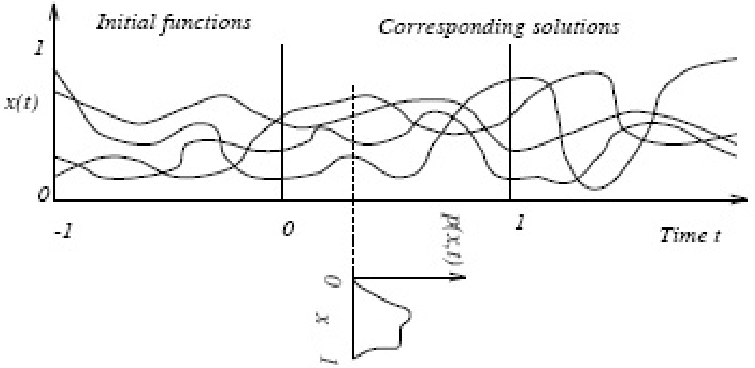
BM: DDE versus RTS results

- Compare numerical DDE results with the RTS results
-

Quantity	DDE (numerical)	RTS (exact)
Bound	$\sim \pm(\sqrt{\beta\gamma} + \gamma)^{-1}$	$\pm\gamma^{-1}$
Correlation	$\sim e^{-\gamma t}$	$e^{-\gamma t}$
Mean μ	~ 0	0
SD σ	$\sim (\beta\gamma)^{-1/2}$	$(\alpha\gamma)^{-1/2}$
Kurtosis γ_2	$\sim -\gamma/\beta$	$-\gamma/\alpha$

- I think the result in red for the DDE is due to numerics
- If $\beta \equiv \alpha$ then the exact results for the random telegraph signal match the numerical results from the differential delay equation
- Lei, MCM, & Tyran-Kamińska (2012) (unpublished, 26 pages)

Q4: How to formulate density evolution under delayed dynamics?



- We are looking at a snapshot of the evolution of all of these trajectories emanating from a whole bunch of initial functions.
- But how is this related to the *density* evolving under the delayed dynamics?

Density evolution in delayed dynamical systems

- We have

$$\frac{dx}{dt} = \epsilon^{-1}\mathcal{F}(x(t), x(t-\tau)), \qquad x(t) = \varphi(t) \quad t \in [-\tau, 0]$$

- and we want to be able to write down

$$\boxed{\text{UNKNOWN OPERATOR ACTING ON DENSITY} = 0.}$$

- If $\mathcal{F}(x, x(t-\tau)) = -x(t) + \mathcal{S}(x(t-\tau))$ so we have

$$\epsilon\frac{dx}{dt} = -x(t) + \mathcal{S}(x(t-\tau)), \qquad x(t) = \varphi(t) \quad t \in [-\tau, 0]$$

  then we should have
  - If $\tau \to 0$ then we should recover the Liouville equation from UNKNOWN OPERATOR
  - If $\epsilon \to 0$ and $t \in \mathbb{N}$ then UNKNOWN OPERATOR should reduce to the Frobenius Perron operator for the map $\mathcal{S}$

Density evolution in DDE's

- DDE induces a flow $\mathcal{T}_t$ on a phase space of continuous functions $C = C([-\tau, 0], \mathbb{R})$, $x_t = \mathcal{T}\varphi$.
- It would seem that the evolution of a density under the action of this semi-group would be given by an extension of FP equation

$$\int_A P^t f(x)\,\mu(dx) = \int_{\mathcal{T}_t^{-1}(A)} f(x)\,\mu(dx) \quad \text{for all measurable } A \subset C$$

- Merely formal, highlights problems. Namely:
  - what is the measure μ on C?
  - what is a density f on C?
  - what does it mean to do integration over subsets of C?
  - what is $\mathcal{T}_t^{-1}$?
  - See MCM & Tyran-Kamińska (2006) (unpublished, 57 pages)

Possible lines of attack

- The method of steps and/or functional iteration
- Hopf functionals
- Approximations using a distribution of delays and the linear chain trick
- However, these don't address the fundamental issues
- We need some way of formulating the problem

Conclusions and implications

- Q1: Chaotic density evolution: Is the sequence

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  possible? Maybe

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  Numerical experiments indicate that really intriguing results show up when one looks at the density of an ensemble of DDE trajectories AT A SINGLE POINT OF TIME (Losson & MCM)

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  - Analytic results (MCM & Tyran-Kamińska) show that you can produce a deterministic Brownian motion
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    Confirmation awaits a formal proof
- Q4: How to develop a theory for the evolution of densities in systems with delays?
  Quite unknown.

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