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CENTRAL LIMIT THEOREMS FOR NON-INVERTIBLE MEASURE PRESERVING MAPS

BY

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Abstract. Using the Perron–Frobenius operator we establish a new functional central limit theorem for non-invertible measure preserving maps that are not necessarily ergodic. We apply the result to asymptotically periodic transformations and give a specific example using the tent map.

1. Introduction. This paper is motivated by the question "How can we produce the characteristics of a Wiener process (Brownian motion) from a semidynamical system?". This question is intimately connected with central limit theorems for non-invertible maps and various invariance principles. Many results on central limit theorems and invariance principles for maps have been proved (see e.g. the surveys by Denker [5] and Mackey and Tyran-Kamińska [17]). These results extend back over some decades, and include the work of Boyarsky and Scarowsky [3], Gouëzel [8], Jabłoński and Malczak [12], Rousseau-Egele [25], and Wong [32] for the special case of maps of the unit interval. Martingale approximations, developed by Gordin [7], were used by Keller [13], Liverani [16], Melbourne and Nicol [19], Melbourne and Török [20], and Tyran-Kamińska [27] to give more general results.

Throughout this paper, $(Y,\mathcal{B},\nu)$ denotes a probability measure space and $T: Y \to Y$ a non-invertible measure preserving transformation. Thus ν is invariant under T, i.e. $\nu(T^{-1}(A)) = \nu(A)$ for all $A \in \mathcal{B}$. The transfer operator $\mathcal{P}_T : L^1(Y,\mathcal{B},\nu) \to L^1(Y,\mathcal{B},\nu)$, by definition, satisfies

$$\int \mathcal{P}_T f(y) g(y)\, \nu(dy) = \int f(y) g(T(y))\, \nu(dy)$$

for all $f \in L^1(Y,\mathcal{B},\nu)$ and $g \in L^\infty(Y,\mathcal{B},\nu)$.

Let $h \in L^2(Y,\mathcal{B},\nu)$ with $\int h(y)\,\nu(dy) = 0$. Define the process $\{w_n(t) : t \in [0,1]\}$ by

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$$w_n(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{[nt]-1} h \circ T^j \quad \text{for } t \in [0,1],\ n \geq 1 \tag{1.1}$$

(the sum from 0 to -1 is set equal to 0), where $[x]$ denotes the integer part of x. For each y, $w_n(\cdot)(y)$ is an element of the Skorokhod space $D[0,1]$ of all functions which are right continuous and have left-hand limits, equipped with the Skorokhod metric

$$\varrho_S(\psi, \widetilde{\psi}) = \inf_{s \in \mathcal{S}} \Big(\sup_{t \in [0,1]} |\psi(t) - \widetilde{\psi}(s(t))| + \sup_{t \in [0,1]} |t - s(t)| \Big), \qquad \psi, \widetilde{\psi} \in D[0,1],$$

where $\mathcal{S}$ is the family of strictly increasing, continuous mappings s of $[0,1]$ onto itself such that $s(0) = 0$ and $s(1) = 1$ [1, Section 14].

Let $\{w(t) : t \in [0,1]\}$ be a standard Brownian motion. Throughout the paper the notation

$$w_n \to^d \sqrt{\eta}\, w,$$

where η is a random variable independent of the Brownian process w, denotes the weak convergence of the sequence w_n in the Skorokhod space $D[0,1]$.

Our main result, which is proved using techniques similar to those of Peligrad and Utev [22] and Peligrad *et al.* [23], is the following:

THEOREM 1. *Let T be a non-invertible measure preserving transformation on the probability space $(Y, \mathcal{B}, \nu)$ and let $\mathcal{I}$ be the σ-algebra of all T-invariant sets. Suppose $h \in L^2(Y, \mathcal{B}, \nu)$ with $\int h(y)\,\nu(dy) = 0$ is such that*

$$\sum_{n=1}^{\infty} n^{-3/2} \Big\| \sum_{k=0}^{n-1} \mathcal{P}_T^k h \Big\|_2 < \infty. \tag{1.2}$$

Then

$$w_n \to^d \sqrt{\eta}\, w, \tag{1.3}$$

where $\eta = E_\nu(\widetilde{h}^2 \,|\, \mathcal{I})$ and $\widetilde{h} \in L^2(Y, \mathcal{B}, \nu)$ is such that $\mathcal{P}_T \widetilde{h} = 0$ and

$$\lim_{n \to \infty} \Big\| \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \widetilde{h}) \circ T^j \Big\|_2 = 0.$$

Recall that T is *ergodic* (with respect to ν) if, for each $A \in \mathcal{B}$ with $T^{-1}(A) = A$, we have $\nu(A) \in \{0,1\}$. Thus if T is ergodic then $\mathcal{I}$ is a trivial σ-algebra, so η in (1.3) is a constant random variable. Consequently, Theorem 1 significantly generalizes [27, Theorem 4], where it was assumed that T is ergodic and there is $\alpha < 1/2$ such that

$$\Big\| \sum_{k=0}^{n-1} \mathcal{P}_T^k h \Big\|_2 = O(n^\alpha)$$

(we use the notation $b(n) = O(a(n))$ if $\limsup_{n \to \infty} b(n)/a(n) < \infty$).

Usually, in proving central limit theorems for specific examples of transformations one assumes that the transformation is mixing. For non-invertible ergodic transformations for which the transfer operator is quasi-compact on some subspace $F \subset L^2(\nu)$ with norm $|\cdot| \geq \|\cdot\|_2$, the central limit theorem and its functional version was given in Melbourne and Nicol [19]. Since quasicompactness implies exponential decay of the L^2 norm, our result applies, thus extending the results of [19] to the non-ergodic case. For examples of transformations in which the decay of the L^2 norm is slower than exponential and our results apply, see [27].

In the case of invertible transformations, non-ergodic versions of the central limit theorem and its functional generalizations were studied by Volný [28–31] using martingale approximations. In a recent review by Merlevède *et al.* [21], the weak invariance principle was studied for stationary sequences $(X_k)_{k\in\mathbb{Z}}$ which, in particular, can be described as $X_k = X_0 \circ T^k$, where T is a measure preserving invertible transformation on a probability space and X_0 is measurable with respect to a σ-algebra $\mathcal{F}_0$ such that $\mathcal{F}_0 \subset T^{-1}(\mathcal{F}_0)$. Choosing a σ-algebra $\mathcal{F}_0$ for a specific example of invertible transformation is not an easy task and the requirement that X_0 is $\mathcal{F}_0$-measurable may sometimes be too restrictive (see [4, 16]). Sometimes, it is possible to reduce an invertible transformation to a non-invertible one (see [20, 27]). Our result in the non-invertible case extends [22, Theorem 1.1], which is also to be found in [21, Theorem 11], where a condition introduced by Maxwell and Woodroofe [18] is assumed. In [27] the condition was transformed to equation (1.2). In the proof of our result we use Theorem 4.2 in Billingsley [1] and approximation techniques which were motivated by [22]. The corresponding maximal inequality in our non-invertible setting is stated in Proposition 1, and its proof, based on ideas of [23], is provided in Appendix 4.4 for completeness. As in [22], the random variable η in Theorem 1 can also be obtained as a limit in L^1, which we state in Appendix 4.4.

The outline of the paper is as follows. After the presentation of some background material in Section 2, we turn to a proof of our main Theorem 1 in Section 3. Section 4 introduces asymptotically periodic transformations as a specific example of a system to which Theorem 1 applies. We analyze the specific example of an asymptotically periodic family of tent maps in Section 4.4.

2. Preliminaries. The definition of the Perron–Frobenius (transfer) operator for T depends on a given σ-finite measure μ on the measure space $(Y, \mathcal{B})$ with respect to which T is non-singular, i.e. $\mu(T^{-1}(A)) = 0$ for all $A \in \mathcal{B}$ with $\mu(A) = 0$. Given such a measure the *transfer operator* $P : L^1(Y, \mathcal{B}, \mu) \to L^1(Y, \mathcal{B}, \mu)$ is defined as follows. For any $f \in L^1(Y, \mathcal{B}, \mu)$, there is a unique element Pf in $L^1(Y, \mathcal{B}, \mu)$ such that

$$\int_A Pf(y)\,\mu(dy) = \int_{T^{-1}(A)} f(y)\,\mu(dy) \quad \text{for all } A \in \mathcal{B}. \tag{2.1}$$

This in turn gives rise to different operators for different underlying measures on $\mathcal{B}$. Thus if ν is invariant for T, then T is non-singular and the transfer operator $\mathcal{P}_T : L^1(Y, \mathcal{B}, \nu) \to L^1(Y, \mathcal{B}, \nu)$ is well defined. Here we write $\mathcal{P}_T$ to emphasize that the underlying measure ν is invariant under T.

