Physica D 53 (1991) 295–318
North-Holland

PHYSICA D

Asymptotic periodicity and banded chaos

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Received 26 July 1990
Revised manuscript received 18 March 1991
Accepted 7 June 1991
Communicated by H. Flaschka

Treating one dimensional maps as dynamical systems, we examine their evolution in terms of density flow. In particular, a type of density evolution known as asymptotic periodicity is studied. Unlike statistically stable (or exact) systems, asymptotically periodic systems do not in general evolve to an invariant density, even though they possess one. Consequently, a nonequilibrium formalism, using several metastable states rather than one invariant density, is examined as an alternative to study the physical properties of asymptotically periodic systems. Asymptotic periodicity is demonstrated for the hat map and the quadratic map at the parameters where these maps generate *banded chaos* or *quasiperiodicity*. Using the ergodic properties of asymptotic periodicity, a compact expression for the time correlation function is obtained that does not make any assumptions about decoupling the periodic and stochastic components of $C(t)$. Finally a generalization to the Boltzmann–Gibbs entropy, known as the conditional entropy, is studied as an index characterizing the asymptotic density sequence that emerges in asymptotically periodic systems.

1. Introduction

Following the numerical (computer generated) demonstration of the existence of highly irregular dynamical behavior in relatively simple nonlinear systems, the study of so-called "chaotic" systems has captured the attention of scientists in the physical, mathematical, biological and social sciences. Since the evolution of a system is most naturally envisioned through a time series in phase space, it is natural that many investigators have focused attention on the irregular behavior displayed by the trajectories of these nonlinear systems.

However, the irregular and apparently unpredictable nature of trajectory evolution in many nonlinear dynamical systems can be greatly simplified if one looks at their behavior in terms of density evolution [19]. This alternative viewpoint (density vs. trajectory) has particular appeal when applying the concepts of nonlinear dynamics to many problems in statistical physics [22], and

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offers an immediate connection with the mathematical discipline of ergodic theory which developed from questions raised by the early work of Boltzmann and Gibbs.

The development of the measure theoretic tools used in the study of density evolution of dynamical systems has seen a steady progression over the past century starting with the work of Gibbs and Boltzmann up to the past decade with work on the properties of invariant measures of one-dimensional systems [3, 9–11, 14, 15, 17, 19, 23]. The work of Gibbs and Boltzmann has been interpreted and extended to lay the foundations of equilibrium statistical mechanics, as well as some of nonequilibrium statistical mechanics.

In the study of complex physical systems, it is sometimes convenient to reduce these to smaller systems with a few degrees of freedom, such as one-dimensional mappings. This type of reduction (where applicable) endows these mappings with a dynamical role analogous to that of the equations of motion of the systems of classical mechanics. Often, like systems in the thermodynamic limit, systems with few degrees of freedom evolve to equilibrium, and their asymptotic properties can be statistically described by their invariant density alone.

Often, however, low-dimensional systems such as one-dimensional mappings do not approach a statistical equilibrium, even though they may possess an invariant density. For such systems the conventional techniques used in statistical physics (averaging over one invariant density) cannot be used to determine physical properties. Indeed a method by which to infer the *physical* properties of low-dimensional nonequilibrium systems, is not totally clear from the existing body of results in ergodic theory.

This paper studies the property of asymptotic periodicity [15, 17, 18, 20, 26] in the density evolution of one-dimensional maps. By appropriate interpretation of measure theoretic results, we present a formalism in which these systems can be viewed as nonequilibrium dynamical systems whose physical properties must be directly associated with the statistical dynamics of several metastable states, rather than one invariant density. In so doing a methodology is developed that associates the mathematical language of measure theory to direct physical properties of asymptotically periodic transformations.

In this section we briefly discuss the trajectory versus density approach and introduce the Frobenius–Perron operator, a linear integral (Markov) operator that governs the flow of densities. Section 2 reviews the dynamical concept of asymptotic periodicity, deriving a general formulation of the autocorrelation function for systems displaying asymptotic periodicity. Using this formulation a general form for the autocorrelation function of the quadratic map is obtained that does not presuppose the decomposition of trajectories into periodic and stochastic components. There we also introduce a generalization of the Boltzmann–Gibbs entropy, known as the conditional entropy, and use it as an index to describe the degree of information required to localize a system in phase space. In sections 3 and 4 we illustrate the behavior of asymptotically periodic systems using two maps which give rise to "banded chaos" [8, 21, 29] for certain parameter values.

The first map we study (section 3) is the hat map

$$S_a(x) = ax, \qquad 0 \le x \le \tfrac{1}{2},$$
$$= a(1-x), \quad \tfrac{1}{2} < x \le 1, \tag{1}$$

where $0 < a \le 2$. For $2^{1/2^{n+1}} < a \le 2^{1/2^n}$ ($n = 0, 1, \dots$), it has been shown [29] that (1) generates banded chaos of period 2^n. The second map (section 4) is the quadratic map

$$S(x) = rx(1-x), \quad 0 < r \le 4, \tag{2}$$

which has been extensively studied because of its wealth of dynamical properties. Jacobson [14] has recently studied the properties of the invariant densities of this map for r near $r = 4$. The quadratic map (2) has also seen extensive use in physics, either as a dynamical system in its own right [2], or as approximations to Poincaré sections of low-dimensional strange attractors of dissipative systems. Examples include Couette flow, Bénard instability [12], and a truncated approximation of the Navier–Stokes equations [6]. For the quadratic map there exists a set of parameters, $\{r_n\}$, where banded chaos of period 2^n emerges [8].

Qualitatively the phenomenon of banded chaos in (1) and (2) is characterized by the emergence of a phase space attractor on $[0,1]$ comprising 2^n bands. A trajectory of the time series of (1), (2) periodically visits each band. However, the motion within each band is chaotic, possessing a positive Lyapunov exponent. More precisely, denoting the ith band by J_i, the trajectory of S^{2^n}: $J_i \to J_i$ is chaotic. Lorenz [21] called this phenomenon *noisy periodicity*. The construction of the bands J_i and the maps S^{2^n}: $J_i \to J_i$ are given in refs. [29, 8], for the hat map and the quadratic map, respectively.

Finally in section 5 we numerically examine the parameter dependence of the limiting conditional entropy of the hat and quadratic maps in the asymptotically periodic regime.

1.1. Trajectory versus density evolution

Knowing the phase space attractor of a chaotic dynamical system, but not the actual solution of the trajectory through it, is reminiscent of the situation encountered when dealing with the N-body problem. To make the study of the evolution of a chaotic dynamical system more tractable, we may argue that a macroscopic state of a system, at a time t, is not in general given by a single point in phase space, but rather a collection, or *ensemble* of points. This ensemble is distributed according to some density, f_t. The evolution of a system, in this formalism, is therefore given by the evolution (or flow) of densities. In this approach, exact values are replaced by ensemble averages or expectations weighted by the phase space density f_t.

For systems of statistical mechanics in the thermodynamic limit (number of particles $N \to \infty$), it is assumed that the evolution of densities attain the density f_z of the canonical ensemble, i.e. $\lim_{t\to\infty} f_t = f_z$. All equilibrium properties of systems in the thermodynamic limit can, in principle, be derived from the stationary density f_z. In a similar fashion one dimensional maps whose density sequence f_t attains an invariant density f^* can be physically described using f^*. The properties of invariant densities of one-dimensional maps has been the subject of considerable study in recent years. Haufbauer and Keller [9–11, 15] have rigorously dealt with the properties of invariant densities of piecewise monotonic transformations as well as the rates of convergence to this invariant density. Also Mackey [22] has examined dynamical systems in a physical context, with the aim of understanding and classifying the conditions under which these systems may display a strong convergence to an equilibrium density and the consequences of this for the evolution of their entropy.

In general, however, for low-dimensional systems such as (1) and (2) the evolution of densities can display several types of behavior, three of which we will review below. In all three types of behavior an invariant density, denoted f^*, exists but f_t need not approach it, i.e. $\lim_{t\to\infty} f_t \neq f^*$. Asymptotically periodic systems are one such class of systems that do not in general attain their invariant density. Hence the properties of their invariant density cannot be used to infer the physical properties of these systems, even after an infinite period of time. As a result, a central question that arises is how one can statistically describe the physical properties of asymptotically periodic systems given that their flow of densities never equilibrates to a stationary density f^*.

1.2. Markov operators and the evolution of densities

The evolution of densities under the action of a dynamical system, S, is described by a Markov operator which we denote by P. Formally any linear operator P^t: $L^1 \to L^1$ that satisfies

$$P^t f > 0 \quad \text{and} \quad \int_X P^t f(x)\,dx = \int_X f(x)\,dx$$

is called a *Markov operator* [19], where X denotes the phase space on which S operates. Throughout this paper we deal with a subset of L^1 functions which are everywhere nonnegative and normalized to one. This is the set of densities and is denoted by D. It is clear that when a Markov operator acts on a density it yields another density. Beginning with an ensemble of phase space points representing some macroscopic state of a system, and distributed according to a density f_0, one unit of time (iteration) later the new density state of the system, f_1, is given by $f_1 = Pf_0$. For deterministic systems P is also known as the *Frobenius–Perron* operator, and if S is one dimensional it is given by

$$Pf(x) = \frac{d}{dx}\int_{S^{-1}([0,x])} f(y)\,dy. \tag{3}$$

Markov operators may also possess a *stationary density* f^*. This density satisfies $Pf^* = f^*$ and may be associated with a state of thermodynamic equilibrium of a dynamical system.

The evolution of densities under (3) characterizes P as well as the dynamical map S [19]. Three general behaviors may be displayed by the sequence $\{P^t f_0\}$, where f_0 represents the density of initial preparation of the system. These are *ergodicity*, *mixing*, and *exactness*. In all three cases the system possesses an invariant density f^*. However, the three behaviors differ in the way the sequence $\{P^t f_0\}$ converges to f^*.

Of the three, exactness implies the strongest form of convergence of $\{P^t f_0\}$. Mathematically a system is said to be exact if and only if

$$\lim_{t\to\infty} \|P^t f_0 - f^*\| = 0$$

as $t \to \infty$, for all initial densities f_0. Exactness may be considered as the analog of an approach to equilibrium from all initial preparations of a system.

Mixing implies a weaker form of convergence of $\{P^t f_0\}$. In particular for any function $\mathscr{F}$ a mixing system satisfies

$$\lim_{t\to\infty} \langle P^t f_0, \mathscr{F}\rangle = \langle f^*, \mathscr{F}\rangle,$$

for all initial densities f_0. Mixing systems spread densities throughout the accessible phase space, as determined by the support of f^*.

Ergodicity implies the weakest form of convergence of $\{P^t f_0\}$. For ergodic systems

$$\lim_{t\to\infty} \frac{1}{t}\sum_{n=1}^{t-1} \langle P^n f_0, \mathscr{G}\rangle = \langle f^*, \mathscr{G}\rangle$$

for all $f_0 \in D$ and any function $\mathscr{G}$. Exactness implies mixing which in turn implies ergodicity. However ergodicity alone does not constrain the sequence $\{P^t f_0\}$ to become asymptotically equal to f^*.

