Deterministic Brownian Motion

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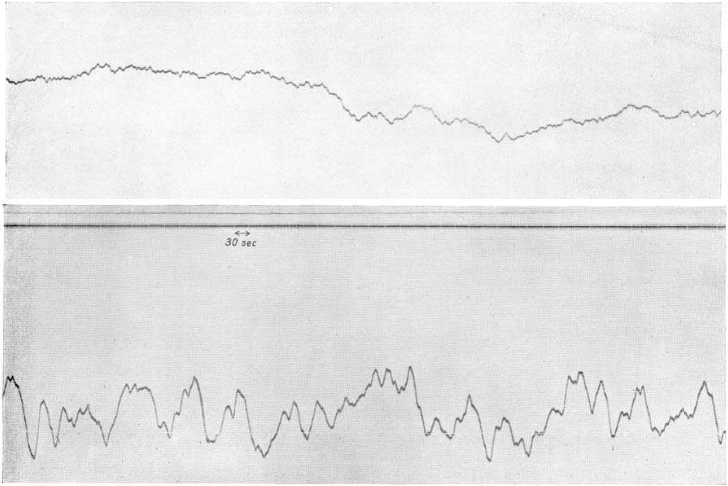


Figure: 30 minute record of mirror position at 760 mm Hg (upper) and 4×10^{-3} mm Hg (lower).

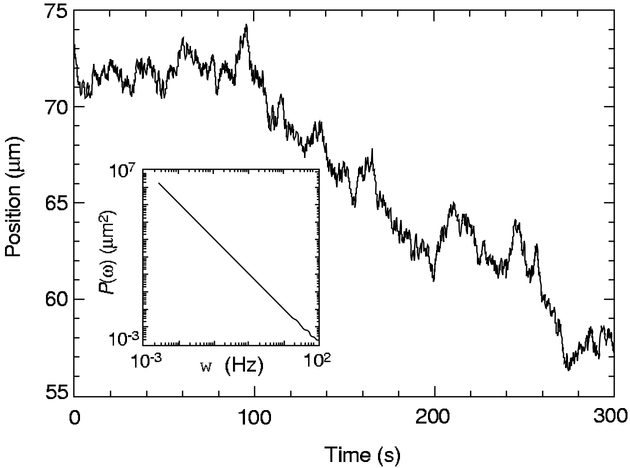


Figure: The position of a 2.5 μm particle in water over a 300 second period with a sampling interval of $\frac{1}{60}$ sec.

Theoretical treatments of Brownian motion

- Thorvald Thiele: 1880
- Louis Bachelier: 1900 PhD thesis on stock and option markets
- Albert Einstein: one of the amazing trio of 1905 papers
- Marian Smoluchowski: 1906
- Predictions of Einstein & Smoluchowski were verified experimentally by Jean Perrin, 1908
- Paul Lévy
- Paul Langevin
- Norbert Wiener
- And the list goes on. This field is now often referred to as the study of stochastic processes and is a mathematically challenging one.

Introduction

- We are accustomed to 'noise' in our world and often invoke stochastic processes to deal with this
- Mathematicians have abstracted this into elaborate and beautifully developed mathematics dealing with 'random' events
- While is clear that the assumption of 'random' events is sufficient to explain aspects of data
- It is by no means clear that it is necessary
- Interesting Question: "Can one produce completely deterministic theories that have the character of randomness that we see in the real world?"

Outline

- The ‘usual’ Brownian Motion
- Solution properties (numerical) of a differential delay equation
- Quasi-Gaussian distributions
- A deterministic quasi-Brownian motion
- The filtered ‘random telegraph signal’
- The mathematical problems preventing analytic proof
- Conclusions and problems

