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Central limit theorem behavior in the skew tent map

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Abstract

In this paper we study and establish central limit theorem behavior in the skew (generalized) tent map transformation T: $Y \to Y$ originally considered by Billings and Bollt [Billings L, Bollt EM. Probability density functions of some skew tent maps. Chaos, Solitons & Fractals 2001; 12: 365–376] and Ito et al. [Ito S, Tanaka S, Nakada H. On unimodal linear transformations and chaos. II. Tokyo J Math 1979; 2: 241–59]. When the measure ν is invariant under T, the transfer operator $\mathscr{P}_T : L^1(\nu) \to L^1(\nu)$ governing the evolution of densities f under the action of the skew tent map, as well as the unique stationary density, are given explicitly for specific transformation parameters. Then, using this development, we solve the Poisson equation $f = \mathscr{P}_T f + \phi$ for two specific integrable observables ϕ and explicitly calculate the variance $\sigma(\phi)^2 = \int_Y \phi^2(y)\nu(\mathrm{d}y)$.

1. Introduction

The statistical properties of uniformly expanding maps on an interval are relatively well understood. Thus, if a map T: $Y \to Y$ is mixing then it has a unique absolutely continuous invariant measure ν by the Birkhoff Ergodic Theorem, i.e., for any observable which is an integrable function $\phi : Y \to \mathbb{R}$ we have

$$\frac{1}{n}\sum_{j=0}^{n-1}\phi(T^j(x)) \to \nu(\phi) = \int_Y \phi(y)\nu(\mathrm{d}y), \quad \text{as} \quad n \to \infty,$$

where the convergence is almost sure. Given a Hölder continuous function $\phi : Y \to \mathbb{R}$, or a function of bounded variation, the sequence $\phi \circ T^j$ has an exponential decay of correlations which gives rise to various probabilistic limit theorems such as the Central Limit Theorem (CLT), i.e., an observable ϕ is said to satisfy CLT if $\nu(\phi) = 0$ and there exists a constant $\sigma = \sigma(\phi) > 0$ such that

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$$\nu\left(x \in Y : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi(T^j(x)) < u\right) \to \frac{1}{\sqrt{2\pi\sigma(\phi)^2}} \int_{-\infty}^{u} \mathrm{e}^{-\frac{t^2}{2\sigma(\phi)^2}} \mathrm{d}t,$$

as $n \to \infty$, for all $u \in \mathbb{R}$. There are various generalizations of the CLT such as almost sure invariance principles (ASIP) [5,11], i.e., there exists $\epsilon > 0$, a sequence of random variables S_n, and a standard Brownian motion w such that S_n has the same distribution as $\sum_{j=0}^{n-1} \phi \circ T^j$, and almost everywhere

$$S_n = \sigma(\phi) w(n) + \mathrm{O}(n^{1/2-\epsilon}) \quad \text{as} \quad n \to \infty.$$

We have

$$\sigma(\phi)^2 = \lim_{n\to\infty} \frac{1}{n} \int_Y \left(\sum_{j=0}^{n-1} \phi(T^j(y))\right)^2 \nu(\mathrm{d}y),$$

which reduces to

$$\sigma(\phi)^2 = \int_Y \phi^2(y)\nu(\mathrm{d}y) + 2\sum_{j=1}^{\infty} \int_Y \phi(y)\phi(T^j(y))\nu(\mathrm{d}y). \tag{1.1}$$

In general it is difficult to calculate $\sigma(\phi)$ directly. Sometimes one may proceed by calculating the correlation function based on determining the eigenvalues and eigenvectors of the Perron–Frobenius operator. However the calculations often become quite cumbersome, even for the apparently simple example of the standard tent map

$$S(y) = a - 1 - a|y|$$

for $a \in (\sqrt{2}, 2]$, studied by a number of authors, specifically [15,4]. For expanding piecewise-linear Markov maps on $[0,1]$ calculations of various statistics, including correlations, are described in [6], but the method used there again requires finding the eigenvalues and eigenvectors of finite-dimensional matrices. In other situations it is easier to find a solution f of the so-called Poisson equation [12]:

$$f = \mathscr{P}_T f + \phi, \tag{1.2}$$

where $\mathscr{P}_T : L^1(\nu) \to L^1(\nu)$ is the transfer operator associated with T. Then the calculation of the variance reduces to

$$\sigma(\phi)^2 = 2\int_Y f(y)\phi(y)\nu(\mathrm{d}y) - \int_Y \phi(y)^2\nu(\mathrm{d}y). \tag{1.3}$$

The definition of the Perron–Frobenius (transfer) operator for T depends on a specific σ-finite measure μ on the measure space $(Y, \mathscr{B})$ with respect to which T is nonsingular, i.e., $\mu(T^{-1}(A)) = 0$ for all $A \in \mathscr{B}$ with $\mu(A) = 0$. Given such a measure the *transfer operator* $P : L^1(\mu) \to L^1(\mu)$ is defined as follows. For any $f \in L^1(\mu)$, there is a unique element Pf in $L^1(\mu)$ such that

$$\int_A Pf(y)\mu(\mathrm{d}y) = \int_{T^{-1}(A)} f(y)\mu(\mathrm{d}y) \quad \text{for all } A \in \mathscr{B}. \tag{1.4}$$

This in turn gives rise to different operators for different underlying measures on $\mathscr{B}$. Thus if μ is the Lebesgue measure on Y then P is called the *Perron–Frobenius operator*. If ν is invariant under T, then T is nonsingular and the transfer operator $\mathscr{P}_T : L^1(\nu) \to L^1(\nu)$ is well defined. Here we write $\mathscr{P}_T$ to emphasize that the underlying measure ν is invariant under T. In particular, the transfer operator $\mathscr{P}_T$ is the Perron–Frobenius operator iff the Lebesgue measure is invariant. If ν is absolutely continuous with respect to the Lebesgue measure, then there is a stationary density g_* such that $Pg_* = g_*$, and

$$\nu(A) = \int_A g_*(y)\mu(\mathrm{d}y), \quad A \in \mathscr{B}.$$

In particular, if $Y = \mathrm{supp}(g_*) = \{y \in Y : g_*(y) > 0\}$ then the transfer operator $\mathscr{P}_T : L^1(\nu) \to L^1(\nu)$ is given by

$$\mathscr{P}_T(f) = \frac{P(fg_*)}{g_*} \quad \text{for} \quad f \in L^1(\nu). \tag{1.5}$$

Now, we can rewrite (1.1) as

$$\sigma(\phi)^2 = \int_Y \phi^2(y)\nu(\mathrm{d}y) + 2\sum_{n=1}^{\infty} \int_Y \mathscr{P}_T^n\phi(y)\phi(y)\nu(\mathrm{d}y).$$

In particular, if ϕ is such that $\mathscr{P}_T\phi \equiv 0$, then $\sigma(\phi)^2 = \int_Y \phi^2(y)\nu(dy)$ (c.f. [10] for examples of such transformations). If ϕ is a function of bounded variation then the function

$$f = \sum_{n=0}^{\infty} \mathscr{P}_T^n \phi,$$

is a well defined element of $L^1(\nu)$ and is a solution of the Poisson equation (1.2), which leads to (1.3).