2. Asymptotic periodicity

Asymptotic periodicity is a type of density evolution that may be displayed by one-dimensional maps. Without loss of generality the phase space of these maps will be taken as $[0,1]$. For asymptotically periodic systems the sequence $\{P^t f_0\}$ satisfies a spectral decomposition theorem [18]. This theorem states that when P is asymptotically periodic, then for any initial density f_0,

$$Pf_0(x) = \sum_{i=1}^{r} \lambda_i(f_0)\, g_i(x) + Qf_0(x), \tag{4}$$

where the functions g_i form a sequence of r densities satisfying $g_i g_j = 0$ if $i \neq j$, $i,j = 1,\dots,r$. This condition implies that the supports of the g_i

densities, denoted supp g_i, are disjoint. Also the g_i satisfy $Pg_i = g_{\alpha(i)}$, where $\alpha(i)$ is a permutation of the numbers $\{1, 2, \ldots, r\}$. For ergodic systems $\alpha(i)$ must be a cyclic permutation [19]. The scaling coefficients $\lambda_i(f_0)$ are linear functionals of the initial density f_0 given by

$$\lambda_i(f_0) = \int_0^1 Z_i(x)\, f_0(x)\, \mathrm{d}x,$$

where $\{Z_i(x)\}$ is a sequence of L^∞ functions. The symbol Q is called the transient operator, and satisfies $\|Q^t f_0\| \to 0$ as $t \to \infty$. The tth iterate, $P^t f_0$, may be written as

$$P^t f_0(x) = \sum_{i=1}^{r} \lambda_i(f_0)\, g_{\alpha^t(i)}(x) + Q^t f_0(x). \tag{5}$$

Allowing the transient operator to decay and noting that the permutation $\alpha(i)$ is invertible we may write

$$P^t f_0(x) = \sum_{i=1}^{r} \lambda_{\alpha^{-t}(i)}(f_0)\, g_i(x). \tag{6}$$

From (6), it is easy to verify that the (necessarily unique) invariant density of ergodic asymptotically periodic systems is given by

$$f^*(x) = \frac{1}{r} \sum_{i=1}^{r} g_i(x). \tag{7}$$

Eq. (6) describes a density evolution that is periodic in time. At a given time $P^t f_0$ may be visualized as a linear combination of the basis states g_i, each scaled by a probabilistic weighting factor $\lambda_{\alpha^{-t}(i)}(f_0)$. Since we are dealing with densities, the $\lambda_i(f_0)$ sum to 1. Moreover (6) shows that asymptotically periodic systems will generally not evolve to the stationary density f^*. Rather $P^t f_0$ will indefinitely cycle through an entire set of densities, each of which depend functionally on the initial density f_0.

Physically we associate the densities in decomposition (6) with a set of *metastable* states, each transformed into another of the sequence under the operation of P, which defines the mechanism by which the system evolves. A real system can only be prepared to be in some state of its phase space distributed according to a density, f_0. If the underlying dynamics of this system are asymptotically periodic however, the question of which density state will asymptotically describe this system is not defined. Physically therefore an asymptotically periodic system must be treated as a purely nonequilibrium system, periodically alternating among the metastable states described in the expansion (6), with some characteristic period. The measurable properties of these dynamical systems can be statistically determined only at discrete times of a cycle of $\{P^t f_0\}$, using as an appropriate weighting density the metastable state in the expansion (6) corresponding to the particular time in the cycle. In particular, the coefficient $\lambda_{\alpha^{-t}(i)}$ gives a measure of the probability of an asymptotically periodic system, described by the map S, being in a basis state g_i at the time t. Scaling of more than one basis state implies that the system has a probability of being in more than one basis state. When only one term is present in the sum of (6), the system will be found in one g_i state at all times.

Expectation values of a measurable quantity, O, at a time t are given by weighting over the density $P^t f_0$. It is clear from (6) that $\langle O \rangle$ will generally be periodic in time. The time dependence of the oscillation is found from (6) to be

$$\langle O \rangle(t) = \sum_{i=1}^{r} \lambda_{\alpha^{-t}(i)}(f_0) \langle O(x) \rangle_i, \tag{8}$$

where

$$\langle O(x) \rangle_i = \int_{\mathrm{supp}\, g_i} O(x)\, g_i(x)\, \mathrm{d}x.$$

From (8) it is clear that the $\lambda_{\alpha^{-t}(i)}(f_0)$ define the measurable properties associated with the metastable state $P^t f_0$.

2.1. The autocorrelation function in asymptotically periodic systems

From the stochastic as well as the periodic properties of the metastable states $\{P^t f_0\}$ defined by (6), it is possible to calculate all nonequilibrium properties of the system represented by an asymptotically periodic map S. In this section the time correlation function is calculated for asymptotically periodic systems using the properties of the decompositions (5) and (6). We assume that we are dealing with an ergodic asymptotically periodic system, so the permutation $\alpha(i)$ is cyclic [19], i.e. $\alpha(i) = (i+1) \bmod r$, where r is the number of elements in the sum of (5) or (6).

Using the properties of the Markov operator, the autocorrelation function can be written as $R_{xx}(\tau) \equiv \langle x_t x_{t+\tau} \rangle$, so the time correlation function is written as $C(\tau) = R_{xx}(\tau) - \langle x \rangle^2$. The autocorrelation function can be written using the properties of the Markov operator as

$$R_{xx}(\tau) = \int_0^1 x P^\tau[x f^*(x)]\,\mathrm{d}x. \tag{9}$$

From the invariant density (7) we have

$$P^\tau[x f^*(x)] = \frac{1}{r} \sum_{i=1}^{r} P^\tau[x g_i(x)]. \tag{10}$$

Also, by (4) we may write

$$P[x g_i(x)] = \lambda_i(x g_i)\, g_{\alpha(i)}(x) + Q[x g_i(x)],$$

where only one term appears since

$$\lambda_i(x g_i) = \int_{\operatorname{supp} g_i(x)} Z_i(x)\,[x g_i(x)]\,\mathrm{d}x. \tag{11}$$

Hence,

$$P^\tau[x g_i(x)] = \lambda_i(x g_i)\, g_{\alpha^\tau(i)}(x) + Q^\tau[x g_i(x)]. \tag{12}$$

Since Markov operators satisfy $\|P^\tau f_0\| = \|f_0\|$, and $\|Q^t(f_0)\| \to 0$ as $t \to \infty$, we have

$$\lambda_i(x g_i) = \langle x \rangle_i. \tag{13}$$

Substituting (13) into (12), using the definition (9) of the autocorrelation, and noting that

$$\langle x \rangle = \frac{1}{r} \sum_{i=1}^{r} \langle x \rangle_i$$

and

$$\langle x \rangle^2 = \frac{1}{r^2} \sum_{i,j=1}^{r} \langle x \rangle_i \langle x \rangle_j, \tag{14}$$

the time correlation function, $C(\tau)$, takes the form

$$C(\tau) = \frac{1}{r} \sum_{i=1}^{r} \left(\langle x \rangle_{\alpha^\tau(i)} - \frac{1}{r} \sum_{j=1}^{r} \langle x \rangle_j \right) \langle x \rangle_i + \sum_{i=1}^{r} \chi_i(\tau), \tag{15}$$

where the $\chi_i(\tau)$ are defined by

$$\chi_i(\tau) = \frac{1}{r} \int_0^1 x Q^\tau[x g_i(x)]\, dx. \tag{16}$$

By the properties of the transient operator Q the terms $\xi_i(\tau) \to 0$ as $\tau \to \infty$.

The first term of (15) is periodic due to the cyclicity of the permutation $\alpha(i)$. To see this recall that $\alpha^\tau(i) = (i+\tau) \bmod r$. Extract the $j = (i+\tau) \bmod r$ term of the sum in the second term of (15) and add it to single term $\langle x \rangle_{\alpha^\tau(i)}$. The first term of (15) can then be rewritten as

$$\frac{1}{r}\sum_{i=1}^{r}\left(\frac{r-1}{r}\langle x\rangle_{\alpha^{\tau}(i)}-\frac{1}{r}\sum_{j,j\neq i+\tau}^{r}\langle x\rangle_{j}\right)\langle x\rangle_{i}. \tag{17}$$

With the aid of the identity

$$\sum_{m=1}^{n-1}\exp\left(2\pi\sqrt{-1}\,\frac{mz}{n}\right)=-1, \tag{18}$$

where n and z are integers, with z nonzero, (17) can be written as

$$\frac{1}{r^2}\sum_{i=1}^{r}\left\{\sum_{j=1}^{r}\langle x\rangle_j \times\left[\sum_{m=1}^{r-1}\exp\left(2\pi\sqrt{-1}\,\frac{m(i+\tau-j)}{r}\right)\right]\right\}\langle x\rangle_i. \tag{19}$$

The sum $i+\tau$ in the exponent of (19) is understood to be modulo r. We make the substitution $k=m+1$ in (19) and use the fact that $\exp[2\pi\sqrt{-1}\ (I \bmod r)/r]=\exp(2\pi\sqrt{-1}\ I/r)$, hence dropping the "modulo" notation. Also, we define the *discrete frequencies* $\omega_j\equiv 2\pi(j-1)/r$. With these substitutions in (15) the time correlation function becomes

$$C(\tau)=\sum_{m=2}^{r}|\psi(\omega_m)|^2\exp\left[\sqrt{-1}\,\omega_m\tau\right]+\sum_{i=1}^{r}\chi(\tau), \tag{20}$$

where

$$\psi(\omega_m)\equiv\frac{1}{r}\sum_{k=1}^{r}\langle x\rangle_k\exp\left[\sqrt{-1}\,\omega_m(k-1)\right]. \tag{21}$$

Note that with the substitution $k=m+1$ in (19) the periodic part of (20) begins at $m=2$.

An interesting property of asymptotically periodic systems becomes apparent from (20). Namely, the correlation function $C(\tau)$ naturally decomposes into periodic and stochastic components. This decoupling of the time correlation function into two independent components can be understood as follows. Asymptotically periodic systems have r disjoint attracting regions of their phase space X whose union is given by

$$\bigcup_{i=1}^{r}\operatorname{supp} g_i.$$

Each of the regions $\operatorname{supp} g_i$ map onto each other cyclically according to $\alpha(i)$. All ensembles of initial conditions will asymptotically map into these regions (i.e. all densities will decompose). Thus a time series will also visit these supports periodically, and we expect a periodic component in the time correlation function. However, iterates of the time series which return into any one of the $\operatorname{supp}\{g_i\}$, are described by a density g_i, and so there must exist a stochastic component of the correlation function (the second term of (20)).

2.2. *The conditional entropy for asymptotically periodic systems*

Assuming the existence of a density f describing a thermodynamic state of a system at a time t, Gibbs introduced the concept of the *index of probability*, given by $-\log f(x)$. Weighting the index of probability by the density f, he introduced what is now known as the *Boltzmann–Gibbs entropy*, given by

$$H(f)=-\int_X f(x)\log f(x)\,\mathrm{d}x.$$

It can be shown [16, 25] that the Boltzmann–Gibbs entropy is the only entropy definition satisfying the property of being an extensive quantity, which a mathematical analog of the thermodynamic entropy should have.

