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Crossover from Geometrical to Stochastic Fractal Statistics for Translationally Invariant Random Distributions of Independent Particles in *n*-Dimensional Euclidean Space

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Abstract—The paper deals with the statistics of translationally invariant random distributions of independent particles in Euclidean space. The geometrical fractal model used in astrophysics and hydrodynamics assumes that the number of particles enclosed in a hypersphere of radius r obeys a Poisson law $\nu^N(N!)^{-1}\exp(-\nu)$, where the average number of particles is a power function of radius r: $\nu \sim r^{d_f}$ and d_f is the dimension of a fractal structure embedded in the Euclidean space considered. A statistical fractal distribution is introduced by assuming that the probability density of the distance between two nearest particles has a long tail of the inverse power law type $\phi_0(r) \sim r^{-(1+H_r)}$ as $r \to \infty$, where $H_r > 0$ is a statistical fractal exponent. The distribution of the number of particles enclosed in a hypersphere of radius r is also Poissonian but the average number of points increases logarithmically rather than algebraically with the radius r: $\nu \sim H_r \ln(r/r_0)$ as $r \to \infty$. The spatial distribution of points corresponding to this statistical fractal model is much rarer than in the geometrical fractal case. An alternative approach is derived by assuming that the probability density of the distance between two particles has a very broad logarithmic tail $\phi_0(r)\,dr \sim d[\ln^m(r/r_0)]/[\ln^m(r/r_0)]^{1+H}$, $H > 0$, $r \to \infty$ where $\ln^m(r/r_0) = \ln\ldots\ln(r/r_0)$ is the mth iterated logarithm of r/r_0. For such a logarithmic statistical fractal the number of particles increases much more slowly with the radius r than in the 'pure' statistical fractal case: $\nu \sim H\ln^{m+1}(r/r_0)$ as $r \to \infty$. By using a heuristic approach two general probabilistic models are derived which include both the geometrical and statistical fractal models as particular cases; these models predict power law dependences of the average number of particles for small systems and logarithmic dependences for large systems, respectively. The significance of the generalized models is elucidated by using the notion of a specific fractal hypervolume $\tilde{\omega}$ which corresponds to a given particle. For the generalized model derived from the pure statistical fractal $\tilde{\omega}$ is made up of two additive contributions: a constant one which determines the behavior of the model for small systems and a linearly increasing contribution with the total fractal hypervolume which is predominant for large systems. A similar structure of $\tilde{\omega}$ exists for the generalized model leading to the logarithmic statistical fractal regime.

1. INTRODUCTION

In the last decade both geometrical [1] and statistical [2] fractals have been extensively studied. Both types of fractals share some common features due to the existence of scaling in the real space [1] or for the tails of the probability densities [2], respectively. However, little is known about the general relationships between the geometrical and statistical fractals.

The aim of this paper is the analysis of a particular aspect related to the interaction between the geometrical and statistical fractals, namely the distribution of independent particles in Euclidean space. This problem arises in a variety of situations, for instance in

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connection with the distribution of vortices in a turbulent fluid [3] or in connection with the distribution of matter in the universe [4]. The common approach to this problem [3, 4] is based on the use of a geometrical fractal model: one assumes that the particles are uniformly and randomly distributed in a fractal structure embedded in the Euclidean space. In this paper we suggest an approach based on the use of a statistical (stochastic) fractal for which the probability density of the empty space around a particle has a long tail. Such a model leads to a distribution of particles which is much rarer than for the geometrical fractal model. We make a connection with the usual geometrical fractal description by constructing two generalized models which include both approaches as particular cases.

The plan of the paper is as follows. In Section 2 we rephrase the geometrical fractal model in a form which is appropriate for a comparison with our own approach. In Section 3 the main properties of the statistical fractal model are analysed. In Sections 4 and 5 we introduce the generalized models and analyse the geometrical fractal and the statistical fractal regimes. Finally in Section 6 the physical significance of the generalized models is analysed.

2. GEOMETRICAL FRACTAL POISSONIAN DISTRIBUTIONS

The uniform Poissonian distribution is commonly used to describe the positions of particles in space. The integer dimension (non-fractal) version of the approach is used in many branches of science ranging from astrophysics [5] and radiophysics [6] to ecology [7]. A usual assumption is that the particles are small enough so that they may be approximated by points of zero dimension. The main assumption of the theory is that the number N of particles enclosed in a hypersphere of radius r in n-dimensional Euclidean space obeys the Poisson statistics

$$P(N) = \nu^N (N!)^{-1} \exp(-\nu), \tag{1}$$

with

$$\nu = \mathcal{N} V \sim r^n, \tag{2}$$

where the average number ν of particles is proportional to the volume V of the hypersphere

$$V(r) = \pi^{n/2}[\Gamma(1 + n/2)]^{-1} r^n. \tag{3}$$

$\mathcal{N}$ is the volume density of particles, assumed constant, and $\Gamma(x)$ is the complete gamma function.