In this paper we consider the skew (or generalized) tent map T on $\mathbb{R}$ defined by

$$T(y) = T_{a,b}(y) = \begin{cases} b - 1 + ay, & y < 0, \\ b - 1 - by, & y \geqslant 0, \end{cases} \tag{1.6}$$

where $a > 0$, $b > 1$. In Section 2 we restrict the map T to an interval Y and describe a set of parameters a, b for which T has a positive invariant density in Y, by showing that it is conjugate to a skew tent map on $[0,1]$, studied by [7,1]. For specific values of a and b the action of the map and the form of its invariant density are described in Section 3. For these values of a and b the transfer operator has a particularly simple form, which is considered in Section 4. For ϕ from a finite dimensional subspace $\mathscr{L}$ of $L^2(\nu)$ the solution of the Poisson equation (1.2) is given in Section 5, where we also derive a closed form expression for $\sigma(\phi)^2$.

Finally, in Section 6, we consider two specific observable functions ϕ and calculate the variance $\sigma(\phi)^2$. In Example 1 we consider the potential $\phi = \log|T'| - \int \log|T'(y)|\nu(\mathrm{d}y)$, while Example 2 considers the deviation away from the mean $\mathfrak{m}$, $\phi(y) = y - \mathfrak{m}$. Examples 3 and 4 give explicit expressions for $\sigma(\phi)^2$ for specific values of the parameters (a, b).

2. The family of skew tent maps

Let T be defined by (1.6) and define an interval Y by $Y = [T^2(0), T(0)] = [-(b-1)^2, b-1]$. Then $T(Y) \subset Y$ if $a + b \geqslant ab$. The Perron–Frobenius operator $P : L^1(\mu) \to L^1(\mu)$ is then given by

$$Pf(y) = \frac{1}{a} f(\psi^-(y)) 1_{[T^3(0), b-1]}(y) + \frac{1}{b} f(\psi^+(y)), \tag{2.1}$$

where ψ^- and ψ^+ are the inverse branches of T given by

$$\psi^-(y) = \frac{y + 1 - b}{a}, \quad \psi^+(y) = -\frac{y + 1 - b}{b} \tag{2.2}$$

and μ is the Lebesgue measure on Y.

The transformation T: $Y \to Y$ is conjugated to the transformation $S_{a,b} : [0,1] \to [0,1]$

$$S_{a,b}(x) = \begin{cases} ax + \frac{a+b-ab}{b}, & 0 \leqslant x \leqslant 1 - \frac{1}{b}, \\ b(1-x), & 1 - \frac{1}{b} \leqslant x \leqslant 1, \end{cases} \tag{2.3}$$

studied by Ito et al. [7], i.e., $T_{a,b}(\varrho(x)) = \varrho(S_{a,b}(x))$ for $x \in [0,1]$, where

$$\varrho(x) = b(b-1)x - (b-1)^2.$$

Define

$$D = \{(a,b) : a > 0, b > 1, a + b \geqslant ab\}, \quad D_1^* = \{(a,b) \in D : a > 1\}.$$

For $m \geqslant 2$ let

$$D_m^* = \left\{ (a,b) \in D : a \leqslant 1,\ \sum_{i=0}^{m-1} a^{-i} < b \leqslant \sum_{i=0}^{m} a^{-i} \right\}.$$

As shown in [7], the transformation $S_{a,b}$ has a strictly positive invariant density in the parameter subset

$$\widetilde{D} = \bigcup_{m=1}^{\infty} \left\{ (a,b) \in D_m^* : a^m b > 1, a + b < a^m b^2 \right\},$$

and this invariant density is given by

$$h_* = \sum_{n=0}^{\infty} C \left(\frac{1}{a}\right)^{N_0(n)} \left(-\frac{1}{b}\right)^{N_1(n)} 1_{[f_{a,b}^n(0), 1]},$$

where C is a normalizing constant and $N_0(n), N_1(n) \geqslant 0$ are sequences of integers [7, Eq. 45].

It should also be noted that the transformation $F_{\alpha,\beta}$ on $[0,1]$ studied by [1]:

$$F_{\alpha,\beta}(x) = \begin{cases} \beta + \frac{1-\beta}{\alpha}x, & 0 \leqslant x \leqslant \alpha, \\ \frac{1-x}{1-\alpha}, & \alpha \leqslant x \leqslant 1, \end{cases} \tag{2.4}$$

reduces to (2.3) if we set

$$a = \frac{1-\beta}{\alpha}, \quad b = \frac{1}{1-\alpha}.$$

3. Characterizing the rising periodic orbits

Consider those values of (a,b) for which $\{T^n(0)\}$ is a periodic orbit with period $K \geqslant 3$ and in which $T^n(0)$ is negative for $n = 2,\ldots,K-1$ and $T^K(0) = 0$. Then we have

$$T^n(0) = (b-1)\big(1 + a + \ldots + a^{n-2} - a^{n-2}b\big) \quad \text{for} \quad n \geqslant 2. \tag{3.1}$$

Thus

$$b = \sum_{l=0}^{K-2} a^{l+2-K}. \tag{3.2}$$

Define the intervals A_l by

$$A_l = [T^{l+1}(0), T^{l+2}(0)] \quad \text{for} \quad l = 1,2,\ldots,K-1. \tag{3.3}$$

Then

$$\mu(A_l) = (b-1)a^{l+1-K} \tag{3.4}$$

and

$$T(A_l) = \begin{cases} A_{l+1}, & l = 1,\ldots,K-2, \\ \bigcup_{i=1}^{K-1} A_i, & l = K-1. \end{cases} \tag{3.5}$$

We have $Y = \bigcup_{l=1}^{K-1} A_l$. The invariant measure ν has a density given by

$$g_* = \sum_{l=1}^{K-1} d_l 1_{A_l}, \tag{3.6}$$

where the d_l, $l = 1,\ldots,K-1$ are the solutions of

$$\begin{cases} d_{K-1} = bd_1, \\ bd_{l-1} + ad_{K-1} = abd_l, \quad l = 2,\ldots,K-1, \\ \sum_{l=1}^{K-1} d_l\mu(A_l) = 1, \end{cases} \tag{3.7}$$

and are given by

$$d_l = \sum_{j=0}^{l-1} a^{-j} d_1, \quad d_1 = \frac{a^{K-2}}{(b-1)\sum_{l=1}^{K-1}\sum_{j=0}^{l-1} a^j}. \tag{3.8}$$

If $a \neq 1$ then from (3.2) it follows that

$$b = \frac{a^{K-1}-1}{a^{K-2}(a-1)}, \quad b - 1 = \frac{a^{K-2}-1}{a^{K-2}(a-1)}, \quad a + b = \frac{a^K - 1}{a^{K-2}(a-1)}. \tag{3.9}$$

Consequently,

$$\frac{d_l}{d_1} = \frac{a^l - 1}{a^{l-1}(a-1)}, \quad d_1 = \frac{(a-1)}{(b-1)(a+b-Ka^{2-K})}. \tag{3.10}$$

Making use of (3.6), (3.10), and (3.4) we obtain

$$v(A_l) = \frac{(b-1)(a^l-1)d_1}{(a-1)a^{K-2}}, \quad l = 1,\ldots,K-1. \tag{3.11}$$