The *Koopman operator* is defined by

$$U_T f = f \circ T$$

for every measurable $f : Y \to \mathbb{R}$. In particular, U_T is also well defined for $f \in L^1(Y, \mathcal{B}, \nu)$ and is an isometry of $L^1(Y, \mathcal{B}, \nu)$ into $L^1(Y, \mathcal{B}, \nu)$, i.e. $\|U_T f\|_1 = \|f\|_1$ for all $f \in L^1(Y, \mathcal{B}, \nu)$. Since the measure ν is finite, we have $L^p(Y, \mathcal{B}, \nu) \subset L^1(Y, \mathcal{B}, \nu)$ for $p \geq 1$. The operator $U_T : L^p(Y, \mathcal{B}, \nu) \to L^p(Y, \mathcal{B}, \nu)$ is also an isometry on $L^p(Y, \mathcal{B}, \nu)$.

The following relations hold between the operators $U_T, \mathcal{P}_T \colon L^1(Y, \mathcal{B}, \nu) \to L^1(Y, \mathcal{B}, \nu)$:

$$\mathcal{P}_T U_T f = f \quad \text{and} \quad U_T \mathcal{P}_T f = E_\nu(f \,|\, T^{-1}(\mathcal{B})) \tag{2.2}$$

for $f \in L^1(Y, \mathcal{B}, \nu)$, where $E_\nu(\cdot \,|\, T^{-1}(\mathcal{B})) : L^1(Y, \mathcal{B}, \nu) \to L^1(Y, T^{-1}(\mathcal{B}), \nu)$ is the operator of conditional expectation. Note that if the transformation T is invertible then $U_T \mathcal{P}_T f = f$ for $f \in L^1(Y, \mathcal{B}, \nu)$.

THEOREM 2. *Let T be a non-invertible measure preserving transformation on the probability space $(Y, \mathcal{B}, \nu)$ and let $\mathcal{I}$ be the σ-algebra of all T-invariant sets. Suppose that $h \in L^2(Y, \mathcal{B}, \nu)$ is such that $\mathcal{P}_T h = 0$. Then*

$$w_n \to^d \sqrt{\eta}\, w,$$

where $\eta = E_\nu(h^2 \,|\, \mathcal{I})$ is a random variable independent of the Brownian motion $\{w(t) : t \in [0,1]\}$.

Proof. When T is ergodic, a direct proof based on the fact that the family

$$\left\{ T^{-n+j}(\mathcal{B}), \frac{1}{\sqrt{n}}\, h \circ T^{n-j} : 1 \leq j \leq n,\, n \geq 1 \right\}$$

is a martingale difference array is given in [17, Appendix A] and uses the martingale central limit theorem (cf. [2, Theorem 35.12]) together with the Birkhoff ergodic theorem. This can be extended to the case of non-ergodic T by using a version of the martingale central limit theorem due to Eagleson [6, Corollary p. 561]. ∎

EXAMPLE 1. We illustrate Theorem 2 with an example. Let $T : [0,1] \to [0,1]$ be defined by

$$T(y) = \begin{cases} 2y, & y \in [0, 1/4), \\ 2y - 1/2, & y \in [1/4, 3/4), \\ 2y - 1, & y \in [3/4, 1]. \end{cases}$$

Observe that the Lebesgue measure on $([0,1], \mathcal{B}([0,1]))$ is invariant for T and that T is not ergodic since $T^{-1}([0,1/2]) = [0,1/2]$ and $T^{-1}([1/2,1]) = [1/2,1]$. The transfer operator is given by

$$\mathcal{P}_T f(y) = \frac{1}{2} f\left(\frac{1}{2} y\right) 1_{[0,1/2)}(y) + \frac{1}{2} f\left(\frac{1}{2} y + \frac{1}{4}\right) + \frac{1}{2} f\left(\frac{1}{2} y + \frac{1}{2}\right) 1_{[1/2,1]}(y).$$

Consider the function

$$h(y) = \begin{cases} 1, & y \in [0, 1/4), \\ -1, & y \in [1/4, 1/2), \\ -2, & y \in [1/2, 3/4), \\ 2, & y \in [3/4, 1]. \end{cases}$$

A straightforward calculation shows that $\mathcal{P}_T h = 0$ and $E_\nu(h^2 \,|\, \mathcal{I}) = 1_{[0,1/2]} + 4 \cdot 1_{[1/2,1]}$. Thus Theorem 2 shows that

$$w_n \to^d \sqrt{E_\nu(h^2 \,|\, \mathcal{I})}\, w.$$

In particular, the one-dimensional distribution of the process $\sqrt{E_\nu(h^2 \,|\, \mathcal{I})}\, w$ has a density equal to

$$\frac{1}{2} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) + \frac{1}{2} \frac{1}{\sqrt{8\pi t}} \exp\left(-\frac{x^2}{8t}\right), \qquad x \in \mathbb{R}.$$

In general, for a given h the equation $\mathcal{P}_T h = 0$ may not be satisfied. Then the idea is to write h as a sum of two functions, one of which satisfies the assumptions of Theorem 2 while the other is irrelevant for the convergence to hold. At least a part of the conclusions of Theorem 1 is given in the following

THEOREM 3 (Tyran-Kamińska [27, Theorem 3]). *Let T be a non-invertible measure preserving transformation on the probability space $(Y, \mathcal{B}, \nu)$. Suppose $h \in L^2(Y, \mathcal{B}, \nu)$ with $\int h(y)\, \nu(dy) = 0$ is such that* (1.2) *holds. Then there exists $\widetilde{h} \in L^2(Y, \mathcal{B}, \nu)$ such that $\mathcal{P}_T \widetilde{h} = 0$ and*

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \widetilde{h}) \circ T^j \to 0$$

in $L^2(Y, \mathcal{B}, \nu)$ as $n \to \infty$.

We will use the following two results for subadditive sequences.

LEMMA 1 (Peligrad and Utev [22, Lemma 2.8]). *Let V_n be a subadditive sequence of non-negative numbers. Suppose that $\sum_{n=1}^{\infty} n^{-3/2} V_n < \infty$. Then*

$$\lim_{m\to\infty} \frac{1}{\sqrt{m}} \sum_{j=0}^{\infty} \frac{V_{m2^j}}{2^{j/2}} = 0.$$

LEMMA 2. *Let V_n be a subadditive sequence of non-negative numbers. Then for every integer $r \geq 2$ there exist two positive constants C_1, C_2 (depending on r) such that*

$$C_1 \sum_{j=0}^{\infty} \frac{V_{r^j}}{r^{j/2}} \leq \sum_{n=1}^{\infty} \frac{V_n}{n^{3/2}} \leq C_2 \sum_{j=0}^{\infty} \frac{V_{r^j}}{r^{j/2}}.$$

Proof. When $r = 2$, the result follows from Lemma 2.7 of [22], the proof of which can be easily extended to the case of arbitrary $r > 2$. ∎

3. Maximal inequality and the proof of Theorem 1. We start by first stating our key maximal inequality which is analogous to Proposition 2.3 in [22].

PROPOSITION 1. *Let n, q be integers such that $2^{q-1} \leq n < 2^q$. If T is a non-invertible measure preserving transformation on the probability space $(Y, \mathcal{B}, \nu)$ and $f \in L^2(Y, \mathcal{B}, \nu)$, then*

$$\Big\| \max_{1\leq k\leq n} \Big| \sum_{j=0}^{k-1} f \circ T^j \Big| \Big\|_2 \leq \sqrt{n}\,(3\|f - U_T \mathcal{P}_T f\|_2 + 4\sqrt{2}\,\Delta_q(f)), \tag{3.1}$$

where

$$\Delta_q(f) = \sum_{j=0}^{q-1} 2^{-j/2} \Big\| \sum_{k=1}^{2^j} \mathcal{P}_T^k f \Big\|_2. \tag{3.2}$$

In what follows we assume that T is a non-invertible measure preserving transformation on the probability space $(Y, \mathcal{B}, \nu)$.