In astrophysical [4] and hydrodynamical [3] contexts it has been suggested that the above theory may be extended to geometrical fractal structures with a fractal dimension d_f embedded in n-dimensional Euclidean space. The main assumption is that the Poisson law (1) is valid with the difference that the average number of particles ν is proportional to r^{d_f}. A common heuristical approach is to introduce a fractal hypervolume $\Omega(r)$ of a hypersphere of radius r [3]

$$\Omega(r) = \pi^{d_f/2}[\Gamma(1 + d_f/2)]^{-1} r^{d_f}. \tag{4}$$

The choice of the proportionality constant in equation (4) has no profound theoretical basis; however, this choice is useful because for $d_f = n$ we have $\Omega(r) = V(r)$. We introduce a fractal density of points

$$\mathcal{N} = d\nu/d\Omega, \tag{5}$$

which is assumed to be constant. It follows that

$$\nu = \Omega\mathcal{N} \sim r^{d_f}. \tag{6}$$

As $d_f \leqslant n$ for this geometrical fractal model the distribution of particles is rarer than in the case of uniform filling of n-dimensional Euclidean space. For instance in astrophysics observational and theoretical evidence has been accumulated [4, 8] that in the near universe the average mass scales with an exponent $2 > d_f > 1$ which is much smaller than $n = 3$.

In order to outline a basic feature of the geometrical fractal model we consider a non-fractal Euclidean model ($d_f = n$) for which the density is not constant but varies with the ray length r according to a power law

$$\mathcal{N}(r) = \mathrm{d}\nu/\mathrm{d}V \sim r^{-b}, \qquad n > b \geqslant 0. \tag{7}$$

The corresponding average number of particles results by integrating (7)

$$\nu(r) \sim r^{n-b}. \tag{8}$$

Apparently such a Euclidean model with variable density is equivalent with a geometrical fractal model with $d_f = n - b$. However, this is not true. The inhomogeneous Euclidean model is not translationally invariant; for it the power law (8) is valid for a single set of concentric hyperspheres whose centres are placed in the point of maximum density $\mathcal{N} \to \infty$. On the contrary for the geometrical fractal model the centre of a hypersphere may be placed anywhere; due to the translational invariance the power law (6) is valid for any position in space.

A main property of the fractal geometrical model is related to the probability density of the empty space corresponding to a particle. We use the notation ϕ_0 for this probability density. ϕ_0 may be expressed in terms of three different variables: the distance r from a particle to its nearest neighbour, or the fractal or Euclidean volumes $\Omega(r)$, $V(r)$ corresponding to this distance, respectively. We have

$$\phi_0(r)\,\mathrm{d}r = \frac{\pi^{d_f/2}\mathcal{N}}{\Gamma(1 + d_f/2)} d_f r^{d_f - 1} \exp\left(-\frac{\mathcal{N}\pi^{d_f/2}}{\Gamma(1 + d_f/2)} r^{d_f}\right) \mathrm{d}r, \tag{9}$$

$$\phi_0(\Omega)\,\mathrm{d}\Omega = \mathcal{N}\exp(-\mathcal{N}\Omega)\,\mathrm{d}\Omega, \tag{10}$$

and

$$\phi_0(V)\,\mathrm{d}V = d_f \frac{\mathcal{N}[\Gamma(1 + n/2)]^{d_f/n}}{n\Gamma(1 + d_f/2)} V^{(d_f/n)-1} \exp\left(-\frac{\mathcal{N}[\Gamma(1 + n/2)]^{d_f/n}}{\Gamma(1 + d_f/2)} V^{d_f/n}\right) \cdot \mathrm{d}V. \tag{11}$$

All positive moments of these probability densities can be computed exactly. We get

$$\langle r^k \rangle = \int_0^\infty r^k \phi_0(r)\,\mathrm{d}r = \frac{[\Gamma(1 + d_f/2)]^{k/d_f}\Gamma(1 + k/d_f)}{\pi^{k/2}\mathcal{N}^{k/d_f}}, \tag{12}$$

$$\langle \Omega^k \rangle = \Gamma(k + 1)\mathcal{N}^{-k}, \tag{13}$$

and

$$\langle V^k \rangle = \frac{[\Gamma(1 + d_f/2)]^{kn/d_f}\Gamma(1 + kn/d_f)}{[\Gamma(1 + n/2)]^k \mathcal{N}^{kn/d_f}}, \tag{14}$$

where k is an arbitrary non-negative number, not necessarily an integer. As all positive moments of the variables r, Ω and V exist and are finite it follows that the probability densities (9)–(11) fall off very fast and the statistical fractal features are missing.