If $a = 1$ then $b = K - 1$,

$$d_l = ld_1, \quad d_1 = \frac{2}{(K-2)(K-1)K}, \tag{3.12}$$

and

$$v(A_l) = (b-1)ld_1, \quad l = 1,\ldots,K-1. \tag{3.13}$$

4. The transfer operator

The transfer operator $\mathcal{P}_T$ on the space $L^1(v)$ is given by the equation

$$\mathcal{P}_T f = \frac{P(fg_*)}{g_*}, \; f \in L^1(v).$$

The values of (a,b) were chosen in Section 3 so $A_l \subset (-\infty,0]$ for $l = 1,\ldots,K-2$ and $A_{K-1} \subset [0,\infty)$. From (3.5) and (3.6) we thus obtain

$$g \circ \psi^- = \sum_{l=1}^{K-2} d_l 1_{A_{l+1}}$$

and

$$g \circ \psi^+ = d_{K-1} \sum_{l=1}^{K-1} 1_{A_l}.$$

Consequently Eq. (2.1) for the transfer operator $\mathcal{P}_T : L^1(v) \to L^1(v)$ has the form

$$\mathcal{P}_T f = f \circ \psi^+ 1_{A_1} + \sum_{l=2}^{K-1} \left(\frac{d_{l-1}}{ad_l} f \circ \psi^- + \frac{d_{K-1}}{bd_l} f \circ \psi^+ \right) 1_{A_l}$$

and can be rewritten as

$$\mathcal{P}_T f = \sum_{l=1}^{K-1} (c_l f \circ \psi^- + (1-c_l) f \circ \psi^+) 1_{A_l}, \tag{4.1}$$

where

$$c_l = \begin{cases} 0, & l = 1 \\ \frac{d_{l-1}}{ad_l}, & l = 2,\ldots,K-1. \end{cases} \tag{4.2}$$

Note that

$$c_l v(A_l) = v(A_{l-1}), \; l = 2,\ldots,K-1. \tag{4.3}$$

Consider the finite dimensional subspace $\mathcal{L}$ of $L^2(v)$ spanned by the elements

$$\theta_j = 1_{A_j} - c_{j+1} 1_{A_{j+1}}, \; j = 1,\ldots,K-2 \tag{4.4}$$

and

$$\eta_j(y) = (y - v_j) 1_{A_j}(y), \; j = 1,\ldots,K-1, \tag{4.5}$$

where v_j is the center of the interval A_j, $j = 1,\ldots,K-1$. From (3.1) and (3.3) we obtain

$$v_j = \begin{cases} \frac{b-1}{2(a-1)}((a+1)a^{j-K+1} - 2), & a \neq 1, \\ \frac{K-2}{2}(2j + 3 - 2K), & a = 1. \end{cases} \tag{4.6}$$

We are going to show that $\mathcal{P}_T(\mathcal{L}) \subset \mathcal{L}$.

To do this we need expressions for $\mathscr{P}_T\theta_j$ and $\mathscr{P}_T\eta_j$. From (3.5) and (4.1) it follows that

$$\mathscr{P}_T\theta_j = \begin{cases} c_{j+1}\theta_{j+1}, & j = 1,\dots,K-3, \\ -c_{K-1}\sum_{l=1}^{K-2}\theta_l, & j = K-2 \end{cases} \tag{4.7}$$

and

$$\mathscr{P}_T\eta_j = c_{j+1}(\psi^- - v_j)1_{A_{j+1}}$$

for $j = 1,\dots,K-2$. Since v_j is the center of the interval $A_j \subset (-\infty, 0]$, we have $v_j < 0$ and

$$\psi^-(y) - v_j = \frac{y - T(v_j)}{a}.$$

However $T(A_j) = A_{j+1}$, so $T(v_j) = v_{j+1}$ and

$$\mathscr{P}_T\eta_j = \frac{c_{j+1}}{a}\eta_{j+1}, \; j = 1,\dots,K-2. \tag{4.8}$$

Since $A_{K-1} \subset [0,1]$ and $T(A_{K-1}) = \bigcup_{l=1}^{K-1}A_l$, we obtain

$$\mathscr{P}_T\eta_{K-1} = \sum_{l=1}^{K-1}(1-c_l)(\psi^+ - v_{K-1})1_{A_l},$$

which leads to

$$\mathscr{P}_T\eta_{K-1} = -\frac{1}{b}\sum_{l=1}^{K-1}(1-c_l)\eta_l - \frac{1}{b}\sum_{l=1}^{K-1}(1-c_l)(v_l - T(v_{K-1}))1_{A_l}.$$

Define

$$\beta_l = \begin{cases} v_1 - T(v_{K-1}), & l = 1, \\ c_l\beta_{l-1} + (1-c_l)(v_l - T(v_{K-1})), & l = 2,\dots,K-1, \end{cases} \tag{4.9}$$

which, after some algebra based on formulas (3.8), (4.2), and (4.6), can be rewritten as

$$\beta_l = \begin{cases} \frac{(b-1)}{2(a-1)}(a^{l-K+2} - a), & a \neq 1, \\ \frac{K-2}{2}(l-K+1), & a = 1, \end{cases} \quad l = 1,\dots,K-1. \tag{4.10}$$

Hence $\beta_{K-1} = 0$ and

$$\mathscr{P}_T\eta_{K-1} = -\frac{1}{b}\sum_{l=1}^{K-1}(1-c_l)\eta_l - \frac{1}{b}\sum_{l=1}^{K-2}\beta_l\theta_l. \tag{4.11}$$

5. Solution of the Poisson equation for functions from $\mathcal{L}$

The action of the operator $\mathscr{P}_T$ on the space $\mathscr{L}$ can be identified with the $(2K-3)\times(2K-3)$ matrix

$$C = \begin{pmatrix} C_0 & C_{01} \\ 0 & C_1 \end{pmatrix}, \tag{5.1}$$

where

$$C_0 = \begin{pmatrix} 0 & & & & -c_{K-1} \\ c_2 & 0 & & & -c_{K-1} \\ & c_3 & \ddots & & \vdots \\ & & \ddots & 0 & -c_{K-1} \\ & & & c_{K-2} & -c_{K-1} \end{pmatrix} \tag{5.2}$$

for $K \geqslant 4$ and $C_0 = (-c_2)$ for $K = 3$,

$$C_{01} = \frac{1}{b}\begin{pmatrix} 0 & & & -\beta_1 \\ & 0 & & -\beta_2 \\ & & \ddots & \vdots \\ & & 0 & -\beta_{K-2} \end{pmatrix}, \tag{5.3}$$

and

$$C_1 = \frac{1}{ab}\begin{pmatrix} 0 & & & & -a \\ bc_2 & 0 & & & -a(1-c_2) \\ & bc_3 & \ddots & & \vdots \\ & & \ddots & 0 & -a(1-c_{K-2}) \\ & & & bc_{K-1} & -a(1-c_{K-1}) \end{pmatrix}, \tag{5.4}$$

where the unspecified elements are all zero. We have

$$(I-C)^{-1} = \begin{pmatrix} (I_0-C_0)^{-1} & (I_0-C_0)^{-1}C_{01}(I_1-C_1)^{-1} \\ 0 & (I_1-C_1)^{-1} \end{pmatrix},$$

where I_0 and I_1 are identity matrices of dimension $K-2$ and $K-1$, respectively.

Let the observable $\phi \in \mathscr{L}$ be of the form

$$\phi = \sum_{l=1}^{K-2}\alpha_l^0\theta_l + \sum_{l=1}^{K-1}\alpha_l^1\eta_l, \tag{5.5}$$

where α_l^0, $l=1,\ldots,K-2$ and α_l^1, $l=1,\ldots,K-1$, are constants. Then the solution of the Poisson equation (1.2) is given by

$$f = \sum_{l=1}^{K-2}\Delta_l^0\theta_l + \sum_{l=1}^{K-1}\Delta_l^1\eta_l, \tag{5.6}$$

where

$$\begin{pmatrix} \Delta_1^0 \\ \vdots \\ \Delta_{K-2}^0 \end{pmatrix} = (I_0-C_0)^{-1}\left\{\begin{pmatrix} \alpha_1^0 \\ \vdots \\ \alpha_{K-2}^0 \end{pmatrix} + C_{01}(I_1-C_1)^{-1}\begin{pmatrix} \alpha_1^1 \\ \vdots \\ \alpha_{K-1}^1 \end{pmatrix}\right\} \tag{5.7}$$

and

$$\begin{pmatrix} \Delta_1^1 \\ \vdots \\ \Delta_{K-1}^1 \end{pmatrix} = (I_1-C_1)^{-1}\begin{pmatrix} \alpha_1^1 \\ \vdots \\ \alpha_{K-1}^1 \end{pmatrix}. \tag{5.8}$$

From now on we assume that ϕ, and the corresponding solution f of the Poisson equation, are respectively given by (5.5) and (5.6). In the next two lemmas, we will provide explicit formulae for Δ_l^1 and Δ_l^0.