Usual Brownian motion

$$\frac{dx}{dt} = v$$
$$m\frac{dv}{dt} = -\gamma v + f(t)$$

- f is a fluctuating "force" given by

$$f(t) = \sigma\xi(t)$$

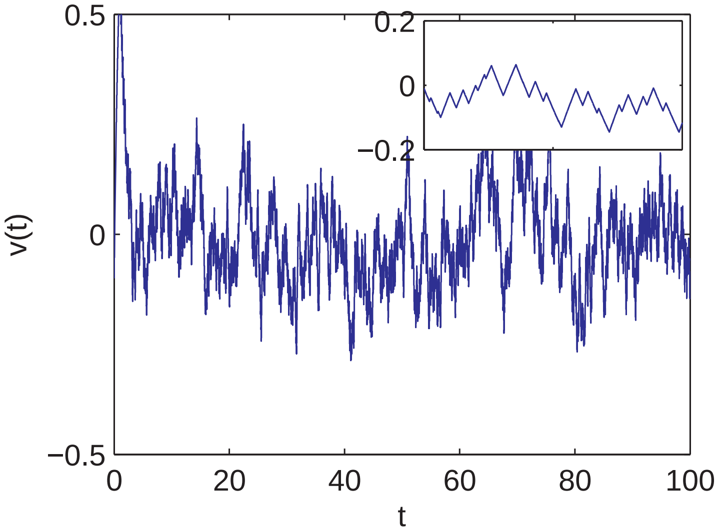
- $\xi = \frac{dw}{dt}$: a 'white noise' (delta correlated) which is the 'derivative' of a Wiener process $w(t)$
- $\xi(t)$: normally distributed with $\mu = 0$, $\sigma = 1$

A differential delay equation: Lei & MCM, 2011

$$
\begin{aligned}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= -\gamma v + \sin(2\pi\beta v(t-1)) \\
v(t) &= \phi(t), \ -1 \le t \le 0
\end{aligned}
$$

- The 'random' force is a rapidly oscillating function with respect to $v(t-1)$

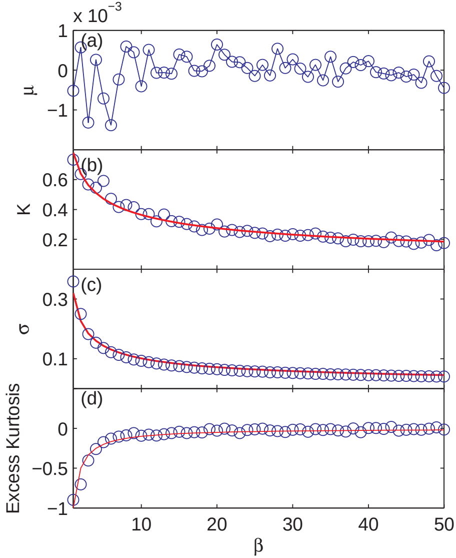
Sample solution



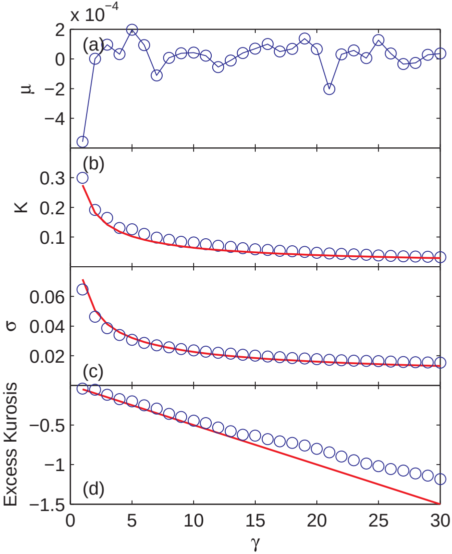
Statistical quantifiers

- Sampling: $\{v_n\}$, where $v_n = v(n \times 10^3 \Delta t)$ and $\Delta t = 0.001$
- Mean: $\mu = \frac{1}{N} \sum_{n=1}^{N} v_n$
- Bound: $K = \max_n |v_n|$
- Standard deviation: $\sigma = \sqrt{\frac{1}{N} \sum_{n=1}^{N} (v_n - \mu)^2}$
- Excess kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4} - 3$ where $\mu_4 = \frac{1}{N} \sum_{n=1}^{N} (v_n - \mu)^4$

Velocity statistics vs. β ($\gamma = 1$)



Velocity statistics vs. γ ($\beta = 20$)