The Boltzmann–Gibbs entropy can be generalized by introducing the *conditional entropy*. If f and g are two densities such that $\operatorname{supp} f \subset \operatorname{supp} g$, then the conditional entropy of the density f with respect to the density g is defined as

$$H_c(f|g) = -\int_X f(x) \log\left(\frac{f(x)}{g(x)}\right) dx. \tag{22}$$

As $H_c(f|g) = 0$ when $f = g$, this implies that the conditional entropy is a measure of how close the functional form of f is to g. Moreover, using an identity known as the Gibbs inequality it can be shown [19] that the conditional entropy satisfies

$$H_c(f|g) \leq 0. \tag{23}$$

When dealing with asymptotically periodic systems the limiting conditional entropy takes on a particularly transparent form, clearly expressing that $\lim_{t\to\infty} H_c(P^t f_0|f^*)$ is dependent on the initial preparation of the system through f_0. To see this, use the invariant density (7) along with the asymptotic decomposition (6), and the orthogonality of the g_i, to obtain

$$\begin{aligned}\lim_{t\to\infty} H_c(P^t f_0|f^*) &= \sum_{i=1}^{r} \int_X \lambda_{\alpha^{-t}(i)}(f_0)\, g_i(x) \log\left[r\lambda_{\alpha^{-t}(i)}(f_0)\right] dx \\ &= \sum_{i=1}^{r} \int_X \lambda_{\alpha^{-t}(i)}(f_0)\, g_i(x) \log \lambda_{\alpha^{-t}(i)}(f_0) \\ &\quad + \log(r)\, dx. \end{aligned} \tag{24}$$

Also since the permutation $\alpha(i)$ is invertible we have

$$\sum_{i=1}^{r} \lambda_{\alpha^{-t}(i)}(f_0) = \sum_{i=1}^{r} \lambda_i(f_0). \tag{25}$$

Thus, defining

$$H_c^{\infty}(P^t f_0|f^*) \equiv \lim_{t\to\infty} H_c(P^t f_0|f^*), \tag{26}$$

we may re-express the limiting conditional entropy as

$$H_c^{\infty}(P^t f_0|f^*) = -\log r - \sum_{i=1}^{r} \lambda_i(f_0) \log \lambda_i(f_0). \tag{27}$$

Noting that the $0 \leq \lambda_i(f_0) \leq 1$ for all i we obtain

$$-\log r \leq H_c^{\infty}(P^t f_0|f^*) \leq 0. \tag{28}$$

When an initial density f_0 is localized over one of $\operatorname{supp} g_i$ then $\{P^t f_0\}$ will asymptotically cycle through the sequence $\{g_i\}$. In this case there is only one component to the spectral decomposition (5) at any time t. According to (27) this situation is one of lowest conditional entropy and $H_c^{\infty}(P^t f_0|f_0) = -\log r$. Physically this implies that the initial ensemble of phase space points, described by f_0, will evolve through $\operatorname{supp} g_i$ in the most localized manner possible. At any time t, only a "pure state" g_i is needed to describe the statistical properties of the system. The metastable states of the expansion (6) are then just the sequence $\{g_i\}$ itself.

In general any f_0 whose support runs over the boundary of some g_i will cause $P^t f_0$ to decompose into a linear combination of several densities g_i. At a time t the members of an ensemble are now less localized, and more information is required to determine their distribution through the phase space. As a result the conditional entropy of a linear combination of several densities g_i is higher than for a single pure state g_i.

It has been shown [27] that if P is a Markov operator then the condition entropy must satisfy

$$H_c(P^t f_0|f^*) \geq H_c(f_0|f^*). \tag{29}$$

By the decomposition (5), the density sequence $\{P^t f_0\}$ settles onto a periodic cycle after a sufficiently long transient. The rate at which the transient decays is controlled by the transient term $Q^t f_0$ in the expansion (5). As a result, since $\|Q^t f_0\| \to 0$ as $t \to \infty$, the conditional entropy of asymptotically periodic systems is a nondecreasing function with an upper bound given by (27), and satisfying

$$\Delta H_c(P^t f_0 | f^*) \geq 0, \tag{30}$$

where Δ denotes the temporal change in H_c. In the special case when f_0 is a linear combination of the states g_i there is no transient term in the expansion (5). In that case the equality in (30) holds.

Eq. (27) shows that the limiting conditional entropy of an asymptotically periodic system approaches a value uniquely determined by the density of the initial preparation of the system, while the iterates $P^t f_0$ remain asymptotically periodic. This implies that all density states within the cycle to which $\{P^t f_0\}$ converges are of the same entropy with respect to the stationary density (7).

3. Asymptotic periodicity in the hat map

While much work has been done on the hat map (1), it has rarely been studied from the point of view of density evolution. In this section it is shown that the hat map is in fact asymptotically periodic, possessing a spectral decomposition of the form (5). Thus it falls within the formalism developed above.

Using eq. (3) it is easy to show that the Frobenius–Perron operator corresponding to the hat map is given by

$$Pf(x) = \frac{1}{a}\left[f\left(\frac{x}{a}\right) + f\left(1 - \frac{x}{a}\right)\right]. \tag{31}$$

To show that the sequence $\{P^t f_0\}$, where P is defined by (31), is asymptotically periodic we use a result from ref. [19; theorem 6.4.1], which guarantees that the Frobenius–Perron operator corresponding to a map S displays asymptotic periodicity when the following set of (sufficient) conditions are satisfied:

(1) There exists a partition $0 = b_0 < b_1 < \ldots < b_m = 1$ of $[0, 1]$ such that for each integer $i = 1, \ldots, m$ the restriction of $S(x)$ to $[b_{i-1}, b_i]$ is a C^2 function.

(2) $|S'(x)| \geq \vartheta > 1$, $x \neq b_i$.

(3) There exists a real constant c such that $|S''(x)|/|S'(x)|^2 \leq c < \infty$, $x \neq b_i$, $i = 0, 1, \ldots, m$.

It is clear that for $1 < a \leq 2$, and for the partition $b_0 = 0 < b_1 = \frac{1}{2} < b_2 = 1$, (1) satisfies these conditions. Thus, the hat map is asymptotically periodic and the evolution of densities via the operator (31) can be expressed through the spectral decomposition (5).

The hat map is also ergodic [13], possessing a unique invariant density f^* of the form (7). Its form has been derived in the parameter window $a_{n+1} = 2^{1/2^{n+1}} < a \leq a_n = 2^{1/2^n}$ [29].

To develop the form of f^* consider $[a_{n+1}, a_n]$, $n = 0, 1, 2, \ldots$. In each of these there exist 2^n subspaces J_l of the unit interval $[0, 1]$. The J_l themselves contain an interval $X_l \subset J_l$, $l = 1, 2, \ldots, 2^n$. On each of the subspaces J_l the map $S_a^{2^n}$: $J_l \to J_l$, $l = 1, \ldots, 2^n$, is conjugate to the map $S_{a^{2^n}}$: $[0, 1] \to [0, 1]$, where $a_1 \leq a^{2^n} \leq a_0$. (Two functions $f(x)$ and $g(x)$ are *conjugate* if there exists a transformation $\Gamma(x)$ such that $g(x) = \Gamma^{-1} \circ f \circ \Gamma(x)$, where the "$\circ$" denotes composition.)

The mapping $S_a^{2^n}$: $J_l \to J_l$ conjugate to $S_{a^{2^n}}$: $[0, 1] \to [0, 1]$ is given by

$$S_a^{2^n}(x) = \Psi_{l,a}^{-1} \circ S_{a^{2^n}} \circ \Psi_{l,a}(x), \quad l = 1, \ldots, 2^n.$$

The transformation $\Psi_{l,a}(x)$ is given by

$$\Psi_{l,a}(x) = \Gamma_{i_n, a^{2^{n-1}}} \circ \Gamma_{i_{n-1}, a^{2^{n-2}}} \circ \ldots \circ \Gamma_{i_1, a}, \tag{32}$$

where

$$l = 1 + i_1 + 2i_2 + \ldots + 2^{n-1} i_n,$$

$$i_k = 0, 1, \quad k = 1, 2, \ldots, n,$$

and

$$\Gamma_{i,a}(x) = \frac{x - x^*}{m_i}, \quad i = 0,1, \tag{33}$$

with

$$m_0 = \delta_1 - x^* \quad \text{and} \quad m_1 = -(\gamma_1 - x^*). \tag{34}$$

The constants δ_1, γ_1 are solutions of $S_a^2(x) = x^*$ where x^* is the nonzero fixed point of (1). The J_l represent miniature replicas of [0, 1] and are given by the set

$$J_l = \{0 \le \Psi_{l,a}(x) \le 1\}, \tag{35}$$

while the union of the X_l define the attracting basin of phase space, as $t \to \infty$, and are given by

$$X_l = \{x: S_{a^{2^n}}^2(\tfrac{1}{2}) \le \Psi_{l,a}(x) \le S_{a^{2^n}}(\tfrac{1}{2})\},$$
$$l = 1,\dots,2^n. \tag{36}$$

Further, the X_l are invariant under the map $S_a^{2^n}$: $J_l \to J_l$ and satisfy S_a: $X_l \to X_{l+1}$, $l = 1 \bmod 2^n, \dots 2^n \bmod 2^n$.

Each of the submaps $S_a^{2^n}$: $J_l \to J_l$ possesses an invariant density which is just a reduced copy of the invariant density of the hat map $S_{a^{2^n}}$: $[0,1] \to [0,1]$, i.e.

$$f_l^*(x, X_l: S_a^{2^n}) = \left|\frac{d\Psi_{l,a}^{-1}(x)}{dx}\right|$$
$$\times f_w^*(\Psi_{l,n}(x), X: S_{a^{2^n}}), \quad l = 1,\dots,2^n, \tag{37}$$

where a^{2^n} lies in the period one window wherein the basin of attraction consists of one band [13]. The notation $f_l(x, X_l: S_a^{2^n})$ defines the invariant density of $S_a^{2^n}$: $J_l \to J_l$, nonzero on X_l. The w subscript on the right-hand side signifies the invariant density of $S_a^{2^n}$: $[0,1] \to [0,1]$, nonzero on X.

Using (37) and the fact that the X_l map into each other cyclically, the invariant density for the period 2^n window of the hat map is the average of the 2^n densities supported on each of the 2^n subspaces X_l, i.e.

$$f_w^*(x, \Sigma_{2^n}, S_a) = \frac{1}{2^n} \sum_{l=1}^{2^n} \left|\frac{d\Psi_{l,a}^{-1}(x)}{dx}\right|$$
$$\times f_w^*(\Psi_{l,a}(x), X: S_{a^{2^n}}), \tag{38}$$

where $\Sigma_{2^n} = \bigcup_{l=1}^{2^n} X_l$. Eq. (38) can now be compared with the general expression for the invariant density for asymptotically periodic systems (7) to conclude that in the period 2^n window the hat map is asymptotically periodic with a period $r = 2^n$. The components of the invariant density (38), multiplied by 2^n, are the components of the density sequence $\{g_l\}$. Hence, the metastable states of the expansion (6) for the system of the hat map are given by

$$\lim_{t \to \infty} P^t f_0(x) = \sum_{l=1}^{2^n} \lambda_{\alpha^{-t}(l)}(f_0) \left|\frac{d\Psi_{l,a}^{-1}(x)}{dx}\right|$$
$$\times f_w^*(\Psi_{l,a}(x), X: S_{a^{2^n}}) \tag{39}$$

for all initial densities f_0. The permutation $\alpha^t(l)$ is given by $\alpha^t(l) = (l + t) \bmod 2^n$. The 2^n scaling coefficients $\lambda_{\alpha^t(l)}(f_0)$ are equal to the fraction of an initial ensemble, distributed according to f_0, that accumulates in the subspace $X_{\alpha^t(l)}$. Furthermore the 2^n coefficients oscillate among themselves with period 2^n.