Lemma 1. *We have*

$$\Delta_l^1 = \frac{\sum_{j=1}^{l} a^j v(A_j)\alpha_j^1}{a^l v(A_l)} - \frac{(a^l+1)\sum_{j=1}^{K-1} a^j v(A_j)\alpha_j^1}{2(a+b)a^{l+K-2}v(A_{K-1})}.$$

Proof. Referring to Appendix A we note that the matrix C_1 is of the form (A.1) with entries given by

$$\gamma_l = \frac{c_l}{a} = \frac{v(A_{l-1})}{av(A_l)},\ l=2,\ldots,K-1,$$
$$\delta_l = \frac{1-c_l}{b},\ l=1,\ldots,K-1.$$

Since

$$\prod_{i=j+1}^{K-1} \gamma_i = \frac{a^j v(A_j)}{a^{K-1} v(A_{K-1})},$$

it follows from (3.11) and (3.13) that

$$\frac{\delta_j v(A_j)}{v(A_{K-1})} = \frac{a^{j+1-K}}{b^2},$$

which leads to

$$\sum_{j=1}^{l} \delta_j \prod_{i=j+1}^{K-1} \gamma_i = \frac{1}{b^2} \begin{cases} \frac{a^{2l}-1}{a^{2(K-2)}(a^2-1)}, & a \neq 1, \\ l, & a = 1, \end{cases}$$

and can be rewritten as

$$\sum_{j=1}^{l} \delta_j \prod_{i=j+1}^{K-1} \gamma_i = \frac{(a^l+1)v(A_l)}{ba^{K-2}(a+1)v(A_{K-1})}.$$

Setting $l = K-1$ in the last formula and adding 1 gives, by (3.9),

$$\det(I_1 - C_1) = \frac{2(a+b)}{b(a+1)},$$

which completes the proof by making use of formula (A.4). □

Lemma 2. *We have*

$$\Delta_l^0 = \frac{\sum_{j=1}^{l} \left(\alpha_j^0 - \beta_j^0\right) v(A_j)}{v(A_l)} - v\left(\bigcup_{j=1}^{l} A_j\right) \frac{\sum_{j=1}^{K-2} \left(\alpha_j^0 - \beta_j^0\right) v(A_j)}{v(A_l)}, \tag{5.9}$$

where

$$\beta_j^0 = \frac{\Delta_{K-1}^1}{b} \beta_j \quad j = 1, \ldots, K-2 \tag{5.10}$$

and β_j is given by (4.10).

Proof. The form of the matrix C_{01} and (5.8) leads to

$$C_{01}(I_1 - C_1)^{-1} \begin{pmatrix} \alpha_1^1 \\ \vdots \\ \alpha_{K-1}^1 \end{pmatrix} = -\frac{\Delta_{K-1}^1}{b} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{k-2} \end{pmatrix} = -\begin{pmatrix} \beta_1^0 \\ \vdots \\ \beta_{K-2}^0 \end{pmatrix}.$$

Consequently,

$$\begin{pmatrix} \Delta_1^0 \\ \vdots \\ \Delta_{K-2}^0 \end{pmatrix} = (I_0 - C_0)^{-1} \begin{pmatrix} \alpha_1^0 - \beta_1^0 \\ \vdots \\ \alpha_{K-2}^0 - \beta_{K-2}^0 \end{pmatrix}.$$

The matrix C_0 is of the form (A.1) with entries given by

$$\gamma_l = c_l = \frac{v(A_{l-1})}{v(A_l)}, \; l = 2, \ldots, K-2,$$

$$\delta_l = c_{K-1} = \frac{v(A_{K-2})}{v(A_{K-1})}, \; l = 1, \ldots, K-2.$$

Since

$$\delta_j \prod_{i=j+1}^{K-2} \gamma_i = \frac{\nu(A_j)}{\nu(A_{K-1})},$$

we have

$$\det(I_0 - C_0) = \frac{1}{\nu(A_{K-1})},$$

and the s_l in (A.2) in this case are given by

$$s_l^0 = \nu\left(\bigcup_{j=1}^{l} A_j\right), \; l = 1, \ldots, K-2,$$

which implies (5.9) using (A.4) with $\tau_j = \alpha_j^0 - \beta_j^0$. $\square$

Finally, the specific representations for ϕ and f allows us to derive the following formula for the variance of ϕ from (1.3).

Proposition 1. *We have*

$$\sigma(\phi)^2 = \sum_{l=1}^{K-2} \left(2\Delta_l^0 - \alpha_l^0\right)\nu(A_l)(e_l - e_{l+1}) + \frac{1}{12}\sum_{l=1}^{K-1}\left(2\Delta_l^1 - \alpha_l^1\right)\alpha_l^1 \nu(A_l)\mu(A_l)^2,$$

where

$$e_l = \alpha_l^0 - c_l \alpha_{l-1}^0, \; l = 1, \ldots, K-1, \quad \text{and} \quad \alpha_0^0 = \alpha_{K-1}^0 = 0. \tag{5.11}$$

Moreover,

$$\sum_{l=1}^{K-2} \Delta_l^0 \nu(A_l)(e_l - e_{l+1}) = \sum_{l=1}^{K-2} e_l(\alpha_l^0 - \beta_l^0)\nu(A_l). \tag{5.12}$$

Proof. Let $\langle\cdot,\cdot\rangle$ denote the scalar product in $L^2(\nu)$. Since $\sigma(\phi)^2 = \langle 2f - \phi, \phi\rangle$, we obtain, by (5.5) and (5.6),

$$\sigma(\phi)^2 = \left\langle \sum_{l=1}^{K-2}\left(2\Delta_l^0 - \alpha_l^0\right)\theta_l + \sum_{l=1}^{K-1}\left(2\Delta_l^1 - \alpha_l^1\right)\eta_l, \sum_{l=1}^{K-2}\alpha_l^0\theta_l + \sum_{l=1}^{K-1}\alpha_l^1\eta_l \right\rangle.$$

The elements η_j, $j = 1, \ldots, K-1$, are orthogonal in $L^2(\nu)$ and

$$\int_Y \eta_j(y)\nu(\mathrm{d}y) = 0, \quad \langle \eta_j, \eta_j\rangle = \frac{1}{12}\nu(A_j)\mu(A_j)^2.$$

From (4.3) we obtain

$$\int_Y \theta_j(y)\nu(\mathrm{d}y) = 0, \quad \langle\theta_j, \theta_j\rangle = \nu(A_j)(1 + c_{j+1}).$$

We also have

$$\langle\theta_j, \theta_l\rangle = \begin{cases} -\nu(A_l), & j = l+1; \\ 0, & \text{otherwise, when } j \neq l, \end{cases}$$

and

$$\eta_j\theta_l = \begin{cases} \eta_l, & j = l; \\ -c_{l+1}\eta_{l+1}, & j = l+1; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\sigma(\phi)^2 = \sum_{l=1}^{K-2}\sum_{j=1}^{K-2}\left(2\Delta_l^0 - \alpha_l^0\right)\alpha_j^0\langle\theta_l, \theta_j\rangle + \sum_{l=1}^{K-1}\left(2\Delta_l^1 - \alpha_l^1\right)\alpha_l^1\langle\eta_l, \eta_l\rangle. \tag{5.13}$$