PROPOSITION 2. *Let $h \in L^2(Y, \mathcal{B}, \nu)$. Define*

$$h_m = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} h \circ T^j \quad \text{and} \quad w_{k,m}(t) = \frac{1}{\sqrt{k}} \sum_{j=0}^{[kt]-1} h_m \circ T^{mj} \tag{3.3}$$

for $m, k \in \mathbb{N}$ and $t \in [0,1]$. Let us take an m such that the sequence $\|\max_{1\leq l\leq k} |w_{k,m}(l/k)|\,\|_2$ is bounded. Then

$$\lim_{n\to\infty} \|\sup_{0\leq t\leq 1} |w_{n,1}(t) - w_{[n/m],m}(t)|\,\|_2 = 0.$$

Proof. Let $k_n = [n/m]$. We have

$$|w_{n,1}(t) - w_{k_n,m}(t)| \le \frac{1}{\sqrt{n}} \Big| \sum_{j=m[k_n t]}^{[nt]-1} h \circ T^j \Big| + \left(\frac{1}{\sqrt{k_n}} - \frac{\sqrt{m}}{\sqrt{n}} \right) \Big| \sum_{j=0}^{[k_n t]-1} h_m \circ T^{mj} \Big|,$$

which leads to the estimate

$$\begin{aligned} (3.4) \qquad & \| \sup_{0\le t\le 1} |w_{n,1}(t) - w_{k_n,m}(t)| \, \|_2 \\ & \le \frac{3m}{\sqrt{n}} \| \max_{1\le l\le n} |h \circ T^l| \|_2 + \left(1 - \sqrt{\frac{k_n m}{n}}\right) \| \max_{1\le l\le k_n} |w_{k_n,m}(l/k_n)| \, \|_2. \end{aligned}$$

Since $h \in L^2(Y, \mathcal{B}, \nu)$ we have

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \| \max_{1\le l\le n} |h \circ T^l| \|_2 = 0.$$

Furthermore, since the sequence $\|\max_{1\le l\le k} |w_{k,m}(l/k)| \, \|_2$ is bounded by assumption, and $\lim_{n\to\infty}(1-\sqrt{k_n m/n}) = 0$, the second term on the right-hand side of (3.4) also tends to zero. ∎

Proof of Theorem 1. From Theorem 3 it follows that there exists $\widetilde{h} \in L^2(Y, \mathcal{B}, \nu)$ such that $\mathcal{P}_T \widetilde{h} = 0$ and

$$(3.5) \qquad \lim_{n\to\infty} \left\| \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \widetilde{h}) \circ T^j \right\|_2 = 0.$$

For each $m \in \mathbb{N}$, define

$$\widetilde{h}_m = \frac{1}{\sqrt{m}} \sum_{j=1}^{m-1} \widetilde{h} \circ T^j \quad \text{and} \quad \widetilde{w}_{k,m}(t) = \frac{1}{\sqrt{k}} \sum_{j=0}^{[kt]-1} \widetilde{h}_m \circ T^{mj}$$

for $k \in \mathbb{N}$ and $t \in [0,1]$. We have $\mathcal{P}_{T^m} \widetilde{h}_m = 0$ for all m. Thus Theorem 2 implies

$$(3.6) \qquad \widetilde{w}_{k,m} \to^d \sqrt{E_\nu(\widetilde{h}_m^2 \,|\, \mathcal{I}_m)}\, w$$

as $k \to \infty$, where $\mathcal{I}_m$ is the σ-algebra of T^m-invariant sets. Proposition 1, applied to T^m and $\widetilde{h}_m$, gives

$$\| \max_{1\le l\le k} |\widetilde{w}_{k,m}(l/k)| \, \|_2 \le 3 \|\widetilde{h}_m\|_2.$$

Therefore, by Proposition 2, we obtain

$$\lim_{n\to\infty} \| \sup_{0\le t\le 1} |\widetilde{w}_{n,1}(t) - \widetilde{w}_{[n/m],m}(t)| \, \|_2 = 0$$

for all $m \in \mathbb{N}$, which implies, by Theorem 4.1 of [1], that the limit in (3.6) does not depend on m and is thus equal to $\sqrt{E_\nu(\widetilde{h}^2 \,|\, \mathcal{I})}\, w$.

To prove (1.3), using Theorem 4.2 of [1] we have to show that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\|\sup_{0\le t\le 1}|w_n(t)-\widetilde{w}_{[n/m],m}(t)|\,\|_2=0. \tag{3.7}$$

Let h_m and $w_{k,m}$ be defined as in (3.3). We

$$\begin{aligned}\|\sup_{0\le t\le 1}|w_n(t)-\widetilde{w}_{[n/m],m}(t)|\,\|_2 &\le \|\sup_{0\le t\le 1}|w_n(t)-w_{[n/m],m}(t)|\,\|_2\\ &\quad+\|\sup_{0\le t\le 1}|w_{[n/m],m}(t)-\widetilde{w}_{[n/m],m}(t)|\,\|_2.\end{aligned} \tag{3.8}$$

Making use of Proposition 1 with T^m and h_m we obtain

$$\|\max_{1\le l\le k}|w_{k,m}(l/k)|\|_2\le 3\|h_m-U_{T^m}\mathcal{P}_{T^m}h_m\|_2+4\sqrt{2}\sum_{j=0}^{\infty}2^{-j/2}\Big\|\sum_{i=1}^{2^j}\mathcal{P}^i_{T^m}h_m\Big\|_2.$$

However,

$$\mathcal{P}_{T^m}h_m=\frac{1}{\sqrt{m}}\sum_{j=0}^{m-1}\mathcal{P}_{T^m}U_{T^j}h=\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\mathcal{P}^j_T h$$

by (2.2), and thus

$$\sum_{j=0}^{\infty}2^{-j/2}\Big\|\sum_{i=1}^{2^j}\mathcal{P}^i_{T^m}h_m\Big\|_2=\frac{1}{\sqrt{m}}\sum_{j=0}^{\infty}2^{-j/2}\Big\|\sum_{i=1}^{m2^j}\mathcal{P}^i_T h\Big\|_2, \tag{3.9}$$

and the series is convergent by Lemma 1, which implies that the sequence $\|\max_{1\le l\le k}|w_{k,m}(l/k)|\,\|_2$ is bounded for all m. From Proposition 2 it follows that

$$\lim_{n\to\infty}\|\sup_{0\le t\le 1}|w_n(t)-w_{[n/m],m}(t)|\,\|_2=0.$$

We next turn to estimating the second term in (3.8). We have

$$\begin{aligned}\|\sup_{0\le t\le 1}|w_{k,m}(t)-\widetilde{w}_{k,m}(t)|\,\|_2 &\le \frac{1}{\sqrt{k}}\Big\|\max_{1\le l\le k}\Big|\sum_{j=0}^{l-1}(h_m-\widetilde{h}_m)\circ T^{mj}\Big|\,\Big\|_2\\ &\le 3\|h_m-\widetilde{h}_m-U_{T^m}\mathcal{P}_{T^m}(h_m-\widetilde{h}_m)\|_2\\ &\quad+4\sqrt{2}\sum_{j=0}^{\infty}2^{-j/2}\Big\|\sum_{i=1}^{2^j}\mathcal{P}^i_{T^m}(h_m-\widetilde{h}_m)\Big\|_2\end{aligned}$$

by Proposition 1. Combining this with (3.9) and the fact that $\mathcal{P}_{T^m}\widetilde{h}_m=0$

leads to the estimate

$$\| \sup_{0\le t\le 1} |w_{k,m}(t)-\widetilde{w}_{k,m}(t)|\, \|_2 \le 3\frac{1}{\sqrt{m}}\Big\| \sum_{j=0}^{m-1} (h-\widetilde{h})\circ T^j \Big\|_2 + \frac{1}{\sqrt{m}}\Big\| \sum_{j=1}^{m} \mathcal{P}_{T^j} h \Big\|_2 + \frac{4\sqrt{2}}{\sqrt{m}} \sum_{j=0}^{\infty} 2^{-j/2} \Big\| \sum_{i=1}^{m2^j} \mathcal{P}_T^i h \Big\|_2,$$

which completes the proof of (3.7), because all terms on the right-hand side tend to zero as $m \to \infty$, by (3.5) and Lemma 1. ∎

4. Asymptotically periodic transformations. The dynamical properties of what are now known as asymptotically periodic transformations seem to have first been studied by Ionescu Tulcea and Marinescu [10]. These transformations form a perfect example of the central limit theorem results we have discussed in earlier sections, and here we consider them in detail.

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Write $L^1(\mu) = L^1(X, \mathcal{A}, \mu)$. The elements of the set

$$D(\mu) = \Big\{ f \in L^1(\mu) : f \ge 0 \text{ and } \int f(x)\,\mu(dx) = 1 \Big\}$$

are called *densities*. Let $T : X \to X$ be a non-singular transformation and $P : L^1(\mu) \to L^1(\mu)$ be the corresponding Perron–Frobenius operator. Then (Lasota and Mackey [15]) (T, μ) is called *asymptotically periodic* if there exists a sequence of densities $g_1, \ldots, g_r$ and a sequence of bounded linear functionals $\lambda_1, \ldots, \lambda_r$ such that

$$\lim_{n\to\infty} \Big\| P^n \Big(f - \sum_{j=1}^{r} \lambda_j(f) g_j \Big) \Big\|_{L^1(\mu)} = 0 \tag{4.1}$$

for all $f \in D(\mu)$. The densities g_j have disjoint supports ($g_i g_j = 0$ for $i \ne j$) and $P g_j = g_{\alpha(j)}$, where α is a permutation of $\{1, \ldots, r\}$.