An explicit expression for $\lambda_1(f_0), \lambda_2(f_0)$ may be obtained for period two asymptotic periodicity on the interval $[a_2, a_1]$. This is done by following how intervals of [0, 1] migrate into J_1 and J_2 (given by (35) with $\Psi_{l,a} = \Gamma_{i,a}$, $i = 0,1$). Suppose an initial density is supported on the entire interval [0, 1]. Consider the two intervals $[0, \gamma_1] \cup [\delta_1, 1]$, where γ_1 and δ_1 were defined above (see fig. 1). To determine how an initial density supported on the sets $[0, \gamma_1] \cup [\delta_1, 1]$ redistributes itself on $J_1 \cup J_2$, segment the sets $[0, \gamma_1]$ and $[\delta_1, 1]$ into an infinite sequence of disjoint subsets. The bounds of these subsets are given by

$$[\gamma_{k+1}, \gamma_k] \quad \text{and} \quad [\delta_k, \delta_{k+1}], \qquad k = 1,2,3,\dots,$$

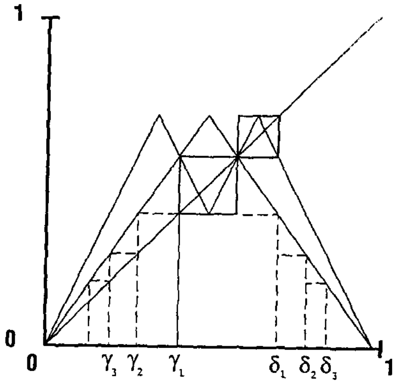

Fig. 1. A partition of the interval [0, 1] into sets $[\gamma_{n+1}, \gamma_n]$, $[\delta_n, \delta_{n+1}]$, $n = 1, 2, 3, \ldots$. By following which of these sets asymptotically maps into the spaces J_0 and J_1 the coefficients $\lambda_1(f_0), \lambda_2(f_0)$ of the decomposition (39) can be determined.

where

$$\gamma_k = \frac{1}{a^{k-1}(a+1)} \quad \text{and} \quad \delta_k = 1 - \gamma_{k+1},$$

$$k = 1, 2, \ldots .$$

We follow what fraction of an initial ensemble of points, distributed according to f_0, winds up in the subspace J_1 and what fraction in J_2. Let us denote these fractions by μ_{J_l}, $l = 1, 2$. Then with the aid of fig. 1 it is not difficult to show that

$$\mu_{J_1} = \sum_{k=1}^{\infty} \left(\int_{\gamma_{2k}}^{\gamma_{2k-1}} f_0(x)\,dx + \int_{\delta_{2k}}^{\delta_{2k+1}} f_0(x)\,dx \right),$$

$$\mu_{J_2} = \sum_{k=1}^{\infty} \left(\int_{\gamma_{2k+1}}^{\gamma_{2k}} f_0(x)\,dx + \int_{\delta_{2k-1}}^{\delta_{2k}} f_0(x)\,dx \right). \tag{40}$$

Therefore, since $\lambda_1(f_0)$ and $\lambda_2(f_0)$ also depend on the density f_0 supported on J_0, J_1 we have

$$\lambda_1(f_0) = \int_{J_2} f_0(x)\,dx + \sum_{k=1}^{\infty} \left(\int_{\gamma_{2k+1}}^{\gamma_{2k}} f_0(x)\,dx + \int_{\delta_{2k-1}}^{\delta_{2k}} f_0(x)\,dx \right),$$

$$\lambda_2(f_0) = \int_{J_1} f_0(x)\,dx + \sum_{k=1}^{\infty} \left(\int_{\gamma_{2k}}^{\gamma_{2k-1}} f_0(x)\,dx + \int_{\delta_{2k}}^{\delta_{2k+1}} f_0(x)\,dx \right), \tag{41}$$

valid for all initial densities f_0.

Fig. 2 illustrates the asymptotic sequence to which the sequence $\{P^t f_0\}$ evolves for three choices of initial density f_0 when $a = \sqrt{2}$. For this choice of a it is easily verified via (41) that the invariant density is

$$f^*(x) = \frac{3 + 2\sqrt{2}}{2} 1_{J_2}(x) + \frac{4 + 3\sqrt{2}}{2} 1_{J_1}(x),$$

where $1_A(x)$ is the indicator function, defined as

$$1_A(x) = 0, \quad x \notin A,$$
$$= 1, \quad x \in A. \tag{42}$$

In figs. 2a and 2b there are no transients ($Qf_0 = 0$) and $P^t f_0$ is periodic from the first iteration. In fig. 2c, however, $P^t f_0$ is periodic only after two transients. The limiting conditional entropy $H_c^{\infty}(P^t f_0 | f^*)$ is lowest in the sequence of fig. 2a, as there $P^t f_0$ cycles through the entire $\{g_l\}$ sequence.

Using eqs. (16)–(21) the correlation function of the hat map may be obtained, recovering the form originally derived in ref. [29].

4. Asymptotic periodicity and the quadratic map

It has been shown [5, 24] that maps with a single quadratic maximum like (2) display period doubling in the number of periodic points as the parameter r is increased. For example, with $1 < r \leq 3$, the trajectories of eq. (2) all converge to

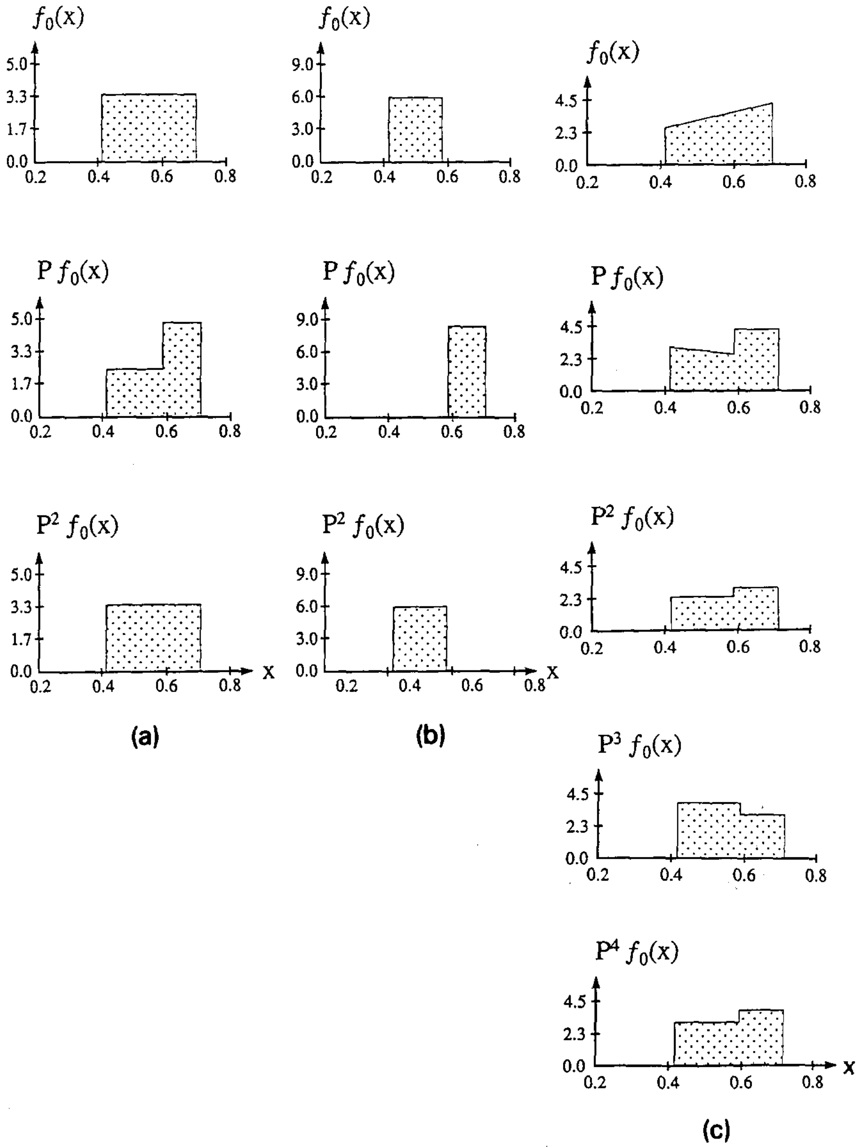

Fig. 2. The evolution of $P^t f_0$ in the period two window of the hat map, with $a = \sqrt{2}$. In (a) f_0 is uniform over $J_1 \cup J_2$. Since the g_i are uniform over J_i, $i = 0, 1$, $P^t f_0$ sets into immediate oscillations without transients. In (b) f_0 is uniform over the subspace J_2. Again $P^t f_0$ sets into immediate oscillations through the states g_1 and g_2. In (c) $f_0(x) = \frac{4}{7}(5 + \sqrt{2})x$, restricted to $J_1 \cup J_2$. Now $P^t f_0$ evolves through two transient densities before settling into a periodic oscillation.

one stable fixed point. Between $3 < r \leq r_c \approx 3.57\ldots$ there is a cascade of parameters which sequentially give rise to first 2 periodic points, then 4, 8, etc. The periodicity of trajectories in each of these intervals is equal to the number of periodic points. At r_c, also known as the accumulation point, trajectories becomes aperiodic with the number of periodic points becoming infinite.

On the other side of the critical parameter, $r_c < r \leq 4$, the quadratic map (and maps like it) contains a spectrum of parameter values, labeled by r_n, $n = 1, 2, \ldots,$ where so-called "banded chaos" has been numerically reported [1, 7]. Recent results on the invariant measures of the quadratic map for r near $r = 4$ have been reported by Jakobson [14]. At these values the unit interval $X = [0, 1]$ partitions into 2^n subintervals, labeled J_l, $l = 1, 2, \ldots, 2^n$. These are such that S^{2^n}: $J_l \to J_l$ maps J_l *onto* J_l. As well each J_l is mapped cyclically through the whole sequence of $\{J_l\}$ after 2^n applications of S. The iterates of a time series are attracted to these J_l subspaces, returning to any J_l every 2^n iterations. These iterates form a sequence with a positive Lyapunov exponent [4] containing no observable periodicity. The recipe by which one obtains the parameter values r_n at which period 2^n banded chaos occurs is given in ref. [8].

The Frobenius–Perron operator corresponding to the quadratic map (2) is

$$Pf(x) = \frac{1}{\sqrt{1 - \frac{4x}{r}}}\left[f\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4x}{r}}\right) + f\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4x}{r}}\right)\right]. \tag{43}$$

Numerically, the iterates of any initial density f_0 supported on $[0, 1]$, acted on by (43), will eventually decompose, so they are supported on the J_l, as illustrated in fig. 3 for r_1 and r_2. Subsequent to the contraction of density supports onto the sequence $\{J_l\}$, the evolution of the sequence $\{P^t f_0\}$ becomes periodic in time. We show here that at $r = r_n$, the numerically observed periodic evolution of ensemble densities, $P^t f_0$, under the action of the operator (43) is, in fact, asymptotically periodic.

4.1. Spectral decomposition of the Frobenius–Perron operator

To prove that the Frobenius–Perron operator of the quadratic map is asymptotically periodic

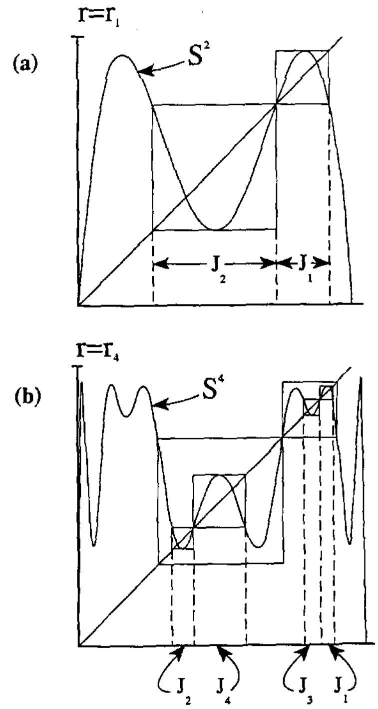

Fig. 3. The two frames show respectively the iterated map S^{2^n}, for $n = 1, 2$, at $r = r_n$, on the unit interval. The bases of the smallest boxes represent the spaces J_1, J_2 in (a) and J_1, J_2, J_3, J_4 in (b).

we first state the following lemma whose proof is easy.