Since $\langle\theta_j,\theta_l\rangle = 0$ for $|j-l|>1$, the first term in (5.13) is equal to

$$\left(2\Delta_1^0-\alpha_1^0\right)\left(\alpha_1^0\langle\theta_1,\theta_1\rangle+\alpha_2^0\langle\theta_1,\theta_2\rangle\right)+\sum_{l=2}^{K-3}(2\Delta_l^0-\alpha_l^0)\sum_{j=l-1}^{l+1}\alpha_j^0\langle\theta_l,\theta_j\rangle$$
$$+\left(2\Delta_{K-2}^0-\alpha_{K-2}^0\right)\left(\alpha_{K-2}^0\langle\theta_{K-2},\theta_{K-2}\rangle+\alpha_{K-3}^0\langle\theta_{K-2},\theta_{K-3}\rangle\right).$$

Note that

$$\alpha_1^0\langle\theta_1,\theta_1\rangle+\alpha_2^0\langle\theta_1,\theta_2\rangle=\alpha_1^0\nu(A_1)(1+c_2)-\alpha_2^0\nu(A_1)=\nu(A_1)(e_1-e_2),$$

where we have used the formula (5.11) for e_1, e_2. Similarly, we have

$$\sum_{j=l-1}^{l+1}\alpha_j\langle\theta_l,\theta_j\rangle=\nu(A_l)(e_l-e_{l+1})$$

for $2\leqslant l\leqslant K-3$ and, since $\alpha_{K-2}c_{K-1}=-e_{K-1}$, we also have

$$\alpha_{K-2}\langle\theta_{K-2},\theta_{K-2}\rangle+\alpha_{K-3}\langle\theta_{K-2},\theta_{K-3}\rangle=\nu(A_{K-2})(e_{K-2}-e_{K-1}),$$

which completes the proof of the first part.

For the proof of (5.12), set

$$\Lambda=\sum_{j=1}^{K-2}\left(\alpha_j^0-\beta_j^0\right)\nu(A_j)=\sum_{j=1}^{K-2}\tau_j\nu(A_j),$$

where $\tau_j=\alpha_j^0-\beta_j^0$, and observe that we defined α_{K-1}^0 in (5.11) to be equal to 0. From (5.10) we also have $\beta_{K-1}^0=0$. Thus we can set $\tau_{K-1}=0$. Substituting (5.9) into the right hand side of (5.12) gives

$$\sum_{l=1}^{K-2}(e_l-e_{l+1})\sum_{j=1}^{l}\tau_j\nu(A_j)-\Lambda\sum_{l=1}^{K-2}(e_l-e_{l+1})\nu\left(\bigcup_{j=1}^{l}A_j\right)$$

which can be rewritten as

$$\sum_{l=1}^{K-2}\left(e_l\sum_{j=1}^{l}\tau_j\nu(A_j)-e_{l+1}\sum_{j=1}^{l+1}\tau_j\nu(A_j)+e_{l+1}\tau_{l+1}\nu(A_{l+1})\right)-\Lambda\sum_{l=1}^{K-2}\left(e_l\nu\left(\bigcup_{j=1}^{l}A_j\right)-e_{l+1}\nu\left(\bigcup_{j=1}^{l+1}A_j\right)+e_{l+1}\nu(A_{l+1})\right)$$
$$=\sum_{l=1}^{K-1}e_l\tau_l\nu(A_l)-e_{K-1}\sum_{j=1}^{K-1}\tau_j\nu(A_j)-\Lambda\sum_{l=1}^{K-1}e_l\nu(A_l)+\Lambda e_{K-1}.$$

By (4.3) and (5.11) we have $e_l\nu(A_l)=\alpha_l^0\nu(A_l)-\alpha_{l-1}^0\nu(A_{l-1})$ for $l\geqslant 2$. Thus

$$\sum_{l=1}^{K-1}e_l\nu(A_l)=\alpha_{K-1}^0\nu(A_{K-1})=0.$$

Since $\sum_{j=1}^{K-1}\tau_j\nu(A_j)=\Lambda$, the proof is complete. □

6. Calculation of the variance for some specific observables

Example 1. In connection with the Ornstein–Weiss formula [14] for the measure theoretic entropy, see also [3,8], the deviation of the entropy from its average value, $\phi=\log|T'|-\int\log|T'(y)|\nu(\mathrm{d}y)$, appears as described in [2]. Thus we first examine the variance of the function

$$\phi_0(y)=\log|T'(y)|-\lambda,$$

where λ is the Lyapunov exponent for T. We have

$$\lambda=\int\log|T'(y)|\nu(\mathrm{d}y)=\nu\left(\bigcup_{j=1}^{K-2}A_j\right)\log a+\nu(A_{K-1})\log b. \tag{6.1}$$

Let

$$\alpha_l^0 = \begin{cases} \log a - \lambda, & l = 1, \\ c_l \alpha_{l-1}^0 + \log a - \lambda, & l = 2, \ldots, K-2, \\ c_{K-1}\alpha_{K-2}^0 + \log b - \lambda, & l = K-1. \end{cases} \tag{6.2}$$

From (4.3) and (6.1) it follows that

$$\alpha_l^0 v(A_l) = \alpha_{l-1}^0 v(A_{l-1}) + (\log a - \lambda) v(A_l), \quad l = 2, \ldots, K-2.$$

Thus

$$\alpha_l^0 v(A_l) = (\log a - \lambda) v\left(\bigcup_{j=1}^{l} A_j\right), \; l = 1, \ldots, K-2$$

and $\alpha_{K-1}^0 = 0$. Hence

$$\phi_0(y) = \sum_{l=1}^{K-2} \alpha_l^0 \theta_l.$$

Using Proposition 1 and Lemma 2 we obtain

$$\sigma(\phi_0)^2 = 2\sum_{l=1}^{K-2} e_l \alpha_l^0 v(A_l) - \sum_{l=1}^{K-2} \alpha_l^0 v(A_l)(e_l - e_{l+1}),$$

where $e_l = \log a - \lambda$ for $l = 1, \ldots, K-2$ and $e_{K-1} = \log b - \lambda$. Thus

$$\sigma(\phi_0)^2 = 2(\log a - \lambda)\sum_{l=1}^{K-2} \alpha_l^0 v(A_l) - \alpha_{K-2}^0 v(A_{K-2}) \log\frac{a}{b}.$$

Since

$$\log a - \lambda = v(A_{K-1}) \log\frac{a}{b},$$

we conclude that

$$\sigma(\phi_0)^2 = v(A_{k-1})^2 \left(\log\frac{a}{b}\right)^2 \left(2\sum_{l=1}^{K-1} v\left(\bigcup_{j=1}^{l} A_j\right) - 1 - \frac{1}{v(A_{K-1})}\right).$$

For the case of $a \neq 1$, from (3.11) it follows that

$$\sum_{l=1}^{K-1} v\left(\bigcup_{j=1}^{l} A_j\right) = \sum_{l=1}^{K-1} \frac{1}{a-1}(av(A_l) - lv(A_1)) = \frac{a}{a-1} - \frac{(K-1)Kv(A_1)}{2(a-1)}, \tag{6.3}$$

and for $a = 1$ we have

$$\sum_{l=1}^{K-1} v\left(\bigcup_{j=1}^{l} A_j\right) = (b-1)d_1 \sum_{l=1}^{K-1} \frac{l(l+1)}{2} = \frac{K+1}{3}. \tag{6.4}$$