If (T, μ) is asymptotically periodic and $r = 1$ in (4.1) then (T, μ) is called *asymptotically stable* or *exact* by Lasota and Mackey [15].

Observe that if (T, μ) is asymptotically periodic then

$$g_* = \frac{1}{r} \sum_{j=1}^{r} g_j$$

is an invariant density for P, i.e. $P g_* = g_*$. The ergodic structure of asymptotically periodic transformations was studied by Inoue and Ishitani [9].

Remark 1. Let $\mu(X) < \infty$. Recall that P is a *constrictive* Perron–Frobenius operator if there exist $\delta > 0$ and $\kappa < 1$ such that for every density

f we have

$$\limsup_{n\to\infty} \int_A P^n f(x)\,\mu(dx) < \kappa$$

for all $A \in \mathcal{A}$ with $\mu(A) \le \delta$. It is known that if P is a constrictive operator then (T, μ) is asymptotically periodic ([15, Theorem 5.3.1], see also Komorník and Lasota [14]), and (T, μ) is ergodic if and only if the permutation $\{\alpha(1), \dots, \alpha(r)\}$ of the sequence $\{1, \dots, r\}$ is cyclical ([15, Theorem 5.5.1]). In this case we call r the *period* of T.

Let (T, μ) be asymptotically periodic and let g_* be an invariant density for P. Let $Y = \operatorname{supp}(g_*) = \{x \in X : g_*(x) > 0\}$, $\mathcal{B} = \{A \cap Y : A \in \mathcal{A}\}$, and

$$\nu(A) = \int_A g_*(x)\,\mu(dx), \qquad A \in \mathcal{A}.$$

The measure ν is a probability measure invariant under T. In what follows we write $L^p(\nu) = L^p(Y, \mathcal{B}, \nu)$ for $p = 1, 2$. The transfer operator $\mathcal{P}_T : L^1(\nu) \to L^1(\nu)$ is given by

$$g_* \mathcal{P}_T(f) = P(f g_*) \quad \text{for } f \in L^1(\nu). \tag{4.2}$$

We now turn to the study of weak convergence of the sequence of processes

$$w_n(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{[nt]-1} h \circ T^j,$$

where $h \in L^2(\nu)$ with $\int h(y)\,\nu(dy) = 0$, by considering first the ergodic and then the non-ergodic case.

4.1. (T, μ) *ergodic and asymptotically periodic.* Let the transformation (T, μ) be ergodic and asymptotically periodic with period r. The unique invariant density of P is given by

$$g_* = \frac{1}{r} \sum_{j=1}^{r} g_j$$

and (T^r, g_j) is exact for every $j = 1, \dots, r$. Let $Y_j = \operatorname{supp}(g_j)$ for $j = 1, \dots, r$. Note that the set $B_j = \bigcup_{n=0}^{\infty} T^{-nr}(Y_j)$ is (almost) T^r-invariant and $\nu(B_j \setminus Y_j) = 0$ for $j = 1, \dots, r$. Since the Y_j are pairwise disjoint, we have

$$E_\nu(f \,|\, \mathcal{I}_r) = \sum_{k=1}^{r} \frac{1}{\nu(Y_k)} \int_{Y_k} f(y)\,\nu(dy)\, 1_{Y_k} \quad \text{for } f \in L^1(\nu),$$

where $\mathcal{I}_r$ is the σ-algebra of T^r-invariant sets. But $\nu(Y_k) = 1/r$, and thus

$$E_\nu(f \,|\, \mathcal{I}_r) = r \sum_{k=1}^{r} \int_{Y_k} f(y)\,\nu(dy)\, 1_{Y_k} = \sum_{k=1}^{r} \int_{Y_k} f(y) g_k(y)\,\mu(dy)\, 1_{Y_k}. \tag{4.3}$$

THEOREM 4. *Suppose that $h \in L^2(\nu)$ with $\int h(y)\,\nu(dy) = 0$ is such that*

$$\sum_{n=1}^{\infty} n^{-3/2} \Big\| \sum_{k=0}^{n-1} \mathcal{P}_T^{rk} h_r \Big\|_2 < \infty, \quad \text{where} \quad h_r = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} h \circ T^k. \tag{4.4}$$

Then

$$w_n \to^d \sigma w,$$

where w is a standard Brownian motion and $\sigma \geq 0$ is a constant. Moreover, if $\sum_{j=1}^{\infty} \int |h_r(y) h_r(T^{rj}(y))|\,\nu(dy) < \infty$ then σ is given by

$$\sigma^2 = r\Big(\int\limits_{Y_1} h_r^2(y)\,\nu(dy) + 2 \sum_{j=1}^{\infty} \int\limits_{Y_1} h_r(y) h_r(T^{rj}(y))\,\nu(dy) \Big). \tag{4.5}$$

Proof. We have $h_r \in L^2(\nu)$ and $\int_Y h_r(y)\,\nu(dy) = 0$. Let

$$w_{k,r}(t) = \frac{1}{\sqrt{k}} \sum_{j=0}^{[kt]-1} h_r \circ T^{rj} \quad \text{for } k \in \mathbb{N},\ t \in [0,1].$$

We can apply Theorem 1 to deduce that

$$w_{k,r} \to^d \sqrt{E_\nu(\widetilde{h}_r^2 \,|\, \mathcal{I}_r)}\, w \quad \text{as } k \to \infty,$$

where $\mathcal{I}_r$ is the σ-algebra of all T^r-invariant sets and

$$E_\nu(\widetilde{h}_r^2 \,|\, \mathcal{I}_r) = \lim_{n \to \infty} \frac{1}{n} E_\nu\Big(\Big(\sum_{j=0}^{n-1} h_r \circ T^{rj} \Big)^2 \,\Big|\, \mathcal{I}_r \Big). \tag{4.6}$$

On the other hand, we also have

$$\sum_{j=0}^{\infty} r^{-j/2} \Big\| \sum_{k=1}^{r^j} \mathcal{P}^{rk} h_r \Big\|_2 = \sum_{j=0}^{\infty} r^{-j/2} \frac{1}{\sqrt{r}} \Big\| \sum_{k=1}^{r^{j+1}} \mathcal{P}^k h \Big\|_2 = \sum_{j=1}^{\infty} r^{-j/2} \Big\| \sum_{k=1}^{r^j} \mathcal{P}^k h \Big\|_2.$$

Thus the series

$$\sum_{n=1}^{\infty} n^{-3/2} \Big\| \sum_{k=0}^{n-1} \mathcal{P}^k h \Big\|_2$$

is convergent by Lemma 2. From Theorem 1 we conclude that there exists $\widetilde{h} \in L^2(\nu)$ such that

$$w_n \to^d \|\widetilde{h}\|_2 w$$

since T is ergodic. But

$$\|\widetilde{h}\|_2 = \sqrt{E_\nu(\widetilde{h}_r^2 \,|\, \mathcal{I}_r)},$$

by Proposition 2. Hence $E_\nu(\widetilde{h}_r^2 \,|\, \mathcal{I}_r)$ is a constant and from (4.3) it follows that for each $k = 1, \ldots, r$ the integral $\int_{Y_k} \widetilde{h}_r^2(y)\,\nu(dy)$ does not depend on k.

Thus

$$\sigma^2 = \|\widetilde{h}\|_2^2 = r \int_{Y_1} \widetilde{h}_r^2(y)\,\nu(dy).$$

Since ν is T^r-invariant, we have

$$\frac{1}{n}\int_{Y_k}\Big(\sum_{j=0}^{n-1} h_r(T^{rj}(y))\Big)^2\,\nu(dy) = \int_{Y_k} h_r^2(y)\,\nu(dy) + 2\,\frac{1}{n}\sum_{l=1}^{n-1}\sum_{j=1}^{l}\int_{Y_k} h_r(y)h_r(T^{rj}(y))\,\nu(dy).$$

By assumption the sequence $(\sum_{j=1}^{n}\int_{Y_k} h_r(y)h_r(T^{rj}(y))\,\nu(dy))_{n\geq 1}$ is convergent to $\sum_{j=1}^{\infty}\int_{Y_k} h_r(y)h_r(T^{rj}(y))\,\nu(dy)$, which completes the proof when combined with (4.6) and (4.3). ■

4.2. (T,μ) *asymptotically periodic but not necessarily ergodic.* Now let us consider (T,μ) asymptotically periodic but not ergodic, so that the permutation α is not cyclical and we can represent it as a product of permutation cycles. Thus we can rephrase the definition of asymptotic periodicity as follows.