Lemma 1. If S is an exact transformation, P is the Frobenius–Perron operator corresponding to S, and $f^*(x)$ is the stationary density of P, then

$$\lim_{t \to \infty} P^t f_0 = K f^*(x)$$

for all $f_0 \in L^1$, $f_0 \geq 0$ where $K = \int_0^1 f_0(x)\, dx$.

Next we state and prove a theorem necessary for a demonstration of the spectral decomposition of the iterates of the Frobenius–Perron operator of the quadratic map.

Theorem 2. Let S be a transformation acting on a phase space $X = [0, 1]$, such that there are N subspaces J_i, $i = 1, \ldots, N$, $J_i \subset X$, which map onto each other cyclically under the action of S. Also, assume that S^N: $J_\nu \to J_\nu$ is exact for some $\nu \in \{1, 2, \dots, N\}$. If

$$\lim_{t\to\infty} \operatorname{supp}\{P^t f_0\} \subset \Sigma \equiv \bigcup_{i=1}^{N} J_i,$$

for all initial densities f_0 then the Markov operator P corresponding to S is asymptotically periodic on X.

Proof. Since the support of any initial density f_0 will eventually contract into Σ, a density, call it $h(x)$, entirely supported in Σ will eventually be obtained. We can write $h(x)$ as

$$h(x) = \sum_{i=1}^{N} \xi_i(x)\, 1_{J_i}(x),$$

where $\xi_i(x) \in L^1$. Denote S^N: $J_\nu \to J_\nu$ by G, and let P_G be the corresponding Frobenius–Perron operator of the map G. Then, since G is exact, by lemma 1 it follows that

$$\lim_{t\to\infty} P_G^t\big[\xi_\nu(x)\, 1_{J_\nu}(x)\big] = \lambda_\nu(\xi_\nu)\, f_{J_\nu}^*(x)\, 1_{J_\nu}(x), \tag{44}$$

where $f_{J_\nu}^*(x) 1_{J_\nu}(x)$ is the stationary density of P_G and $\lambda_\nu(\xi_\nu) = \int_{J_\nu} \xi_\nu(x)\,\mathrm{d}x$.

Alternatively, if τ is an integer, we can write (44) as

$$\lim_{\tau\to\infty} P^{\tau N}\big[\xi_\nu(x)\, 1_{J_\nu}(x)\big] = \lambda_\nu(\xi_\nu)\, f_{J_\nu}^*(x)\, 1_{J_\nu}(x). \tag{45}$$

Hence, there must exist a sequence of densities $\{f_{J_j}^*\}$, supported on J_j, $j = 1, 2, \dots, N$, such that

$$P f_{J_j}^*(x) = f_{J_{\alpha(j)}}^*(x) \quad \text{where} \quad \alpha(j) = (j+1) \bmod N. \tag{46}$$

Since all subspaces map onto each other cyclically, (45) and (46) imply that

$$\lim_{\tau\to\infty} P^{N\tau}\big[\xi_j(x)\, 1_{J_j}\big] = \lambda_j(\xi_j)\, f_{J_j}^*(x)\, 1_{J_j}(x) \quad \text{for all } j = 1, \dots, N, \tag{47}$$

where $\lambda_j(\xi_j) = \int_{J_j} \xi_j(x)\,\mathrm{d}x$. Therefore, combining our assumptions and eq. (47) gives

$$\lim_{t\to\infty} P^t f_0(x) = \sum_{j=1}^{N} \lambda_j(\xi_j)\, f_{J_{\alpha^t(j)}}^*(x), \qquad \alpha^t(j) = (j+t) \bmod N. \tag{48}$$

Hence $\{P^t f_0(x)\}$ is asymptotically periodic, and the densities g_j are given by $g_j = f_{J_j}^*$, $j = 1, \dots, N$. Note that eq. (47) implies that the $f_{J_j}^*$ are the stationary densities of the submaps $S_{J_j}^N$: $J_l \to J_l$. □

Using theorem 2 one can prove the following theorem.

Theorem 3. At the parameter values $r = r_n$, the evolution the iterates, $P^t f_0$, of the Frobenius–Perron operator (43) corresponding to the quadratic map (2) is asymptotically periodic for all $f_0 \in D$.

Proof. The proof of theorem 3 is twofold. It must first be shown that when $r = r_n$, the 2^n subregions J_l attract all ensembles of initial conditions regardless of how they are initially distributed over the phase space [0, 1]. (This also holds exactly for an infinity of other r values where banded chaos exists with period $p2^n$, $p = 1, 2, \dots$.) This can be done by a geometrical argument. We start first with period two asymptotic periodicity (at $r = r_1$), showing that all initial conditions find their way into J_1, or J_2. For period four asymptotic periodicity ($r = r_2$) each of J_1 and J_2 split into two subintervals. Working now with the map S^2: $J_l \to J_l$, $l = 1, 2$, we show that all initial conditions having found their way into J_1 or J_2 will now migrate into one of the four subintervals born out of the period two J_1, J_2 subspaces. Generalizing the procedure thus allows us to satisfy the first requirement of theorem 2 for any r_n.

The second part of the proof entails showing that the map $G = S^{2^n}$: $J_{2^n} \to J_{2^n}$ is exact, where J_{2^n} is the (unique) chaotic band containing the critical point $x = \frac{1}{2}$. In ref. [23] a set of conditions is given for a unimodal map S, which imply the existence of a unique invariant density supported on a set $\mathscr{K}$. The set $\mathscr{K}$ is itself composed of the union of M_0 disjoint subintervals k_i. Further, because of the existence of a unique density, for all initial densities f_0 the iterates $\{P^t f_0\}$ of the Markov operator corresponding to S will be asymptotically supported on the intervals, $k_i \in \mathscr{K}$, $i = 1, \ldots, M_0$. Additionally, for each k_i we have that $S^{M_0}(k_i) = k_i$, and that S^{M_0}: $k_i \to k_i$ is exact on k_i. Now, since G is by its construction an *onto* map on J_{2^n} [7], it can be shown that the set $\mathscr{K}$ actually consists of a single banded interval. Hence $k_i = \mathscr{K} = J_{2^N}$, and so G is exact on J_{2^n}.

Thus by theorem 2 the Frobenius–Perron operator (43) for the quadratic map is asymptotically periodic at the parameter values $r = r_n$. As a result the iterates $P^t f_0$ are spectrally decomposable into a linear combination of 2^n densities g_l, i.e.

$$P^t f_0(x) = \sum_{l=1}^{2^n} \lambda_{\alpha^{-t}(l)}(f_0)\, g_l(x) + Q^{t-1} f_0(x), \tag{49}$$

where, by theorem 2 the g_l are stationary densities of the submaps S^{2^n}: $J_l \to J_l$, $l = 1, \ldots, 2^n$, and $\operatorname{supp} g_l = J_l$. □

4.2. The sequence $\{P^t f_0\}$ and the coefficients $\lambda_i(f_0)$

The parameter values $r = r_n$ define a reverse sequence to the period doubling sequence, for $r \le r_c$, described earlier [1]. For the latter sequence, we talk of a period doubling in the number of periodic points. When $r = r_n$, however, periodic points are replaced by chaotic bands and going from r_n to r_{n+1} involves a doubling in the number of bands. As the motion in these bands is chaotic, the map (2) at $r = r_n$ must be described statistically. The last section showed that a density g_l can be assigned to each one of the chaotic bands. The time dependent evolution of an *ensemble* of phase space trajectories is asymptotically described by a set of metastable densities $\{P^t f_0\}$, which is composed of different linear combination in the g_l states. The scaling coefficients of these linear combinations (decompositions) are fixed by the initial density of the ensemble, f_0. Changing from $P^t f_0$ to $P^{t+1} f_0$ entails a "shuffling" of the scaling coefficients onto different g_l. Therefore, each density of the asymptotic sequence $\{P^t f_0\}$ can be represented by some cyclic permutation of the sequence $\{\lambda_1(f_0), \lambda_2(f_0), \ldots, \lambda_{2^n}(f_0)\}$. Each $\lambda_l(f_0)$ is equal to the fraction of the initial ensemble, distributed according to f_0, that asymptotically becomes distributed over the subspace J_l at time t.

As with the hat map the scaling coefficients $\lambda_1(f_0), \lambda_2(f_0)$ can be analytically determined for period two asymptotic periodicity for the quadratic map when $r = r_1$, and the attracting phase space consists of the subspaces J_1 and J_2. These are disjoint and connected at the fixed point of (2), and S: $J_1 \to J_2$, S: $J_2 \to J_1$. The coefficients $\lambda_1(f_0), \lambda_2(f_0)$ may be obtained for any arbitrary density f_0 supported on the phase space $X = [0, 1]$. Analogous to the hat map, define a sequence of points

$$\gamma_{n+1} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\gamma_n}{r_1}} \quad \text{and} \quad \delta_n = 1 - \gamma_{n+1}, \tag{50}$$

where $\gamma_1 = \frac{1}{4} r_1^2 (1 - \frac{1}{4} r_1)$. By following the fraction of initial points, lying in $[\gamma_{n+1}, \gamma_n]$ and $[\delta_n, \delta_{n+1}]$ and distributed according to f_0, that flows into $J_1 \cup J_2$, we obtain

$$\begin{aligned} \lambda_1(f_0) &= \int_{J_2} f_0(x)\,\mathrm{d}x \\ &\quad + \sum_{k=1}^{\infty} \left(\int_{\gamma_{2k+1}}^{\gamma_{2k}} f_0(x)\,\mathrm{d}x + \int_{\delta_{2k-1}}^{\delta_{2k}} f_0(x)\,\mathrm{d}x \right), \\ \lambda_2(f_0) &= \int_{J_1} f_0(x)\,\mathrm{d}x \\ &\quad + \sum_{k=1}^{\infty} \left(\int_{\gamma_{2k}}^{\gamma_{2k-1}} f_0(x)\,\mathrm{d}x + \int_{\delta_{2k}}^{\delta_{2k+1}} f_0(x)\,\mathrm{d}x \right). \end{aligned} \tag{51}$$

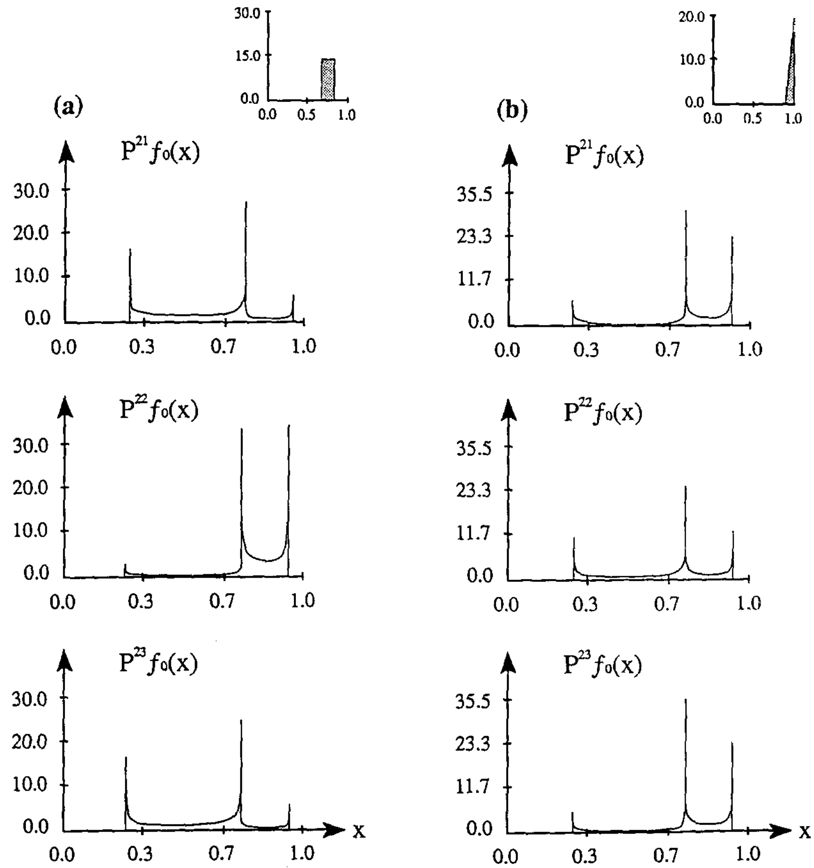

Fig. 4. A numerical illustration of one periodic cycle of the asymptotic sequence $\{P^t f_0\}$ for the parameter $r = r_1 = 3.678573508$. A transient of 20 densities has been discarded, and the iterates $P^{21}f_0$, $P^{22}f_0$, and $P^{23}f_0$ are shown. Since $P^{21}f_0 = P^{23}f_0$, the sequence $\{P^t f_0\}$ asymptotically repeats with period two. In (a) the initial density f_0, shown in the inset, is uniform over $[0.7, 0.8]$. In (b) $f_0(x) = 200(x - 0.9)$ over $[0.9, 1]$.