Example 2. Now consider

$$\phi(y) = y - \mathfrak{m}, \; y \in Y,$$

where

$$\mathfrak{m} = \sum_{l=1}^{K-1} \int_{A_l} y v(\mathrm{d}y) = \sum_{l=1}^{K-1} v_l v(A_l). \tag{6.5}$$

It follows directly from (3.11), (3.13) and (4.6), that

$$\mathfrak{m} = \begin{cases} \frac{b(a+b)(b-1)^2 d_1}{2(a-1)} - \frac{b-1}{a-1}, & a \neq 1, \\ \frac{(K-2)(7-2K)}{6}, & a = 1. \end{cases} \tag{6.6}$$

First we are going to rewrite ϕ as a function from $\mathscr{L}$. We have

$$\phi = \sum_{l=1}^{K-1} \eta_l + \sum_{l=1}^{K-1} (v_l - \mathfrak{m}) 1_{A_l}.$$

Define

$$\alpha_l = \begin{cases} v_1 - \mathfrak{m}, & l = 1, \\ c_l \alpha_{l-1} + v_l - \mathfrak{m}, & l = 2, \ldots, K-1. \end{cases} \tag{6.7}$$

From (4.3) and (6.5) it follows that $\alpha_{K-1} = 0$. Thus

$$\phi = \sum_{l=1}^{K-1} \eta_l + \sum_{l=1}^{K-2} \alpha_l \theta_l. \tag{6.8}$$

Proposition 1 leads to

$$\sigma(\phi)^2 = \frac{(b-1)(a - 7a^2 + 3b - 4ab + a^2 b)}{6(a-1)^2} \left(\mathfrak{m} + \frac{b-1}{a-1} \right) + \frac{2a + 1 - (K-1)K\nu(A_1)}{a-1} \left(\mathfrak{m} + \frac{b-1}{a-1} \right)^2 \quad \text{for} \quad a \neq 1 \tag{6.9}$$

and

$$\sigma(\phi)^2 = \frac{(K-2)^3 (4K^2 - 13K + 13)}{3^3 \cdot 20} \quad \text{for} \quad a = 1. \tag{6.10}$$

The detailed calculations are given in Appendix B.

Example 3. Again consider $\phi(y) = y - \mathfrak{m}$ and assume that $K = 3$. Then

$$b = \frac{a+1}{a}, \quad \nu(A_1) = \frac{1}{a+2}, \quad \nu(A_2) = \frac{a+1}{a+2}.$$

We have

$$\mathfrak{m} = \frac{a^2 + a - 1}{2a^2(a+2)}.$$

and

$$\sigma(\phi)^2 = \frac{(a+1)(a + a^2 + 1)}{12a^3(a+2)^3}.$$

The Lyapunov exponent is equal to

$$\lambda = \frac{1}{a+2} \log a + \frac{a+1}{a+2} \log \frac{a+1}{a}$$

and

$$\sigma(\phi_0)^2 = \frac{a(a+1)}{(a+2)^3} \left(\log \frac{a^2}{a+1} \right)^2.$$

Note that $\sigma(\phi_0) = 0$ iff $a = \left(1 + \sqrt{5}\right)/2$. Let us write

$$a = \frac{1-\alpha}{\alpha}, \ \alpha \in (0, 1).$$

The graphs of $\sigma(\phi)$ and $\sigma(\phi_0)$ as functions of α are shown in Fig. 1. In this case $T_{a,b}$ is conjugated to the map $F_{\alpha,\alpha}$ in (2.4), studied in [13].

Example 4. When $\phi(y) = y - \mathfrak{m}$ and $b = a$ we have $a > 1$ and we must have $a \in \left(\sqrt{2}, 2\right]$ since $(a, b) \in D^*$. For a given $K \geqslant 3$, $a = s_K$ is a solution of the equation

$$s^K - 2s^{K-1} + 1 = 0.$$

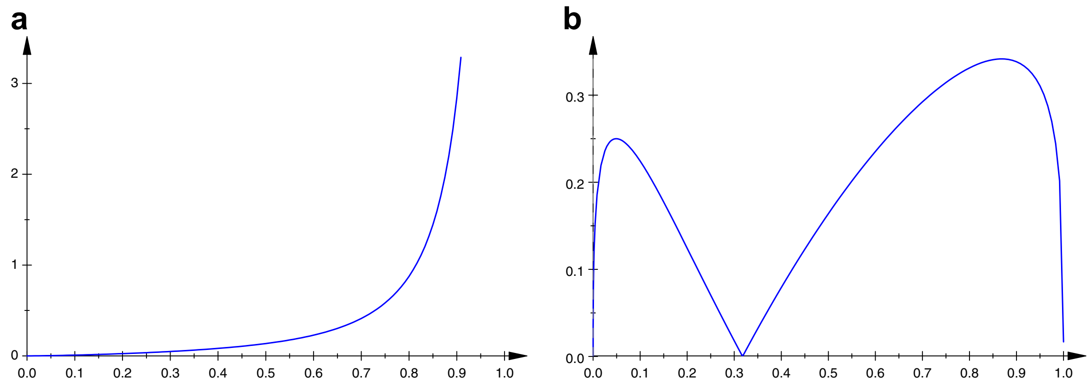

Fig. 1. Plots of σ as a function of α for Example 3 (a) $\sigma(\phi)$ as a function of $\alpha \in (0,1)$ (b) $\sigma(\phi_0)$ as a function of $\alpha \in (0,1)$.

Thus the sequence (s_K) is increasing and converges to 2 as $K \to \infty$. The first two terms are equal to

$$s_3 = \frac{1+\sqrt{5}}{2}, \quad s_4 = \frac{1+\sqrt[3]{19-3\sqrt{33}}+\sqrt[3]{19+3\sqrt{33}}}{3}.$$

Let σ_K denote the variance for the function ϕ as in Example 2 with $a = s_K$. From (6.9) it follows that

$$\sigma_K^2 = s_K^3 d_1 \left(\frac{1}{6}(s_K^2 - 11 s_K + 4) + (2 s_K + 1) v(A_{K-1}) - K(K-1) v(A_1) v(A_{K-1})\right).$$

Since $v(A_{K-1}) \to 1/2$ and $K(K-1)v(A_1) \to 0$ as $K \to \infty$, we have $\sigma_K^2 \to 1/3$, which is the variance for $\phi(y) = y$ in the case of the symmetric tent map $T(y) = 1 - 2|y|$ on the interval $[-1, 1]$.

If a is in the interval $\left(1, \sqrt{2}\right]$ then there is a band splitting [15] and T is no longer exact. In [9] it is shown how to relate the variance of ϕ for $a \in (2^{1/2^{m+1}}, 2^{1/2^m}]$ to the variance of ϕ for a^{2^m}, which belongs to $\left(\sqrt{2}, 2\right]$.