Let there exist a sequence of densities

$$g_{1,1},\dots,g_{1,r_1},\dots,g_{l,1},\dots,g_{l,r_l} \tag{4.7}$$

and a sequence of bounded linear functionals $\lambda_{1,1},\dots,\lambda_{1,r_1},\dots,\lambda_{l,1},\dots,\lambda_{l,r_l}$ such that

$$\lim_{n\to\infty}\Big\|P^n\Big(f-\sum_{i=1}^{l}\sum_{j=1}^{r_i}\lambda_{i,j}(f)g_{i,j}\Big)\Big\|_{L^1(\mu)} = 0 \quad \text{for all } f\in L^1(\mu), \tag{4.8}$$

where the densities $g_{i,j}$ have mutually disjoint supports and, for each i, $Pg_{i,j} = g_{i,j+1}$ for $1\leq j\leq r_i-1$, and $Pg_{i,r_i} = g_{i,1}$. Then

$$g_i^* = \frac{1}{r_i}\sum_{j=1}^{r_i} g_{i,j}$$

is an invariant density for P and (T, g_i^*) is ergodic for every $i = 1,\dots,l$. Let g_* be a convex combination of g_i^*, i.e.

$$g_* = \sum_{i=1}^{l}\alpha_i g_i^*$$

where $\alpha_i \geq 0$ and $\sum_{i=1}^{l}\alpha_i = 1$. For simplicity, assume that $\alpha_i > 0$.

Let $Y_i = \operatorname{supp}(g_i^*)$ and $Y_{i,j} = \operatorname{supp}(g_{i,j})$, $j = 1, \dots, r_i$, $i = 1, \dots, l$. If $\mathcal{I}$ is the σ-algebra of all T-invariant sets, then

$$E_\nu(f \mid \mathcal{I}) = \sum_{i=1}^{l} \frac{1}{\nu(Y_i)} \int_{Y_i} f(y)\, \nu(dy)\, 1_{Y_i} = \sum_{i=1}^{l} \int_{Y_i} f(y) g_i^*(y)\, \mu(dy)\, 1_{Y_i}.$$

Now, if $\mathcal{I}_r$ is the σ-algebra of all T^r-invariant sets with $r = \prod_{i=1}^{l} r_i$, then

$$E_\nu(f \mid \mathcal{I}_r) = \sum_{i=1}^{l} \frac{r_i}{\nu(Y_i)} \sum_{j=1}^{r_i} \int_{Y_{i,j}} f(y)\, \nu(dy)\, 1_{Y_{i,j}}$$

for $f \in L^1(\nu)$, which leads to

$$E_\nu(f \mid \mathcal{I}_r) = \sum_{i=1}^{l} \sum_{j=1}^{r_i} \int_{Y_{i,j}} f(y) g_{i,j}(y)\, \mu(dy)\, 1_{Y_{i,j}}.$$

Using similar arguments to those in the proof of Theorem 4 we obtain

THEOREM 5. *Suppose that $h \in L^2(\nu)$ with $\int h(y)\, \nu(dy) = 0$ is such that condition* (4.4) *holds. Then*

$$w_n \to^d \eta w,$$

where w is a standard Brownian motion and $\eta \ge 0$ is a random variable independent of w. Moreover, if $\sum_{j=1}^{\infty} \int |h_r(y) h_r(T^{rj}(y))|\, \nu(dy) < \infty$ then η is given by

$$\eta = \sum_{i=1}^{l} \frac{r_i}{\nu(Y_i)} \Big(\int_{Y_{i,1}} h_r^2(y)\, \nu(dy) + 2 \sum_{j=1}^{\infty} \int_{Y_{i,1}} h_r(y) h_r(T^{rj}(y))\, \nu(dy) \Big) 1_{Y_i}.$$

REMARK 2. Observe that condition (4.4) holds if

$$\sum_{n=1}^{\infty} \frac{\|\mathcal{P}_T^{rn} h_r\|_2}{\sqrt{n}} < \infty.$$

The operator $\mathcal{P}_T$ is a contraction on $L^\infty(\nu)$. Therefore

$$\|\mathcal{P}_T^n f\|_2 \le \|f\|_\infty^{1/2} \|\mathcal{P}_T^n f\|_1^{1/2} \quad \text{for } f \in L^\infty(\nu),\ n \ge 1,$$

which allows us to easily check condition (4.4) for specific examples of transformations T. It should also be noted that, by (4.2), we have

$$\|\mathcal{P}_T^n f\|_1 = \|P^n(f g_*)\|_{L^1(\mu)} \quad \text{for } f \in L^1(\nu),\ n \ge 1.$$

4.3. *Piecewise monotonic transformations.* Let X be a totally ordered, order complete set (usually X is a compact interval in $\mathbb{R}$). Let $\mathcal{B}$ be the σ-algebra of Borel subsets of X and let μ be a probability measure on X.

Recall that a function $f : X \to \mathbb{R}$ is said to be of *bounded variation* if

$$\operatorname{var}(f) = \sup \sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)| < \infty,$$

where the supremum is taken over all finite ordered sequences (x_j) with $x_j \in X$. The bounded variation norm is given by

$$\|f\|_{\mathrm{BV}} = \|f\|_{L^1(\mu)} + \operatorname{var}(f)$$

and it makes $\mathrm{BV} = \{f : X \to \mathbb{R} : \operatorname{var}(f) < \infty\}$ into a Banach space.

Let $T : V \to X$ be a continuous map, $V \subset X$ be open and dense with $\mu(V) = 1$. We call (T, μ) a *piecewise uniformly expanding map* if:

(1) There exists a countable family $\mathcal{Z}$ of closed intervals with disjoint interiors such that $V \subset \bigcup_{Z \in \mathcal{Z}} Z$ and for any $Z \in \mathcal{Z}$ the set $Z \cap (X \setminus V)$ consists exactly of the endpoints of Z.
(2) For any $Z \in \mathcal{Z}$, $T_{|Z \cap V}$ admits an extension to a homeomorphism from Z to some interval.
(3) There exists a function $g : X \to [0, \infty)$, with bounded variation, $g_{|X \setminus V} = 0$ such that the Perron–Frobenius operator $P : L^1(\mu) \to L^1(\mu)$ is of the form

$$Pf(x) = \sum_{z \in T^{-1}(x)} g(z) f(z).$$

(4) T is expanding: $\sup_{x \in V} g(x) < 1$.

The following result is due to Rychlik [26]:

THEOREM 6. *If (T, μ) is a piecewise uniformly expanding map then it satisfies* (4.8) *with $g_{i,j} \in \mathrm{BV}$. Moreover, there exist constants $C > 0$ and $\theta \in (0, 1)$ such that, for every function f of bounded variation and all $n \geq 1$,*

$$\|P^{rn} f - Q(f)\|_{L^1(\mu)} \leq C\theta^n \|f\|_{\mathrm{BV}},$$

where $r = \prod_{i=1}^{l} r_i$ and

$$Q(f) = \sum_{i=1}^{l} \sum_{j=1}^{r_i} \int_{Y_{i,j}} f(x)\, \mu(dx)\, g_{i,j}.$$

This result and Remark 2 imply

COROLLARY 1. *Let (T, μ) be a piecewise uniformly expanding map and ν an invariant measure which is absolutely continuous with respect to μ. If h is a function of bounded variation with $E_\nu(h \,|\, \mathcal{I}) = 0$ then* (4.4) *holds.*

REMARK 3. AFU-maps (**u**niformly expanding maps satisfying **A**dler's condition with a **f**inite image condition, which are interval maps with a finite number of indifferent fixed points), studied by Zweimüller [35], are asymptotically periodic when they have an absolutely continuous invariant probability measure. However, the decay of the L^1 norm may not be exponential. For Hölder continuous functions h one might use the results of Young [34] to obtain bounds on this norm and then apply our results.

4.4. *Calculation of variance for the family of tent maps using Theorem 4.* Let T be the generalized tent map on $[-1,1]$ defined by

$$T_a(x) = a - 1 - a|x| \quad \text{for } x \in [-1,1], \tag{4.9}$$

where $a \in (1,2]$. The Perron–Frobenius operator $P : L^1(\mu) \to L^1(\mu)$ is given by

$$Pf(x) = \frac{1}{a}\left(f(\psi_a^-(x)) + f(\psi_a^+(x))\right)1_{[-1,a-1]}(x), \tag{4.10}$$

where ψ_a^- and ψ_a^+ are the inverse branches of T_a:

$$\psi_a^-(x) = \frac{x+1-a}{a}, \qquad \psi_a^+(x) = -\frac{x+1-a}{a}, \tag{4.11}$$

and μ is the normalized Lebesgue measure on $[-1,1]$.

Ito *et al.* [11] have shown that the tent map (4.9) is ergodic, thus having a unique invariant density g_a. Provatas and Mackey [24] have proved the asymptotic periodicity of (4.9) with period $r = 2^m$ for

$$2^{1/2^{m+1}} < a \le 2^{1/2^m} \quad \text{for } m = 0, 1, \ldots.$$

Thus, for example, (T,μ) has period 1 for $2^{1/2} < a \le 2$, period 2 for $2^{1/4} < a \le 2^{1/2}$, period 4 for $2^{1/8} < a \le 2^{1/4}$, etc.