Fig. 4 illustrates the asymptotic evolution of $P^t f_0$ after 20 transients, for $r = r_1$. In fig. 4a the initial density is uniform on the region of $J_1 \cup J_2$ given by $[0.7, 0.8]$. Fig. 4b shows an asymptotic cycle of $P^t f_0$ with $f_0(x) = 200(x - 0.9)$ supported on $[0.9, 1]$. Fig. 5a illustrates $P^t f_0$ when $r = r_2$, with 40 transients discarded and the initial density f_0 uniform on $[0.5, 0.85]$. Fig. 5b shows one period 4 cycle of $P^t f_0$ with $f_0(x) = 200(x - 0.91)$ supported on $[0.9, 1]$.

When $r = r_n$, the density analog of a periodic orbit as executed by one initial condition is of particular interest. This happens when $\operatorname{supp} f_0 = J_l$, $l = 1, \dots, 2^n$. Then $P^t f_0$ asymptotically cycles through the sequence $\{g_1, g_2, \dots, g_{2^n}\}$. As (27) reveals this particular choice of f_0 allows the system to attain the lowest possible conditional entropy: $H_c^{\infty}(P^t f_0 | f^*) = -n \log(2)$. Any other choice of f_0 will generally involve several g_l densities in the decomposition of $P^t f_0$, and thus give a higher asymptotic conditional entropy of the system. Fig. 6a shows $P^t f_0$ after 30 transients with $r = r_1$, and with the initial density $f_0(x) = 4.52(x - 0.3)^{1/3}$, supported on $[0.3, 0.7] \subset J_2$ (the subspace containing the critical point). Fig. 6b depicts $P^t f_0$ after 63 transients when $r = r_2$, with $f_0(x) = 19.64(x - 0.41)^{1/2}$, supported on $[0.41, 0.59] \subset J_4$.

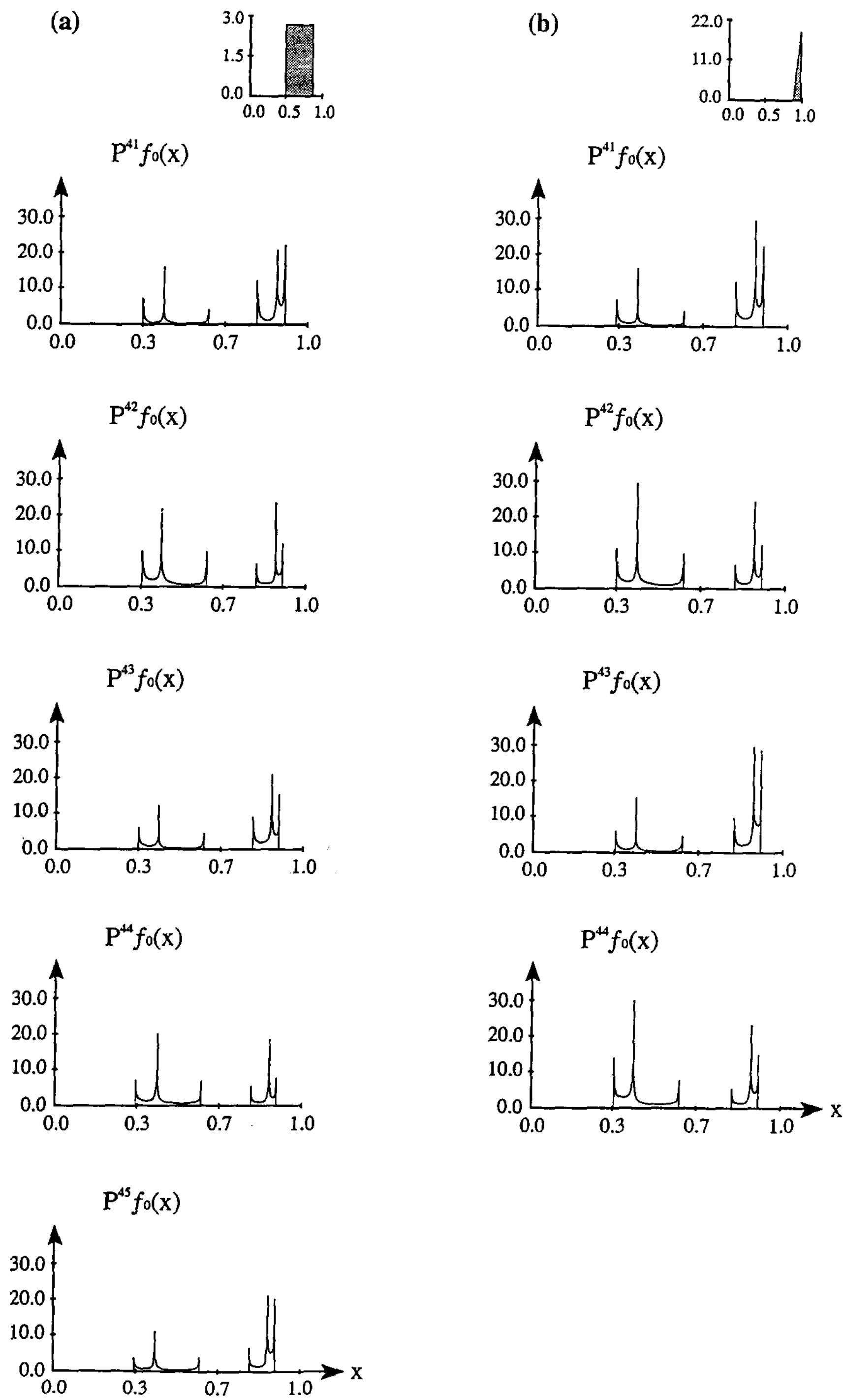

Fig. 5. Two period four cycles of the asymptotic sequence $\{P^t f_0\}$ for the parameter $r = r_2 = 3.592572184$. In this figure 40 transients have been discarded and the iterates $P^{41}f_0$, $P^{42}f_0$, $P^{43}f_0$, $P^{44}f_0$ and $P^{45}f_0$ are shown. Since $P^{41}f_0 = P^{45}f_0$, the sequence $\{P^t f_0\}$ asymptotically repeats with period four. In (a) the initial density (inset) f_0 is uniform over [0.5, 0.85]. In (b) $f_0(x) = 200(x - 0.9)$ over [0.9, 1].

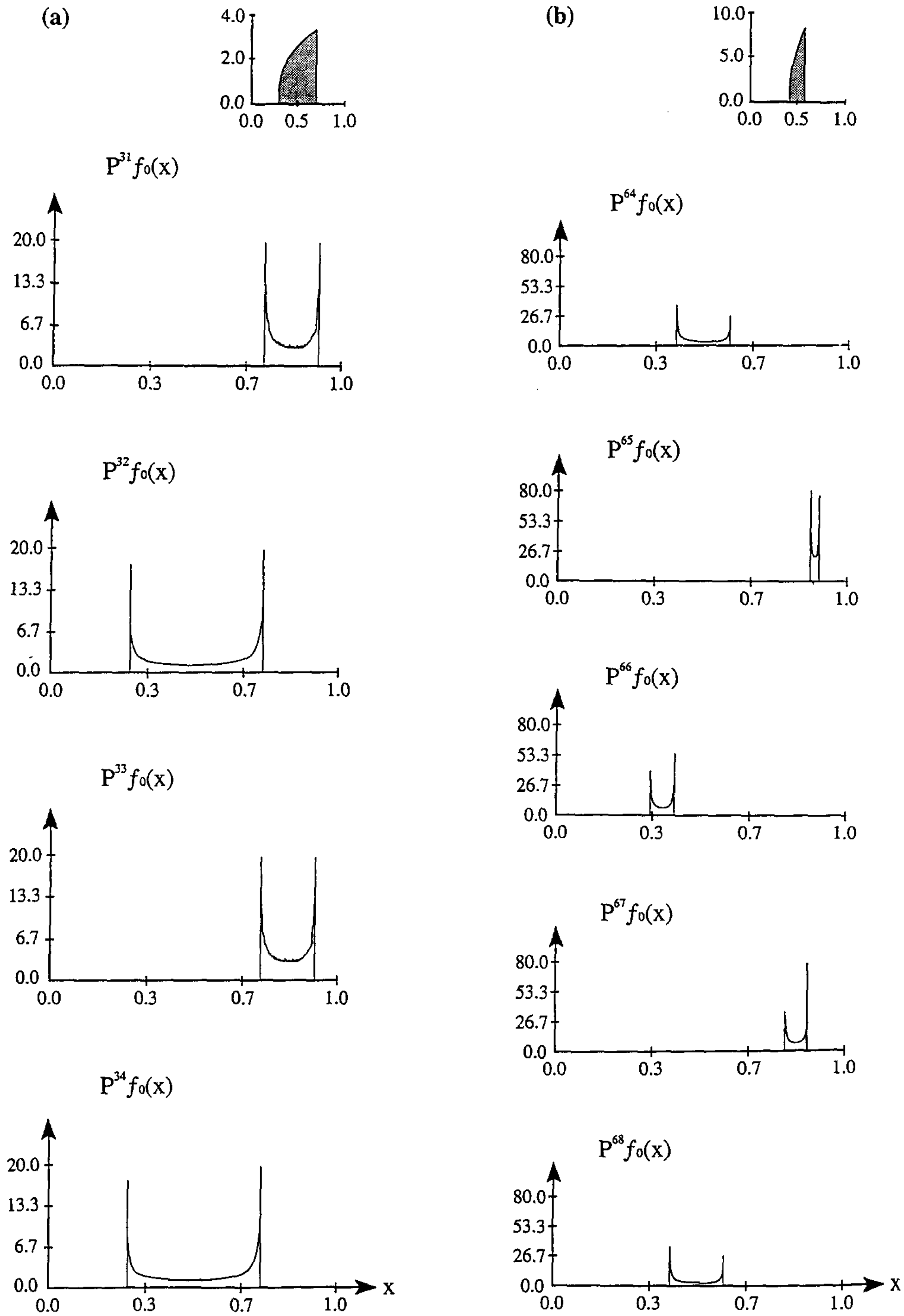

Fig. 6. A period two and period four cycle of the sequence $\{P^t f_0\}$ for $r = r_1 = 3.678573508$ and $r = r_2 = 3.592572184$, respectively. In (a) 30 transients have been discarded and the iterates $P^{31}f_0$ through $P^{34}f_0$ are shown. f_0 (inset) is supported on the subset of J_2, [0.3, 0.7], and has the form $f_0(x) = 4.52(x - 0.30)^{1/3}$. In (b) 63 transients have been discarded and the iterates $P^{64}f_0$ through $P^{68}f_0$ are shown. f_0 has the form $f_0(x) = 19.64(x - 0.41)^{1/2}$.