Statistics: Dependence on β, γ from numerics

- Bound

$$K(\beta,\gamma) = \frac{1}{\sqrt{\gamma}(0.68\sqrt{\beta} + 0.60\sqrt{\gamma})}$$

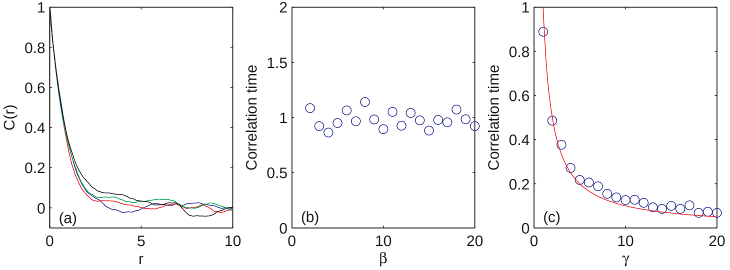
- Standard deviation

$$\sigma(\beta,\gamma) = \frac{0.32}{\sqrt{\beta\gamma}}$$

- Excess kurtosis

$$\gamma_2(\beta,\gamma) = -\frac{\gamma}{\beta}$$

Correlation function and times



- $$C(r) = \lim_{T\to\infty} \frac{\int_0^T v(t)v(t+r)dt}{\int_0^T v(t)^2 dt} \simeq e^{-r/t_0} \simeq e^{-\gamma r}$$

Quasi-Gaussian distributions

- Quasi-Gaussian with $\mu = 0$ and $\sigma = 1$ defined by

$$p(v; 0, 1, K_0) = \begin{cases} \mathcal{C} e^{-v^2/2}, & |v| \le K_0 \\ 0, & \text{other wise} \end{cases}$$

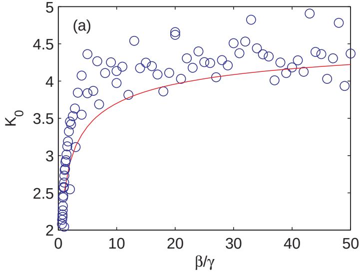
- Normalize velocities to give

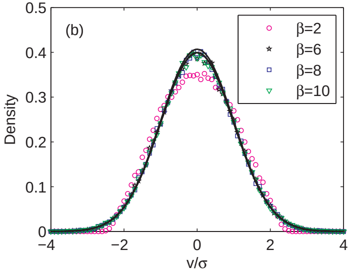
$$\zeta_n = \frac{v_n}{\sigma(\beta, \gamma)}$$

- Sequence $\{\zeta_n\}$ has mean $\mu = 0$, standard deviation $\sigma = 1$, and is bounded (numerically) by

$$K_0 = \frac{K(\beta, \gamma)}{\sigma(\beta, \gamma)} \simeq \frac{\sqrt{\beta/\gamma}}{0.21\sqrt{\beta/\gamma} + 0.19}$$

Quasi-Gaussian simulations of v from DDE



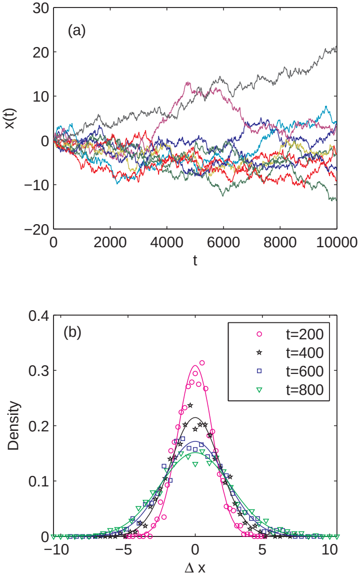


Deterministic Brownian Motion

- Since numerically the velocity is like a ‘quasi-Gaussian noise’
- now we construct a quasi-Brownian motion
- using the full system

$$\begin{cases} \dfrac{dx}{dt} & = & v \\ \dfrac{dv}{dt} & = & -\gamma v + \sin(2\pi\beta v(t-1)) \end{cases}$$

Deterministic BM simulations of x



A differential nonlinearity delay equation

Replace the sin function with a piecewise constant nonlinearity:

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\gamma v + 2\left[H(\sin(2\pi\beta v(t-1)) - \frac{1}{2})\right] \\ v(t) &= \phi(t), \; -1 \le t \le 0 \end{aligned}$$

- H is the Heavyside step function
- Now the 'random' force is discontinuous
- Solutions are piecewise exponentials, increasing and decreasing
- All of the statistical results are the same!
- I think that the only really important thing is the rapidly oscillating nature of the nonlinearity

How can we go about understanding these results analytically?

$$\frac{dx}{dt} = v$$
$$m\frac{dv}{dt} = -\gamma v + f(t)$$

- $f(t) = m\kappa \sum_{n=0}^{\infty} \xi(t)\delta(t - n\tau)$,
- ξ: a "highly chaotic" deterministic variable generated by
- $\xi(t) = T(\xi(t - \tau))$,
- where T is an *exact* map or semi-dynamical system, e.g. the tent map on $[-1, 1]$
- so the results of the simulation are (in principle) understood analytically since the numerical computations are replicating a discrete time (high-dimensional) map
- but we still can't go to the continuous situation.

Random Telegraph Signal: Does this illuminate things?

- Consider a signal $\xi(t)$ that switches between $+1$ and -1 'randomly'
- $+1 \longrightarrow -1$ with transition probability $k_d \Delta t + o(\Delta t)$
- $-1 \longrightarrow +1$ with transition probability $k_u \Delta t + o(\Delta t)$
- This is the random telegraph signal
- Fully characterized analytically

Linear dichotomous flow (LDF)

$$\frac{dv}{dt} = -\gamma v + \xi$$

- The 'random' force ξ is the random telegraph signal
- Pick $k_d = k_u \equiv \alpha$
- Solutions are continuous and consist of segments that are piecewise exponentials, increasing and decreasing
- State space is $V = \left(-\frac{1}{\gamma}, \frac{1}{\gamma}\right)$
- Stationary density is

$$p_*(x) = \frac{\gamma}{B\left(\frac{1}{2}, \frac{\alpha}{\gamma}\right)} \left(1 - \gamma^2 x^2\right)^{\alpha/\gamma - 1} \to \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}, \quad \frac{\alpha}{\gamma} \to \infty$$

DDE versus LDF results

- Compare numerical DDE results with the LDF results
-

Quantity	DDE (numerical)	LDF (exact)
Bound	$\sim \pm(\sqrt{\beta\gamma} + \gamma)^{-1}$	$\pm\gamma^{-1}$
Correlation	$\sim e^{-\gamma t}$	$e^{-\gamma t}$
Mean μ	~ 0	0
SD σ	$\sim (\beta\gamma)^{-1/2}$	$(\alpha\gamma)^{-1/2}$
Kurtosis γ_2	$\sim -\gamma/\beta$	$-\gamma/\alpha$

- I think the result in red for the DDE is due to numerics
- If $\beta \equiv \alpha$ then the exact results for the linear dichotomous flow match the numerical results from the differential delay equation

Density evolution: ODE's

If

$$\frac{dx_i}{dt} = \mathcal{F}_i(x) \qquad i = 1, \dots, d,$$

the evolution of $f(t, x) \equiv P^t f_0(x)$ is governed by the generalized Liouville equation:

$$\frac{\partial f}{\partial t} = -\sum_i \frac{\partial (f \mathcal{F}_i)}{\partial x_i}$$

Density evolution in stochastic systems

For stochastic differential equations

$$\frac{dx_i}{dt} = \mathcal{F}_i(x) + \sigma(x)\xi_i, \qquad i = 1, \dots, d$$

$f(t,x) \equiv P^t f_0(x)$ satisfies the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = -\sum_i \frac{\partial (f \mathcal{F}_i)}{\partial x_i} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 (\sigma^2 f)}{\partial x_i \partial x_j}$$