Let $Y = \operatorname{supp}(g_a)$ and $\nu_a(dy) = g_a(y)\mu(dy)$. For all $1 < a \le 2$ we have $T_a(A) = A$ with $A = [T_a^2(0), T_a(0)]$ and $g_a(x) = 0$ for $x \in [-1,1] \setminus A$. If $\sqrt{2} < a \le 2$ then g_a is strictly positive in A, thus $Y = A$ in this case. For $a \le \sqrt{2}$ we have $Y \subset A$. The transfer operator $\mathcal{P}_a\colon L^1(\nu_a) \to L^1(\nu_a)$ is given by

$$\mathcal{P}_a f = \frac{P(fg_a)}{g_a} \quad \text{for } f \in L^1(\nu_a),$$

where P is the Perron–Frobenius operator (4.10).

If h is a function of bounded variation on $[-1,1]$ with $\int_{-1}^1 h(y)\,\nu_a(dy) = 0$ and

$$w_n(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{[nt]-1} h \circ T_a^j,$$

then there exists a constant $\sigma(h) \ge 0$ such that

$$w_n \to^d \sigma(h) w,$$

where w is a standard Brownian motion. In particular, we are going to study $\sigma(h)$ for the specific example of $h = h_a$ for $a \in (1, 2]$, where

$$h_a(y) = y - \mathfrak{m}_a, \quad y \in [-1, 1], \quad \text{and} \quad \mathfrak{m}_a = \int_{[-1,1]} y g_a(y)\, dy.$$

PROPOSITION 3. *Let $m \geq 1$ and $2^{1/2^{m+1}} < a \leq 2^{1/2^m}$. Then*

$$\sigma(h_a) = \frac{\sigma(h_{a^{2^m}})a(a-1)}{\sqrt{2^m}\, a^{2^m}(a^{2^m} - 1)} \prod_{k=0}^{m-1} (a^{2^k} - 1)^2, \tag{4.12}$$

where

$$\sigma(h_{a^{2^m}})^2 = 2\int h_{a^{2^m}}(y) f_{a^{2^m}}(y)\, \nu_{a^{2^m}}(dy) - \int h^2_{a^{2^m}}(y)\, \nu_{a^{2^m}}(dy), \tag{4.13}$$

$$f_{a^{2^m}} = \sum_{n=0}^{\infty} \mathcal{P}^n_{a^{2^m}} h_{a^{2^m}}.$$

In general, an explicit representation for (4.13) is not known. Hence, before turning to a proof of Proposition 3, we first give the simplest example in which $\sigma(h_{a^{2^m}})^2$ can be calculated exactly.

EXAMPLE 2. For $a = 2$ the invariant density for the transformation T_a is $g_2 = \frac{1}{2} \cdot 1_{[-1,1]}$ and the transfer operator $\mathcal{P}_2 : L^1(\nu_2) \to L^1(\nu_2)$ has the same form as P in (4.10):

$$\mathcal{P}_2 f = \frac{1}{2}\,(f \circ \psi_2^- + f \circ \psi_2^+).$$

Since $\int_{-1}^1 y\, dy = 0$, we have $h_2(y) = y$. We also have $\mathcal{P}_2 h_2 = 0$. Thus

$$\sigma(h_2)^2 = \frac{1}{2} \int_{-1}^{1} y^2\, dy = 1/3$$

and Proposition 3 gives $\sigma(h_a)$ for $a = 2^{1/2^m}$, $m \geq 1$.

We now summarize some properties of the tent map [33], which will allow us to prove Proposition 3. Let $I_0 = [x^*(a), x^*(a)(1 + 2/a)]$ and $I_1 = [-x^*(a), x^*(a)]$, where $x^*(a)$ is the fixed point of T_a other than -1, i.e.

$$x^*(a) = \frac{a-1}{a+1}.$$

Define transformations $\phi_{ia} : I_i \to [-1, 1]$ by

$$\phi_{1a}(x) = -\frac{1}{x^*(a)}\, x \quad \text{and} \quad \phi_{0a}(x) = \frac{a}{x^*(a)}\, x - a - 1.$$

We have

$$\phi_{1a}^{-1}(x) = -x^*(a)x \quad \text{and} \quad \phi_{0a}^{-1}(x) = \frac{x^*(a)}{a}\,(x + a + 1). \tag{4.14}$$

Then for $1 < a \le \sqrt{2}$ the map $T_a^2 : I_i \to I_i$ is conjugate to $T_{a^2} : [-1,1] \to [-1,1]$:

$$T_{a^2} = \phi_{ia} \circ T_a^2 \circ \phi_{ia}^{-1}, \tag{4.15}$$

and the invariant density of T_a is given by

$$g_a(y) = \frac{1}{2x^*(a)} \left(a g_{a^2}(\phi_{0a}(y)) 1_{I_0}(y) + g_{a^2}(\phi_{1a}(y)) 1_{I_1}(y) \right). \tag{4.16}$$

Lemma 3. *If $a \in (1, \sqrt{2}]$ then*

$$\mathfrak{m}_a = \frac{a-1}{2a} - \frac{(a-1)x^*(a)}{2a} \mathfrak{m}_{a^2} \tag{4.17}$$

and

$$(h_a + h_a \circ T_a) \circ \phi_{0a}^{-1} = \frac{(1-a)x^*(a)}{a} h_{a^2}. \tag{4.18}$$

Proof. Equation (4.17) follows from (4.16) and (4.14), while (4.18) is a direct consequence of the definition of ϕ_{0a}^{-1}, the fact that $I_0 \subset [0,1]$, and (4.17). ∎

Let $m \ge 1$. For $2^{1/2^{m+1}} < a \le 2^{1/2^m}$ there exist 2^m disjoint intervals in which g_a is strictly positive and they are defined by

$$Y_j^m = \Phi_{jm}^{-1}([T_{a^{2^m}}^2(0), T_{a^{2^m}}(0)]),$$

where

$$\Phi_{jm} = \phi_{i_m a^{2^{m-1}}} \circ \phi_{i_{m-1} a^{2^{m-2}}} \circ \cdots \circ \phi_{i_2 a^2} \circ \phi_{i_1 a}$$

and $j = 1 + i_1 + 2i_2 + \cdots + 2^{m-1} i_m$, $i_k = 0, 1$, $k = 1, \ldots, m$. We have $T_a(Y_j^m) = Y_{j+1}^m$ for $1 \le j \le 2^m - 1$ and $T_a(Y_{2^m}^m) = Y_1^m$. In particular,

$$Y_1^{m+1} = \phi_{0a}^{-1}(Y_1^m) \quad \text{for } m \ge 0, \tag{4.19}$$

where $Y_1^0 = [T_{a^2}^2(0), T_{a^2}(0)]$.

Lemma 4. *Define*

$$h_{r,a} = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} h_a \circ T_a^k \quad \textit{for } r \ge 1,\ a \in (1,2]. \tag{4.20}$$

Let $m \ge 0$ and $r = 2^m$. If $2^{1/4r} < a \le 2^{1/2r}$ then

$$\int_{Y_1^{m+1}} h_{2r,a}(y) h_{2r,a}(T_a^{2rn}(y))\, \nu_a(dy) = \frac{(1-a)^2 x^*(a)^2}{2^2 a^2} \int_{Y_1^m} h_{r,a^2}(y) h_{r,a^2}(T_{a^2}^{rn}(y))\, \nu_{a^2}(dy) \tag{4.21}$$

for all $n \ge 0$.