4.3. Correlations of the quadratic map and the Frobenius–Perron operator

Using the asymptotic periodicity of the Frobenius–Perron operator (43) we can derive (to first order) the time correlation function of (2), for $r = r_n$. In so doing no assumptions are made about the quasiperiodicity of orbits, or the decoupling of the time correlation into periodic and stochastic components as has been done previously [8]. These properties come naturally from (20) and (21), by virtue of asymptotic periodicity at $r = r_n$.

The conventions we use to label the subspaces J_l are as follows. Let L be the set of intervals to the left of the critical point, $x = \frac{1}{2}$, and not containing it. R denotes the set of intervals to the right of the critical point and not containing it, while C consists of those intervals of X containing the critical point, $x = \frac{1}{2}$. Call the branch of the quadratic map to the left of the critical point S_L and that to the right S_R. Then,

$$S_L^{-1}(x) = \frac{1}{2}\left[1 - \sqrt{\left(1 - \frac{4x}{r}\right)}\right]$$

$$\text{and} \quad S_R^{-1}(x) = \frac{1}{2}\left[1 + \sqrt{\left(1 - \frac{4x}{r}\right)}\right]. \tag{52}$$

The counterimage of all initial conditions within L will be mapped according to S_L^{-1}, while those within R have counterimages given by mapping via S_R^{-1}. Points on C will thus have a counterimage given by mapping according to both S_L^{-1} and S_R^{-1}. Thus, a density f supported on L, R or C will map, respectively, according to

$$\begin{aligned} Pf(x) &= \left|\frac{dS_L^{-1}(x)}{dx}\right| f(S_L^{-1}(x)), \quad \text{supp}\{f\} \in \text{L}, \\ &= \left|\frac{dS_R^{-1}(x)}{dx}\right| f(S_R^{-1}(x)), \quad \text{supp}\{f\} \in \text{R}, \\ &= \left|\frac{dS^{-1}(x)}{dx}\right| \left[f(S_L^{-1}(x)) + f(S_R^{-1}(x))\right], \\ &\qquad \text{supp}\{f\} \in \text{C}, \end{aligned} \tag{53}$$

where $dS^{-1}/dx = dS_R^{-1}/dx = dS_L^{-1}/dx$. The last component of (53) is just the operator (43). At $r = r_n$ there are 2^n g_l densities satisfying $Pg_l = g_{l+1}$, $l = 1 \bmod 2^n, \ldots, 2^n \bmod 2^n$. From (53) the relation among the g_l is given by

$$\begin{aligned} g_{l+1}(x) &= \left|\frac{dS_L^{-1}(x)}{dx}\right| g_l(S_L^{-1}(x)), \quad J_l \in \text{L}, \\ &= \left|\frac{dS_R^{-1}(x)}{dx}\right| g_l(S_R^{-1}(x)), \quad J_l \in \text{R}, \\ &= \left|\frac{dS^{-1}(x)}{dx}\right| \left[g_l(S_L^{-1}(x)) + g_l(S_R^{-1}(x))\right], \\ &\qquad J_l = J_{2^n} \in \text{C}. \end{aligned} \tag{54}$$

The stochastic component of the time correlation function is examined first, and then the simpler periodic component. The mth moment of x weighted with the density g_l is defined by $\langle x^m \rangle_l = \int_{\text{supp}\{g_l\}} x^m g_l(x)\,dx$, while the variance of x with respect to $g_l(x)$ is written as $\text{var}(x)_l = \langle x^2 \rangle_l - \langle x \rangle_l^2$.

Substituting $xg_l(x)$ for $f(x)$ into the operator equations (53), and then using (54), gives

$$\begin{aligned} P[xg_l(x)] &= S_L^{-1}(x) g_{l+1}(x), \quad J_l \in \text{L}, \\ &= S_R^{-1}(x) g_{l+1}(x), \quad J_l \in \text{R}, \\ &= \tfrac{1}{2} g_1(x) + \frac{1}{2r_n}\left[g_{2^n}(S_R^{-1}(x)) \right. \\ &\quad \left. - g_{2^n}(S_L^{-1}(x))\right], \quad J_l = J_{2^n}. \end{aligned} \tag{55}$$

However, by (12) and (13) the important quantity for the stochastic component of the time correlation function is $Q(xg_l) = P(xg_l - \langle x \rangle_l g_l)$. Hence, from (55) we have

$$\begin{aligned} &P[xg_l(x) - \langle x \rangle_l g_l] \\ &\quad = \left[S_L^{-1}(x) - \langle x \rangle_l\right] g_{l+1}, \quad J_l \in \text{L}, \\ &\quad = \left[S_R^{-1}(x) - \langle x \rangle_l\right] g_{l+1}, \quad J_l \in \text{R}, \\ &\quad = \left(\tfrac{1}{2} - \langle x \rangle_{2^n}\right) g_1 + \Delta[g_{2^n}], \quad J_l = J_{2^n}, \end{aligned} \tag{56}$$

where $\Delta[g_{2^n}]$ is defined as

$$\Delta[g_{2^n}] \equiv \frac{1}{2r_n}\left[g_{2^n}(S_R^{-1}(x)) - g_{2^n}(S_L^{-1}(x))\right]. \tag{57}$$

At this point some approximations are made. It can be shown [8] that except for the central portion of the map S, supported on J_{2^n}, all other portions supported on J_l ($l = 1,2,\dots,2^n-1$) approach linear transformations as $r_n \to r_\infty$. Therefore, assuming that all g_l for $l = 1,2,\dots,2^n-1$ map according to a linear transformation we may write

$$\langle x \rangle_{l+1} \approx r_n \langle x \rangle_l (1 - \langle x \rangle_l). \tag{58}$$

Figs. 4–6 suggest that $\langle x \rangle_{2^n} \approx \frac{1}{2}$. The g_{2^n} components of the densities they depict are approximately symmetrical about $x = \frac{1}{2}$. This makes the $\Delta[g_{2^n}]$ term zero to first order. Thus, to first order the third component of (56) may be set equal to zero. Also, expanding $S_R^{-1}(x)$ and $S_L^{-1}(x)$ to first order and substituting (58) into (56) yields

$$\begin{aligned} Q(xg_l) &\approx \frac{(x - \langle x \rangle_{l+1})g_{l+1}}{r_n\sqrt{1 - \frac{4\langle x \rangle_{l+1}}{r_n}}}, \quad J_l \in L, \\ &\approx \frac{-(x - \langle x \rangle_{l+1})g_{l+1}}{r_n\sqrt{1 - \frac{4\langle x \rangle_{l+1}}{r_n}}}, \quad J_l \in R, \\ &\approx 0, \quad J_l = J_{2^n}. \end{aligned} \tag{59}$$

Eq. (59) shows that the function xg_l will approach its asymptotic value of $\langle x \rangle_l g_l$ after approximately $(2^n - l)$ iterations. This behavior causes the time correlation to drop to zero after 2^n iterations. However, this result is only approximate. We will see below that the numerically computed time correlation function actually decays more slowly than this approximation predicts.

Since $Qg_l = 0$, take as a set of basis functions the set $E_l(x) = (x - \langle x \rangle_l)g_l(x)$. With respect to the functions E_l, the stochastic operator Q may be represented as

$$\begin{aligned} [Q] &= \frac{\delta_{l,j-1}}{r_n\sqrt{1 - \frac{4\langle x \rangle_{l+1}}{r_n}}}, \quad J_l \in L, \\ &= \frac{-\delta_{l,j-1}}{r_n\sqrt{1 - \frac{4\langle x \rangle_{l+1}}{r_n}}}, \quad J_l \in R, \\ &= 0, \quad J_l = J_{2^n}, \end{aligned} \tag{60}$$

where $[Q]$ denotes matrix representation and $\delta_{i,j}$ denotes the Kronecker delta. Thus, following a similar procedure as for the hat map (see ref. [29]), $[Q]^t$ is found to be

$$\begin{aligned} [Q]^t &= \frac{1}{r_n^t}(-1)^{NR(i,i+t-1)}\rho(t,i)\delta_{i,j-t}, \\ &\qquad 1 \le t \le 2^n - 1, \\ &= 0, \quad t > 2^n - 1, \end{aligned} \tag{61}$$

where

$$\rho(t,i) = \prod_{j=i}^{i+t-1} \frac{1}{\sqrt{1 - \frac{4\langle x \rangle_{j+1}}{r_n}}}. \tag{62}$$

To complete the analysis of the stochastic component of the time correlation function the second term of eq. (15) must be evaluated. Hence,

$$\begin{aligned} &\sum_{l=1}^{2^n} \int_0^1 xQ^t(xg_l)\,\mathrm{d}x \\ &= \frac{1}{r_n^t} \sum_{l=1}^{2^n - t} (-1)^{NR(l,l+t-1)}\rho(t,l) \int_0^1 xE_{t+l}\,\mathrm{d}x. \end{aligned} \tag{63}$$

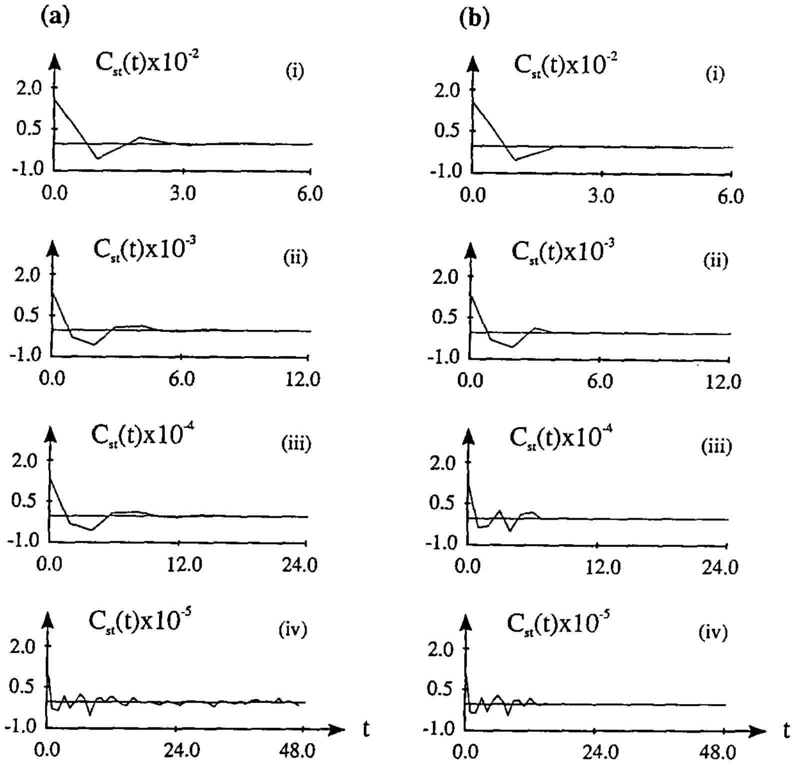

Fig. 7. (a) The stochastic component of the time correlation function $C_{st}(t)$ calculated numerically for r_1, r_2, r_3 and r_4, respectively. Three periods 2^n time units in length are spanned. (b) shows, for the same parameters, $C_{st}(t)$ as calculated by eq. (64).