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Appendix A. A matrix formula

Let γ_i, δ_i, $i = 1, \ldots, n$ be given. Consider a square $n \times n$-matrix of the form

$$\widetilde{C} = \begin{pmatrix} 0 & & & & -\delta_1 \\ \gamma_2 & 0 & & & -\delta_2 \\ & \gamma_3 & \ddots & & \vdots \\ & & \ddots & 0 & -\delta_{n-1} \\ & & & \gamma_n & -\delta_n \end{pmatrix}. \tag{A.1}$$

Then

$$\det(I - \widetilde{C}) = 1 + \sum_{j=1}^{n} \delta_j \prod_{i=j+1}^{n} \gamma_i,$$

where I is the identity matrix of dimension n and the matrix $(I - \widetilde{C})^{-1}$ is given by

$$\begin{pmatrix} 1-s_1 & -\frac{s_1}{\gamma_2} & -\frac{s_1}{\gamma_2\gamma_3} & & & -\frac{s_1}{\prod_{i=2}^{n}\gamma_i} \\ (1-s_2)\gamma_2 & 1-s_2 & -\frac{s_2}{\gamma_3} & & & -\frac{s_2}{\prod_{i=3}^{n}\gamma_i} \\ (1-s_3)\gamma_2\gamma_3 & (1-s_3)\gamma_3 & 1-s_3 & -\frac{s_3}{\gamma_4} & & -\frac{s_3}{\prod_{i=4}^{n}\gamma_i} \\ \vdots & & & & & \vdots \\ (1-s_{n-1})\prod_{i=2}^{n-1}\gamma_i & (1-s_{n-1})\prod_{i=3}^{n-1}\gamma_i & & & 1-s_{n-1} & -\frac{s_{n-1}}{\gamma_n} \\ (1-s_n)\prod_{i=2}^{n}\gamma_i & (1-s_n)\prod_{i=3}^{n}\gamma_i & & & (1-s_n)\gamma_n & 1-s_n \end{pmatrix},$$

where

$$s_l = \frac{\sum_{j=1}^{l}\delta_j\prod_{i=j+1}^{n}\gamma_i}{\det(I-\widetilde{C})}. \tag{A.2}$$

Let $\tau = (\tau_1,\ldots,\tau_n)^T$ be an arbitrary vector. Then

$$(I-\widetilde{C})^{-1}\begin{pmatrix}\tau_1\\ \tau_2\\ \vdots\\ \tau_n\end{pmatrix} = \begin{pmatrix}\Delta_1\\ \Delta_2\\ \vdots\\ \Delta_n\end{pmatrix}, \tag{A.3}$$

where

$$\Delta_l = (1-s_l)\sum_{j=1}^{l}\tau_j\prod_{i=1+j}^{l}\gamma_i - s_l\sum_{j=l+1}^{n}\frac{\tau_j}{\prod_{i=1+l}^{j}\gamma_i}$$

which can be rewritten as

$$\Delta_l = \frac{\sum_{j=1}^{l}\tau_j\prod_{i=1+j}^{n}\gamma_i}{\prod_{i=1+l}^{n}\gamma_i} - \frac{s_l\sum_{j=1}^{n}\tau_j\prod_{i=1+j}^{n}\gamma_i}{\prod_{i=1+l}^{n}\gamma_i}. \tag{A.4}$$

Appendix B. Calculation of $\sigma(\phi)^2$ in Example 2

Lemma 1 with $\alpha_j^1 \equiv 1$ yields

$$\Delta_l^1 = \frac{d_l}{2d_1}. \tag{B.1}$$

Making use of the relations (3.4), (3.11), (3.13), and

$$(2\Delta_l^1 - 1)\mu(A_l) = \frac{v(A_{l-1})}{ad_1},$$

we obtain

$$\frac{1}{12}\sum_{l=1}^{K-1}(2\Delta_l^1-1)v(A_l)\mu(A_l)^2 = \frac{1}{12d_1}\sum_{l=2}^{K-1}v(A_{l-1})v(A_l)\mu(A_l) = \frac{ab(a+b)(b-1)^4 d_1}{12(a^2+a+1)}. \tag{B.2}$$

From Proposition 1 it follows that

$$\sum_{l=1}^{K-2}(2\Delta_l^0-\alpha_l)v(A_l)(e_l-e_{l+1})=2\sum_{l=1}^{K-2}e_l(\alpha_l-\beta_l^0)v(A_l)-\sum_{l=1}^{K-2}\alpha_l v(A_l)(e_l-e_{l+1}),$$

where $\beta_l^0=\beta_l/2$ since $\Delta_{K-1}^1=b/2$, by (3.7) and (B.1), and the e_l defined in (5.11) are given by $e_l=v_l-\mathfrak{m}$. We have $e_l-e_{l+1}=v_l-v_{l+1}$ and from (4.6) we obtain

$$v_l-v_{l+1}=-\frac{1}{2}(b-1)(a+1)a^{l-K+1}. \tag{B.3}$$

Thus, if $a\neq 1$ then

$$e_l=v_l-\mathfrak{m}=-\frac{1}{a-1}(v_l-v_{l+1})-\left(\mathfrak{m}+\frac{b-1}{a-1}\right).$$

Consequently,

$$\sum_{l=1}^{K-2}(2\Delta_l^0-\alpha_l)v(A_l)(e_l-e_{l+1})=-\frac{a+1}{a-1}\Lambda_1+\frac{1}{a-1}\Lambda_2-\left(\mathfrak{m}+\frac{b-1}{a-1}\right)\Lambda_0, \tag{B.4}$$

where

$$\Lambda_0=\sum_{l=1}^{K-2}(2\alpha_l-\beta_l)v(A_l), \tag{B.5}$$

$$\Lambda_1=\sum_{l=1}^{K-2}\alpha_l v(A_l)(v_l-v_{l+1}), \tag{B.6}$$

$$\Lambda_2=\sum_{l=1}^{K-2}\beta_l v(A_l)(v_l-v_{l+1}). \tag{B.7}$$

To compute Λ_1 as defined in (B.6), note that from (6.7) we obtain

$$\alpha_l v(A_l)=\sum_{j=1}^{l}v(A_j)\left(v_j+\frac{b-1}{a-1}\right)-\left(\mathfrak{m}+\frac{b-1}{a-1}\right)v\left(\bigcup_{j=1}^{l}A_j\right),$$

which, by (4.6), leads to

$$\alpha_l v(A_l)=\frac{b-1}{2(a-1)}(a+1)a^{1-K}\sum_{j=1}^{l}a^j v(A_j)-\left(\mathfrak{m}+\frac{b-1}{a-1}\right)v\left(\bigcup_{j=1}^{l}A_j\right). \tag{B.8}$$

By (3.11), we have

$$\sum_{j=1}^{l}a^j v(A_j)=\frac{a(a^{l+1}-1)v(A_l)}{a^2-1},\ \ l=1,\dots,K-1. \tag{B.9}$$

Consequently,

$$\alpha_l v(A_l)=\frac{b-1}{2(a-1)^2}a^{2-K}(a^{l+1}-1)v(A_l)-\left(\mathfrak{m}+\frac{b-1}{a-1}\right)v\left(\bigcup_{j=1}^{l}A_j\right). \tag{B.10}$$

From (B.10) and (B.3) we obtain

$$\sum_{l=1}^{K-2}\alpha_l v(A_l)(v_l-v_{l+1})=-\frac{(b-1)^2(a+1)}{4(a-1)^2}a^{3-2K}\sum_{l=1}^{K-2}a^l(a^{l+1}-1)v(A_l)-\left(\mathfrak{m}+\frac{b-1}{a-1}\right)\sum_{l=1}^{K-2}v\left(\bigcup_{j=1}^{l}A_j\right)(v_l-v_{l+1}).$$