Proof. First observe that

$$h_{2r,a} = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} h_{2,a} \circ T_a^{2k}. \tag{4.22}$$

Let $n \geq 0$. Since $\phi_{0a}^{-1}(\phi_{0a}(y)) = y$ for $y \in [-1, 1]$, a change of variables using (4.19) and (4.16) gives

$$\int\limits_{Y_1^{m+1}} h_{2r,a}(y) h_{2r,a}(T_a^{2rn}(y))\, \nu_a(dy) \tag{4.23}$$
$$= \frac{1}{2} \int\limits_{Y_1^m} h_{2r,a}(\phi_{0a}^{-1}(y)) h_{2r,a}(T_a^{2rn}(\phi_{0a}^{-1}(y)))\, \nu_{a^2}(dy).$$

We have $T_a^{2k} \circ \phi_{0a}^{-1} = \phi_{0a}^{-1} \circ T_{a^2}^k$ for all $k \geq 0$ by (4.15). Thus $T_a^{2rn} \circ \phi_{0a}^{-1} = \phi_{0a}^{-1} \circ T_{a^2}^{rn}$ and from (4.22) it follows that

$$h_{2r,a} \circ \phi_{0a}^{-1} = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} h_{2,a} \circ \phi_{0a}^{-1} \circ T_{a^2}^k.$$

By Lemma 3 we obtain

$$h_{2,a} \circ \phi_{0a}^{-1} = \frac{(1-a)x^*(a)}{\sqrt{2}a}\, h_{a^2}.$$

Hence

$$h_{2r,a} \circ \phi_{0a}^{-1} = \frac{(1-a)x^*(a)}{\sqrt{2}a}\, h_{r,a^2},$$

which, when substituted into equation (4.23), completes the proof. ∎

Proof of Proposition 3. First, we show that if $m \geq 1$ and $2^{1/2^{m+1}} < a \leq 2^{1/2^m}$ then

$$\sigma(h_a) = \frac{\sigma(h_{a^{2^m}})}{\sqrt{2^m}\, a^{2^m - 1}} \prod_{k=0}^{m-1} x^*(a^{2^k})(a^{2^k} - 1). \tag{4.24}$$

Let $m \geq 1$ and $2^{1/2^{m+1}} < a \leq 2^{1/2^m}$. Since the transformation T_a is asymptotically periodic with period 2^m, Theorem 4 gives

$$\sigma(h_a)^2 = 2^m \Big(\int\limits_{Y_1^m} h_{2^m,a}^2(y)\, \nu_a(dy) + 2 \sum_{j=1}^{\infty} \int\limits_{Y_1^m} h_{2^m,a}(y) h_{2^m,a}(T_a^{2^m j}(y))\, \nu_a(dy) \Big).$$

We have $a^2 \in (2^{1/2^m}, 2^{1/2^{m-1}}]$ and the transformation T_{a^2} is asymptotically periodic with period $r = 2^{m-1}$. From (4.21) with $r = 2^{m-1}$ and Theorem 4 it follows that

$$\sigma(h_a)^2 = \frac{(a-1)^2 x^*(a)^2}{2a^2}\, \sigma(h_{a^2})^2.$$

Thus equation (4.24) follows immediately by an induction argument on m. Finally, for each $k = 0, \dots, m-1$ we have

$$x^*(a^{2^k})(a^{2^k}-1) = \frac{a^{2^k}-1}{a^{2^k}+1}(a^{2^k}-1) = \frac{(a^{2^k}-1)^3}{a^{2^{k+1}}-1}$$

and equation (4.12) holds. Since $a^{2^m} > \sqrt{2}$ the function $f_{a^{2^m}}$ is well defined and

$$\int h_{a^{2^m}}(y) f_{a^{2^m}}(y)\, \nu_{a^{2^m}}(dy) = \sum_{n=0}^{\infty} \int h_{a^{2^m}}(y) h_{a^{2^m}}(T^n_{a^{2^m}}(y))\, \nu_{a^{2^m}}(dy),$$

which completes the proof. ■

Appendix A. Proof of the maximal inequality

Proof of Proposition 1. We will prove (3.1) inductively. If $n = 1$ and $q = 1$ then we have

$$\|f\|_2 \le \|f - U_T\mathcal{P}_T f\|_2 + \|U_T\mathcal{P}_T f\|_2 = \|f - U_T\mathcal{P}_T f\|_2 + \Delta_1(f)$$

by the invariance of ν under T. Now assume that (3.1) holds for all $n < 2^{q-1}$. Fix n, $2^{q-1} \le n < 2^q$. By the triangle inequality

$$\max_{1\le k\le n} \Big|\sum_{j=0}^{k-1} f \circ T^j\Big| \le \max_{1\le k\le n} \Big|\sum_{j=0}^{k-1} (f - U_T\mathcal{P}_T f)\circ T^j\Big| + \max_{1\le k\le n} \Big|\sum_{j=0}^{k-1} U_T\mathcal{P}_T f \circ T^j\Big|. \tag{A.1}$$

We first show that

$$\Big\| \max_{1\le k\le n} \Big|\sum_{j=0}^{k-1} (f - U_T\mathcal{P}_T f)\circ T^j\Big| \Big\|_2 \le 3\sqrt{n}\, \|f - U_T\mathcal{P}_T f\|_2. \tag{A.2}$$

Observe that

$$\max_{1\le k\le n} \Big|\sum_{j=0}^{k-1} (f - U_T\mathcal{P}_T f)\circ T^j\Big| \le \Big|\sum_{j=0}^{n-1} (f - U_T\mathcal{P}_T f)\circ T^j\Big| + \max_{1\le k\le n} \Big|\sum_{j=1}^{k} (f - U_T\mathcal{P}_T f)\circ T^{n-j}\Big|.$$

Since $\mathcal{P}_T(f - U_T\mathcal{P}_T f) = 0$, we see that

$$\Big\|\sum_{j=0}^{n-1} (f - U_T\mathcal{P}_T f)\circ T^j\Big\|_2 = \sqrt{n}\, \|f - U_T\mathcal{P}_T f\|_2.$$

For every n the family $\{\sum_{j=1}^{k}(f-U_T\mathcal{P}_Tf)\circ T^{n-j} : 1\le k\le n\}$ is a martingale with respect to $\{T^{-n+k}(\mathcal{B}) : 1 \le k \le n\}$. Thus by the Doob maximal inequality

$$\Big\| \max_{1\le k\le n}\Big|\sum_{j=1}^{k}(f-U_T\mathcal{P}_Tf)\circ T^{n-j}\Big|\Big\|_2 \le 2\Big\|\sum_{j=1}^{n}(f-U_T\mathcal{P}_Tf)\circ T^{n-j}\Big\|_2 = 2\sqrt{n}\,\|f-U_T\mathcal{P}_Tf\|_2,$$

which completes the proof of (A.2).

Now consider the second term on the right-hand side of (A.1). Writing $n=2m$ or $n=2m+1$ yields

$$\max_{1\le k\le n}\Big|\sum_{j=0}^{k-1}U_T\mathcal{P}_Tf\circ T^{j}\Big| \le \max_{1\le l\le m}\Big|\sum_{j=0}^{l-1}f_1\circ T^{2j}\Big| + \max_{0\le l\le m}\Big|U_T\mathcal{P}_Tf\circ T^{2l}\Big|, \tag{A.3}$$

where $f_1 = U_{T^2}\mathcal{P}_Tf + U_T\mathcal{P}_Tf$. To estimate the norm of the second term on the right-hand side of (A.3), observe that

$$\max_{0\le l\le m}|U_T\mathcal{P}_Tf\circ T^{2l}|^2 \le \sum_{l=0}^{m}|U_T\mathcal{P}_Tf\circ T^{2l}|^2,$$

which leads to

$$\|\max_{0\le l\le m}|U_T\mathcal{P}_Tf\circ T^{2l}|\,\|_2 \le \sqrt{m+1}\,\|\mathcal{P}_Tf\|_2, \tag{A.4}$$

since ν is invariant under T. Further, since $m<2^{q-1}$, the measure ν is invariant under T^2, and $f_1\in L^2(Y,\mathcal{B},\nu)$, we can use the induction hypothesis. We thus obtain

$$\Big\|\max_{1\le l\le m}\Big|\sum_{j=0}^{l-1}f_1\circ T^{2j}\Big|\Big\|_2 \le \sqrt{m}\,(3\|f_1-U_{T^2}\mathcal{P}_{T^2}f_1\|_2 + 4\sqrt{2}\,\Delta_{q-1}(f_1)).$$

We have $f_1-U_{T^2}\mathcal{P}_{T^2}f_1 = U_T\mathcal{P}_Tf - U_{T^2}\mathcal{P}_{T^2}f$, by (2.2), which implies

$$\|f_1-U_{T^2}\mathcal{P}_{T^2}f_1\|_2 \le \|\mathcal{P}_Tf\|_2 + \|\mathcal{P}_{T^2}f\|_2 \le 2\|\mathcal{P}_Tf\|_2,$$

since $\mathcal{P}_T$ is a contraction. We also have

$$\begin{aligned}\Delta_{q-1}(f_1) &= \sum_{j=0}^{q-2}2^{-j/2}\Big\|\sum_{k=1}^{2^j}\mathcal{P}_{T^2}^{k}f_1\Big\|_2 = \sum_{j=0}^{q-2}2^{-j/2}\Big\|\sum_{k=1}^{2^j}\mathcal{P}_{T}^{2k}f_1\Big\|_2\\ &= \sum_{j=0}^{q-2}2^{-j/2}\Big\|\sum_{k=1}^{2^j}\mathcal{P}_{T}^{2k}(U_{T^2}\mathcal{P}_Tf + U_T\mathcal{P}_Tf)\Big\|_2\\ &= \sum_{j=0}^{q-2}2^{-j/2}\Big\|\sum_{k=1}^{2^{j+1}}\mathcal{P}_{T}^{k}f\Big\|_2 = \sqrt{2}\,(\Delta_q(f) - \|\mathcal{P}_Tf\|_2).\end{aligned}$$