Noting that $\int_0^1 xE_{t+l}\,dx = \text{var}(x)_{t+l}$, we obtain

$$C_{st}(t) \approx \frac{1}{2^n}\sum_{i=1}^{2^n} \text{var}(x)_i, \quad t=0,$$

$$\approx \frac{1}{2^n r_n^t}\sum_{i=1}^{2^n-t} (-1)^{NR(i,i+t-1)} \rho(t,i)\,\text{var}(x)_{t+i},$$

$$t = 1,\dots,2^n-1,$$

$$\approx 0, \qquad t > 2^n - 1. \qquad (64)$$

In fig. 7, the stochastic component of the time correlation function (64) is compared (fig. 7b) with the numerically computed one (fig. 7a) for $r = r_n$ with $n = 1,2,3,4$. The numerical evaluation spans 3 periods of length 2^n. The agreement is good over one period. However, the real correlation continues past 2^n iterations.

The correlation function also contains a periodic component due to the oscillatory motion of orbits through the various subspaces J_l. From (20) and (21) its form is

$$C_p(t) = \sum_{j=2}^{2^n} \left|\psi(\omega_j)\right|^2 e^{i\omega_j t}, \quad t = 1,2,\dots, \qquad (65)$$

where

$$\psi(\omega_j) = \sum_{l=1}^{2^n} \langle x\rangle_l\, e^{i\omega_j(l-1)}.$$

The expression of the mean values of x with respect to the g_l may be calculated numerically. Fig. 8 shows the graph of (65) for the same values of r_n, $n = 1,2,3$. The periodic component of the correlation function for r_n, $n = 1,2,3$, calculated

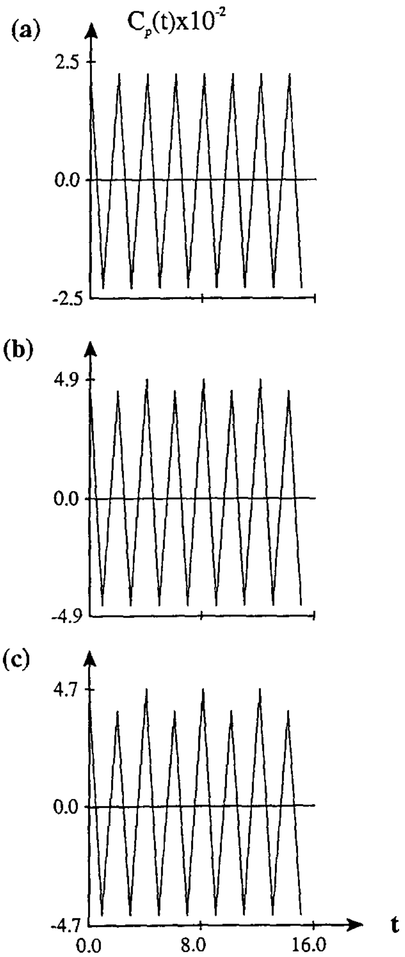

Fig. 8. The periodic component of the correlation function for the parameters $r = r_1, r_2, r_3$ in (a) through (c), respectively. The period eight cycle in (c) cannot be distinguished at the resolution shown. Likewise for period 16 and higher.

numerically from $C(\tau) = \langle x_{t+\tau} x_t \rangle - \langle x \rangle^2$ is identical to that shown in fig. 8.

5. The limiting conditional entropy of maps (1) and (2)

The functional dependence of $H_c^\infty(P^t f_0 | f^*)$ ($H_c^\infty(f_0)$ for short) on the initial state f_0 can be illustrated analytically for the systems (1) and (2) when they generate period two asymptotic periodicity. In particular consider a class of initial densities given by

$$f_0(x) = \frac{1}{\xi}, \quad x \in [\gamma_1, \gamma_1 + \xi],$$
$$= 0, \quad \text{otherwise}, \tag{66}$$

where γ_1 comes from the sequences $\{\gamma_i\}$ used in deriving (41) for the hat map or (51) for the quadratic map. Let ξ vary freely such that

$$\gamma_1 + \xi = \delta_{2m-1} + \sigma, \quad 0 \le \sigma \le \delta_{2m} - \delta_{2m-1},$$
$$= \delta_{2m} + \sigma, \quad 0 \le \sigma \le \delta_{2m+1} - \delta_{2m}, \tag{67}$$

where $m = 1, 2, \ldots$.

Substituting the sequence (66) into the eqs. (41) for $\lambda_1(f_0)$, $\lambda_2(f_0)$ of the hat map we obtain

$$\lambda_1(f_0) = \frac{a}{(a+1)[1+\kappa(m)\sigma]} + \frac{\kappa(m)\sigma}{\kappa(m)\sigma+1},$$
$$\lambda_2(f_0) = \frac{1}{(a+1)[\kappa(m)\sigma+1]}, \tag{68}$$

where

$$\kappa(m) = \frac{a^{2m-1}(a+1)}{a^{2m}-1},$$

for the first case of (67). Likewise for the second interval of variation of ξ in (67) the scaling coefficients are given by

$$\lambda_1(f_0) = \frac{\eta_1(m)}{(a+1)[1+\sigma\kappa(m+\frac{1}{2})]},$$
$$\lambda_2(f_0) = \frac{a\eta_2(m)}{(a+1)[1+\sigma\kappa(m+\frac{1}{2})]}$$
$$+ \frac{\sigma\kappa(m+\frac{1}{2})}{1+\sigma\kappa(m+\frac{1}{2})}, \tag{69}$$

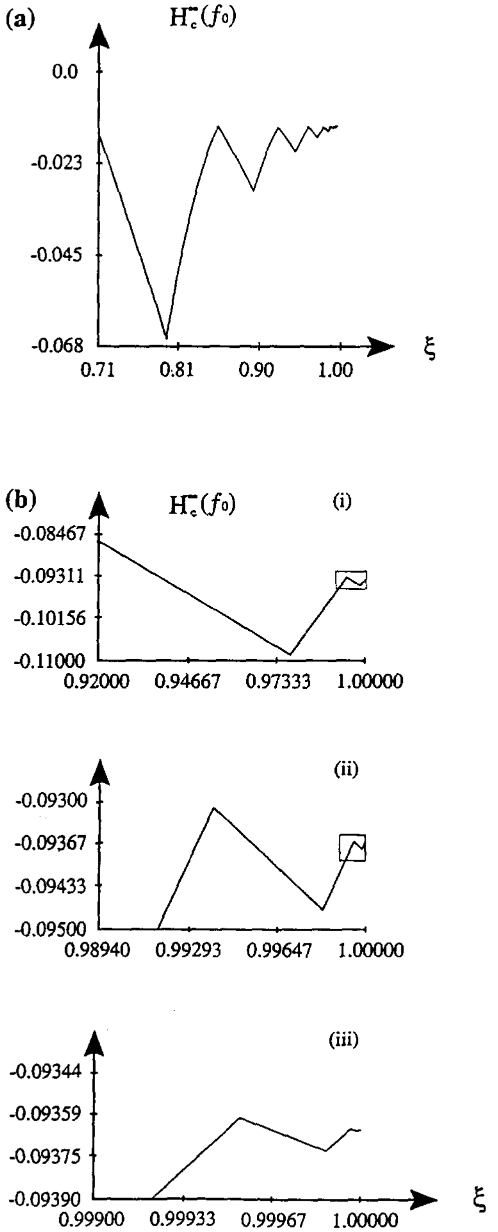

Fig. 9. (a) The limiting conditional entropy, $H_c^\infty(f_0)$, versus the spreading parameter ξ for the hat map at $a=\sqrt{2}$. ξ is equal to the width of the support of an initial density f_0 which is uniform over $[\gamma_1, \gamma_1+\xi]$. The local maxima in the figure correspond to $\gamma_1+\xi=\delta_{2m-1}$ and are all equal. The sequences $\{\gamma_i\}$ and $\{\delta_i\}$ are described in the text. (b) shows a graph of the limiting conditional entropy $H_c^\infty(f_0)$ versus ξ for the quadratic map at $r=r_1$. The parameter ξ plays the same role as in (a). Variations in $H_c^\infty(f_0)$ occur over smaller ξ scale for the quadratic map. (b)(ii) is a blow-up of the inset box in frame (b)(i). (b)(iii) is a blow-up of the inset box in (b)(ii).

where

$$\eta_2(m)=\frac{a^{2m}-1}{a^{2m+1}-1} \quad \text{and} \quad \eta_1(m)=\frac{a^{2m+2}-1}{a^{2m+1}-1}.$$

A plot of $H_c^\infty(f_0)$ using (68) and (69) is shown in fig. 9a. A remarkable feature of fig. 9a occurs when f_0 is such that $\gamma_1+\xi=\delta_{2m-1}$. At these values of ξ, (68) simplifies to

$$\lambda_1(f_0)=\frac{a}{a+1} \quad \text{and} \quad \lambda_2(f_0)=\frac{1}{a+1},$$

making the points $\gamma_1+\xi$ isoentropic. For these values of ξ the asymptotic decomposition of $P^t f_0$ is identical. For example when $a=\sqrt{2}$ the limiting conditional entropy becomes $H_c^\infty(f_0)\approx -0.0148$. Note also the local minima that develop in the limiting conditional entropy as the spreading parameter ξ increases.

A similar comparison of the limiting conditional entropy can be made for the asymptotic periodicity of the quadratic map at $r=r_1$. The same set of initial densities defined by (66) is considered, except now γ_i and δ_i are those used for the scaling coefficients (51) of the quadratic map, at $r=r_1$. Fig. 9b is the analogous plot of fig. 9a for the quadratic map. Note that for the quadratic map the values $\gamma_1+\xi=\delta_{2m-1}$ do not define isoentropic points, although $H_c^\infty(f_0)$ does converge to a value of about $0.093\ldots$ as $\gamma_1+\xi\to 1$. Moreover, a zig-zag pattern similar to fig. 9a emerges but on a much smaller scale, as shown by the insets.

Can the zig-zag pattern in the conditional entropy of fig. 9b act like a signature of maps with a quadratic maximum? Universal properties of maps with a quadratic maximum have been examined for both the Lyapunov exponent and for power spectra [28]. Maps with a quadratic maximum lead, for certain parameter ranges, to banded chaos. Thus asymptotic periodicity and the analysis of section 2 should be applicable to these maps. It is then reasonable to expect that a sequence of initial densities f_0 as defined by (66) should lead to conditional entropies with similar scaling properties as that of fig. 9b.

6. Summary

The notion of associating the state of a low-dimensional dynamical system with a phase space density was proposed. In this formalism the evolution of a system is equivalent to the evolution of an ensemble of phase space points collectively distributed according to the density $P^t f_0$, where P is the Markov operator characterizing the dynamics of the system. A large class of one-dimensional maps "equilibrate" and can asymptotically be described by an invariant density. Another class of dynamical systems which also possess an invariant density are known as asymptotically periodic maps. These systems, however, do not in general attain this invariant density. Hence, physically measurable quantities, defined on the space on which these maps operate, cannot be estimated on the basis of the invariant density. Indeed since the flow of densities $\{P^t f_0\}$ continually cycles according to eq. (6), physical observables as well as statistical states of asymptotically periodic systems can at most be considered as being metastable in time. A description of physical properties can be defined only at discrete times in the cycle, through the appropriate metastable state of (6) associated with the corresponding time in the cycle. In principle a complete determination of the dynamics of asymptotically periodic systems only requires knowledge of the initial density of preparation f_0 and the "pure" states g_i in the decomposition (6).

Asymptotic periodicity was numerically illustrated in two maps, (1) and (2), at the parameter values where they generate banded chaos. Using a general formulation of the autocorrelation function of asymptotically periodic systems, we calculated the autocorrelation function for the quadratic map without assuming the dissociation of trajectories into a periodic and stochastic component. Also the conditional entropy of these maps was studied at parameters where they display asymptotic periodicity, indicating clearly its dependence on the initial density of preparation of the system, f_0.

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