Stationary densities

- A density f_* such that $P_S^t f_* = f_*$ is a *stationary density* (fixed point) of P_S
- For the system of ordinary differential equations, f_* is given by the solution of

$$\sum_i \frac{\partial(f_* F_i)}{\partial x_i} = 0,$$

- For a system of stochastic differential equations, f_* is the solution of

$$-\sum_i \frac{\partial(f_* F_i)}{\partial x_i} + \frac{1}{2}\sum_{i,j} \frac{\partial^2(\sigma^2 f_*)}{\partial x_i \partial x_j} = 0.$$

The problem with DDE's & density evolution

- If x evolves under the action of dynamics described by a differential delay equation (DDE)

$$\frac{dx(t)}{dt} = \mathcal{F}(x(t), x_\tau(t)), \quad x_\tau(t) \equiv x(t - \tau)$$

  then we would like to know how some "density" of the variable x will evolve in time

- We would like to be able to write down an equation like

$$\text{UNKNOWN OPERATOR (DENSITY)} = 0$$

- Unfortunately we don't know how to do this
- Why?

Formal 'Transfer operator'

- $$\frac{dx(t)}{dt} = \mathcal{F}(x(t), x_\tau(t)), \; x(t') \equiv \phi(t') \forall t' \in [-\tau, 0]$$
  induces a flow $\mathcal{T}_t$ on a
- phase space of continuous functions $C = C([-\tau, 0], \mathbb{R})$
- $\{\mathcal{T}_t : t \geq 0\} : C \to C$ is a strongly continuous semigroup

Formal 'Transfer operator'

- Evolution of a density given by

$$\int_A P^t f(x)\, \mu(dx) = \int_{\mathcal{T}_t^{-1}(A)} f(x)\, \mu(dx)\, \forall \text{ mble } A \subset C.$$

- Writing of transfer ('Frobenius-Perron') operator $P^t : L^1(C) \to L^1(C)$ merely formal & highlights problems that we face.
  - What is the measure μ on the space C?
  - What is a density f on C?
  - What does it mean to do integration over subsets of C?
  - How would you actually figure out what $\mathcal{T}_t^{-1}$ is?

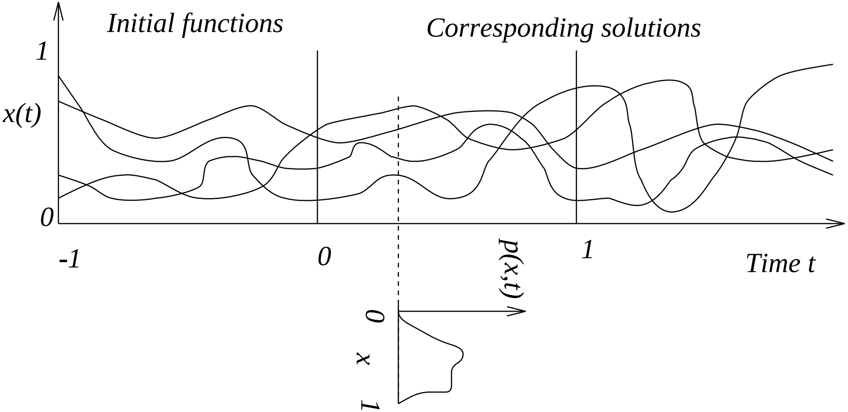
Densities and DDE's: The basic problem

- Differential delay equations–infinite dimensional systems
- Must specify an *initial function* on an interval $[-\tau, 0]$

$$\frac{dx(t)}{dt} = \mathcal{F}(x(t), x_\tau(t)), \; x(t') \equiv \phi(t') \forall t' \in [-\tau, 0]$$

- How to define a density in an infinite dimensional space?
- If we can figure out how to define a density on this space
- how can we relate it to what we actually measure in the laboratory?

What do we really measure?



Conclusions

- From numerical experiments it appears that one can produce a quasi-random process with deterministic dynamics
- The numerical DDE results match the analytic predictions for a filtered random telegraph signal
- However, confirmation awaits a formal proof and the problems involved are legion
- If true, it is no longer necessary to postulate random processes to explain 'noise' in data
- The results suggest that all events in the natural world can be explained using completely deterministic dynamics (*i.e.* deterministic dynamics are sufficient)
- Implications for the interpretation of quantum mechanics and "free will" (but that is another talk–probably over a cold beer!)

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