Therefore

$$\Big\| \max_{1\le l\le m} \Big|\sum_{j=0}^{l-1} f_1\circ T^{2j}\Big| \Big\|_2 \le \sqrt{m}\,(8\Delta_q(f) - 2\|\mathcal{P}_T f\|_2),$$

which combined with (A.1) through (A.4) and the fact that $\sqrt{m+1} \le \sqrt{2m} \le \sqrt{n}$ leads to

$$\begin{aligned}\Big\| \max_{1\le k\le n} \Big|\sum_{j=1}^{k} f\circ T^{n-j}\Big| \Big\|_2 &\le 3\sqrt{n}\,\|f - U_T\mathcal{P}_T f\|_2 + \sqrt{m+1}\,\|\mathcal{P}_T f\|_2 \\ &\quad + \sqrt{2m}\,(4\sqrt{2}\,\Delta_q(f) - \sqrt{2}\|\mathcal{P}_T f\|_2) \\ &\le \sqrt{n}\,(3\|f - U_T\mathcal{P}_T f\|_2 + 4\sqrt{2}\,\Delta_q(f)). \ \blacksquare\end{aligned}$$

Appendix B. The limiting random variable η. Finally, we give a series expansion of $E_\nu(\widetilde{h}^2\,|\,\mathcal{I})$ in Theorem 1 in terms of h and iterates of T.

PROPOSITION 4. *Suppose $h \in L^2(Y,\mathcal{B},\nu)$ with $\int h(y)\,\nu(dy) = 0$ is such that*

$$\sum_{j=0}^{\infty} 2^{-j/2}\Big\|\sum_{k=1}^{2^j} \mathcal{P}_T^k h\Big\|_2 < \infty. \tag{B.1}$$

Then the following limit exists in L^1:

$$\lim_{n\to\infty} \frac{E_\nu(S_n^2\,|\,\mathcal{I})}{n} = E_\nu(h^2\,|\,\mathcal{I}) + \sum_{j=0}^{\infty} \frac{E_\nu(S_{2^j}S_{2^j}\circ T^{2^j}\,|\,\mathcal{I})}{2^j}, \tag{B.2}$$

where $\mathcal{I}$ is the σ-algebra of all T-invariant sets and $S_n = \sum_{j=0}^{n-1} h\circ T^j$, $n\in\mathbb{N}$. Moreover, if $\widetilde{h} \in L^2(Y,\mathcal{B},\nu)$ is such that $\mathcal{P}_T\widetilde{h} = 0$ and

$$\Big\|\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}(h-\widetilde{h})\circ T^j\Big\|_2 \to 0 \qquad \text{as } n\to\infty$$

then

$$E_\nu(\widetilde{h}^2\,|\,\mathcal{I}) = \lim_{n\to\infty}\frac{E_\nu(S_n^2\,|\,\mathcal{I})}{n}. \tag{B.3}$$

Proof. We first prove that the series on the right-hand side of (B.2) is convergent in $L^1(Y,\mathcal{B},\nu)$. Since $\mathcal{I}\subset T^{-2^j}(\mathcal{B})$ for all j, we see that

$$E_\nu(S_{2^j}S_{2^j}\circ T^{2^j}\,|\,\mathcal{I}) = E_\nu(E_\nu(S_{2^j}S_{2^j}\circ T^{2^j}\,|\,T^{-2^j}(\mathcal{B}))\,|\,\mathcal{I}).$$

As $S_{2^j}\circ T^{2^j}$ is $T^{-2^j}(\mathcal{B})$-measurable and integrable we have

$$E_\nu(S_{2^j}S_{2^j}\circ T^{2^j}\,|\,T^{-2^j}(\mathcal{B})) = S_{2^j}\circ T^{2^j}\,E_\nu(S_{2^j}\,|\,T^{-2^j}(\mathcal{B})).$$

However, $E_\nu(S_{2^j}\,|\,T^{-2^j}(\mathcal{B})) = U_T^{2^j}\mathcal{P}_T^{2^j}S_{2^j}$ from (2.2). Consequently,

$$E_\nu(S_{2^j}S_{2^j}\circ T^{2^j}\,|\,\mathcal{I}) = E_\nu\Big(S_{2^j}\sum_{k=1}^{2^j}\mathcal{P}_T^k h\,\Big|\,\mathcal{I}\Big). \tag{B.4}$$

Since the conditional expectation operator is a contraction in L^1, we have

$$\|E_\nu(S_{2^j}S_{2^j}\circ T^{2^j}\,|\,\mathcal{I})\|_1 \le \Big\|S_{2^j}\sum_{k=1}^{2^j}\mathcal{P}_T^k h\Big\|_1,$$

which, by the Cauchy–Schwarz inequality, leads to

$$\|E_\nu(S_{2^j}S_{2^j}\circ T^{2^j}\,|\,\mathcal{I})\|_1 \le \|S_{2^j}\|_2\Big\|\sum_{k=1}^{2^j}\mathcal{P}_T^k h\Big\|_2.$$

Since $\|S_{2^j}\|_2 \le \|\max_{1\le l\le 2^j}|S_l|\,\|_2$, the sequence $\|S_{2^j}\|_2/2^{j/2}$ is bounded, by (B.1), Lemma 2, and Proposition 1. Hence

$$\sum_{j=0}^{\infty}\frac{\|S_{2^j}\|_2\|\sum_{k=1}^{2^j}\mathcal{P}_T^k h\|_2}{2^j} \le C\sum_{j=0}^{\infty}\frac{\|\sum_{k=1}^{2^j}\mathcal{P}_T^k h\|_2}{2^{j/2}} < \infty,$$

which proves the convergence in L^1 of the series in (B.2).

We now prove the equality in (B.2). Since

$$\begin{aligned} S_{2^m}^2 &= (S_{2^{m-1}} + S_{2^{m-1}}\circ T^{2^{m-1}})^2 \\ &= S_{2^{m-1}}^2 + S_{2^{m-1}}^2\circ T^{2^{m-1}} + 2S_{2^{m-1}}S_{2^{m-1}}\circ T^{2^{m-1}}, \end{aligned}$$

we obtain

$$E_\nu(S_{2^m}^2\,|\,\mathcal{I}) = 2E_\nu(S_{2^{m-1}}^2\,|\,\mathcal{I}) + 2E_\nu(S_{2^{m-1}}S_{2^{m-1}}\circ T^{2^{m-1}}\,|\,\mathcal{I}),$$

which leads to

$$\frac{E_\nu(S_{2^m}^2\,|\,\mathcal{I})}{2^m} = E_\nu(h^2\,|\,\mathcal{I}) + \sum_{j=0}^{m-1}\frac{E_\nu(S_{2^j}S_{2^j}\circ T^{2^j}\,|\,\mathcal{I})}{2^j}.$$

Thus the limit on the left-hand side of (B.2) exists for the subsequence $n = 2^m$ and the equality holds. An analysis similar to that in the proof of Proposition 2.1 of [22] shows that the whole sequence is convergent, which completes the proof of (B.2).

We now turn to the proof of (B.3). Let $\widetilde{h}$ be such that $\mathcal{P}_T\widetilde{h} = 0$. Define $\widetilde{S}_n = \sum_{j=0}^{n-1}\widetilde{h}\circ T^j$. Substituting $\widetilde{h}$ into (B.1) and (B.4) gives

$$E_\nu(\widetilde{h}^2\,|\,\mathcal{I}) = \lim_{n\to\infty}\frac{E_\nu(\widetilde{S}_n^2\,|\,\mathcal{I})}{n}.$$

We have

$$\left\|\frac{E_\nu(\widetilde{S}_n^2\,|\,\mathcal{I})}{n}-\frac{E_\nu(S_n^2\,|\,\mathcal{I})}{n}\right\|_1 \le \left\|\frac{\widetilde{S}_n^2}{n}-\frac{S_n^2}{n}\right\|_1 \le \left\|\frac{\widetilde{S}_n}{\sqrt{n}}-\frac{S_n}{\sqrt{n}}\right\|_2\left\|\frac{\widetilde{S}_n}{\sqrt{n}}+\frac{S_n}{\sqrt{n}}\right\|_2$$

by the Hölder inequality, which implies (B.3) when combined with the equality

$$\Big\|\sum_{j=0}^{n-1}\widetilde{h}\circ T^j\Big\|_2=\sqrt{n}\,\|\widetilde{h}\|_2,$$

and the assumption

$$\left\|\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}(h-\widetilde{h})\circ T^j\right\|_2\to 0 \quad \text{as } n\to\infty. \ \blacksquare$$

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