We have

$$\begin{aligned}\sum_{l=1}^{K-2}a^l(a^{l+1}-1)v(A_l)&=\frac{(b-1)a^{2-K}a(a^{K-2}-1)(a^{K-1}-1)(a^K-1)d_1}{(a-1)^2(a^2+a+1)}=\frac{a^{2K-3}(a-1)b(a+b)(b-1)^2d_1}{a^2+a+1}\\&=\frac{2a^{2K-3}(a-1)^2}{a^2+a+1}\left(\mathfrak{m}+\frac{b-1}{a-1}\right)\end{aligned}$$

and

$$\sum_{l=1}^{K-2} v\left(\bigcup_{j=1}^{l} A_j\right)(v_l - v_{l+1}) = \sum_{l=1}^{K-1} v(A_l)v_l - v_{K-1}\sum_{l=1}^{K-1} v(A_l) = \mathfrak{m} - v_{K-1} = \mathfrak{m} + \frac{b-1}{a-1} - \frac{(b-1)(a+1)}{2(a-1)}.$$

Consequently,

$$\Lambda_1 = -\left(\frac{(a+1)(b-1)^2}{2(a^2+a+1)} - \frac{(b-1)(a+1)}{2(a-1)}\right)\left(\mathfrak{m} + \frac{b-1}{a-1}\right) - \left(\mathfrak{m} + \frac{b-1}{a-1}\right)^2. \tag{B.11}$$

Next we compute Λ_2 as defined in (B.7). From (4.10) and (B.3) we obtain

$$\begin{aligned}\sum_{l=1}^{K-2} \beta_l v(A_l)(v_l - v_{l+1}) &= -\frac{a(b-1)^2(a+1)}{4(a-1)} \sum_{l=1}^{K-2} \left(a^{l-K+1} - 1\right)a^{l-K+1} v(A_l) \\ &= -\frac{(a+1)(b-1)^3 d_1}{4(a-1)^2 a^{3K-5}} \sum_{l=1}^{K-2} \left(a^l - a^{K-1}\right)\left(a^{2l} - a^l\right) \\ &= \frac{(a+1)(b-1)^3 d_1 a^2 (a^{K-2}-1)(a^{K-1}-1)(a^K-1)}{4(a-1)^2 a^{3K-5}(a^2-1)(a^2+a+1)} = \frac{ab(a+b)(b-1)^4 d_1}{4(a^2+a+1)},\end{aligned}$$

which gives

$$\Lambda_2 = \frac{a(a-1)(b-1)^2}{2(a^2+a+1)}\left(\mathfrak{m} + \frac{b-1}{a-1}\right). \tag{B.12}$$

It only remains to compute Λ_0 as defined in (B.5). To do this observe that

$$\Lambda_0 = 2\sum_{l=1}^{K-1} \alpha_l v(A_l) - \sum_{l=1}^{K-1} \beta_l v(A_l)$$

since $\alpha_{K-1} = \beta_{K-1} = 0$. By (4.10) and (B.9), we have

$$\begin{aligned}\sum_{l=1}^{K-1} \beta_l v(A_l) &= \frac{b-1}{2(a-1)}\left(a^{2-K}\sum_{l=1}^{K-1} a^l v(A_l) - a\right) = \frac{b-1}{2(a-1)}\left(a^{2-K}\frac{a(a^K-1)v(A_{K-1})}{a^2-1} - a\right) \\ &= \frac{b-1}{2(a-1)}\left(\frac{ab(a+b)(b-1)d_1}{a+1} - a\right),\end{aligned}$$

which leads to

$$\sum_{l=1}^{K-1} \beta_l v(A_l) = \frac{a}{a+1}\left(\mathfrak{m} + \frac{b-1}{a-1}\right) - \frac{a(b-1)}{2(a-1)}. \tag{B.13}$$

From (B.10) it follows that

$$\sum_{l=1}^{K-1} \alpha_l v(A_l) = \frac{(b-1)a^{2-K}}{2(a-1)^2}\left(a\sum_{l=1}^{K-1} a^l v(A_l) - 1\right) - \left(\mathfrak{m} + \frac{b-1}{a-1}\right)\sum_{l=1}^{K-1} v\left(\bigcup_{j=1}^{l} A_j\right).$$

We have

$$\frac{(b-1)a^{2-K}}{2(a-1)^2} a \sum_{l=1}^{K-1} a^l v(A_l) = \frac{(b-1)a^{2-K}a^2(a^K-1)v(A_{K-1})}{2(a-1)^2(a^2-1)} = \frac{a^2 b(a+b)(b-1)^2 d_1}{2(a-1)(a^2-1)} = \frac{a^2}{a^2-1}\left(\mathfrak{m} + \frac{b-1}{a-1}\right).$$

This and (6.3) imply that

$$\sum_{l=1}^{K-1} \alpha_l v(A_l) = \frac{a^2}{a^2-1}\left(\mathfrak{m} + \frac{b-1}{a-1}\right) - \frac{(b-1)(a+b-ab)}{2(a-1)^2} - \left(\frac{a}{a-1} - \frac{(K-1)Kv(A_1)}{2(a-1)}\right)\left(\mathfrak{m} + \frac{b-1}{a-1}\right),$$

which combined with (B.13) gives

$$\Lambda_0 = -\left(\frac{a-(K-1)Kv(A_1)}{a-1}\right)\left(\mathfrak{m} + \frac{b-1}{a-1}\right) + \frac{a(b-1)}{2(a-1)} - \frac{(b-1)(a+b-ab)}{(a-1)^2}. \tag{B.14}$$

Substituting Λ_0, Λ_1, and Λ_2 into the right-hand side of (B.4) we obtain

$$-\frac{a+1}{a-1}\Lambda_1 + \frac{1}{a-1}\Lambda_2 - \Lambda_0\left(\mathfrak{m} + \frac{b-1}{a-1}\right) = \frac{a+1}{a-1}\left(\mathfrak{m} + \frac{b-1}{a-1}\right)^2 + \left(\frac{(a+1)^2(b-1)^2}{2(a-1)(a^2+a+1)} - \frac{(a+1)^2(b-1)}{2(a-1)^2}\right) \times \left(\mathfrak{m} + \frac{b-1}{a-1}\right) + \frac{a(b-1)^2}{2(a^2+a+1)}\left(\mathfrak{m} + \frac{b-1}{a-1}\right) + \frac{a-(K-1)Kv(A_1)}{a-1}\left(\mathfrak{m} + \frac{b-1}{a-1}\right)^2 + \left(\mathfrak{m} + \frac{b-1}{a-1}\right) \times \left(-\frac{a(b-1)}{2(a-1)} + \frac{(b-1)(a+b-ab)}{(a-1)^2}\right).$$

After some algebra we conclude that

$$\sum_{j=1}^{K-2}\left(2\Delta_j^0 - \alpha_j\right)v(A_j)(v_j - v_{j+1}) = \frac{(a+1)(b-1)(b-ab-a^2-2a^3)}{2(a-1)^2(a^2+a+1)}\left(\mathfrak{m} + \frac{b-1}{a-1}\right) + \frac{2a+1-(K-1)Kv(A_1)}{a-1}\left(\mathfrak{m} + \frac{b-1}{a-1}\right)^2,$$

which together with (B.2) and Proposition 1 gives (6.9).

Now, if $a = 1$ then

$$v_l - \mathfrak{m} = \frac{(K-2)(3l+1-2K)}{3} \quad \text{and} \quad \alpha_l = \frac{(K-2)(l+1)(l+1-K)}{3}.$$

Hence

$$\sum_{l=1}^{K-2}\alpha_l v(A_l)(v_l - v_{l+1}) = \frac{(K-2)^3(K+1)}{18}$$

and

$$\sum_{l=1}^{K-2}(v_l - m)(2\alpha_l - \beta_l)v(A_l) = \frac{(K-2)^3(4K+13)(K+1)}{3^3 \cdot 20},$$

which together with (B.2) and Proposition 1 gives (6.10).

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