3. THE STATISTICAL FRACTAL BEHAVIOUR

In this section we introduce a new type of model by assuming that the probability densities of the empty space have long tails of the power law type:

$$\phi_0(r)\,\mathrm{d}r \sim r^{-(1+H_r)}\,\mathrm{d}r \qquad \text{as} \qquad r \to \infty, \tag{15}$$

$$\phi_0(\Omega)\,\mathrm{d}\Omega \sim \Omega^{-(1+H_\Omega)}\,\mathrm{d}\Omega \qquad \text{as} \qquad \Omega \to \infty, \tag{16}$$

and

$$\phi_0(V)\,\mathrm{d}V \sim V^{-(1+H_V)}\,\mathrm{d}V \qquad \text{as} \qquad V \to \infty, \tag{17}$$

where H_r, H_Ω, H_V are positive scaling exponents. From the definitions of $\Omega(r)$ and $V(r)$ we obtain

$$r = \pi^{-1/2}[\Omega\Gamma(1 + d_f/2)]^{1/d_f}, \tag{18}$$

and

$$V = [\Gamma(1 + d_f/2)]^{n/d_f}[\Gamma(1 + n/2)]^{-1}\Omega^{n/d_f}. \tag{19}$$

By using equations (15)–(19) we get

$$H_\Omega = H_r/d_f, \qquad H_V = H_r/n, \tag{20}$$

that is, only one of the three scaling exponents H_r, H_Ω, H_V is independent.

The variables r, Ω, and V may take any value between 0 and ∞ and thus the probability densities $\phi_0(r)$, $\phi_0(\Omega)$ and $\phi_0(V)$ obey the normalization conditions

$$\int_0^\infty \phi_0(r)\,\mathrm{d}r = 1, \qquad \int_0^\infty \phi_0(\Omega)\,\mathrm{d}\Omega = 1, \qquad \int_0^\infty \phi_0(V)\,\mathrm{d}V = 1. \tag{21}$$

It follows that the inverse power laws cannot be true as $r, \Omega, V \to 0$. If they were valid as $r, \Omega, V \to 0$ the normalization conditions (21) would be violated.

As before we assume that the particles are independent so that the number of particles enclosed in a region of fractal volume Ω obeys the Poisson statistics (1) where the average number of particles $\nu(\Omega)$ should be determined from the asymptotical laws (15)–(17). In order to evaluate $\nu(\Omega)$ we introduce the probability $g(\Omega)$ that in the fractal hypervolume Ω surrounding a particle there are no other particles. In terms of $g(\Omega)$ and $\phi_0(\Omega)$ we can write the following probability balance equations

$$-\frac{\partial g(\Omega)}{\partial \Omega} = \frac{\partial \nu}{\partial \Omega} g(\Omega), \tag{22}$$

$$\phi_0(\Omega)\,\mathrm{d}\Omega = g(\Omega)\frac{\partial \nu}{\partial \Omega}\,\mathrm{d}\Omega. \tag{23}$$

Here and in the following we assume that equations (22) and (23) are valid for any fractal hypervolume, not necessarily only for the fractal hypervolume enclosed by a hypersphere of radius r. The initial condition corresponding to (22) is

$$g(0) = 1. \tag{24}$$

By integrating (22) with the initial condition (24) and inserting the result into (23) we obtain

$$g(\Omega) = \exp[-\nu(\Omega)], \tag{25}$$

$$\phi_0(\Omega) = [\partial\nu(\Omega)/\partial\Omega]\exp[-\nu(\Omega)]. \tag{26}$$

In order to evaluate the asymptotic behaviour of $\nu(\Omega)$ we compute from equation (16) the probability $g(\Omega)$ as $\Omega \to \infty$:

$$g(\Omega) = \int_{\Omega}^{\infty} \phi_0(\Omega)\,d\Omega \sim \frac{1}{H_\Omega}\Omega^{-H_\Omega} \qquad \text{as} \qquad \Omega \to \infty, \tag{27}$$

From equations (16) and (27) we come to

$$\phi_0(\Omega) \sim g(\Omega)H_\Omega/\Omega \qquad \text{as} \qquad \Omega \to \infty, \tag{28}$$

from which, by comparison with (23) we come to a differential equation in $\nu(\Omega)$:

$$\partial\nu(\Omega)/\partial\Omega = H_\Omega/\Omega \qquad \text{as} \qquad \Omega \to \infty. \tag{29}$$

By integrating (29) we get the sought for expression for $\nu(\Omega)$

$$\nu(\Omega) = H_\Omega \ln(\Omega/\Omega_0) \qquad \text{as} \qquad \Omega \to \infty. \tag{30}$$

where Ω_0 is an integration constant having the dimension of a fractal hypervolume. By expressing equation (30) in terms of r and V we obtain

$$\nu(r) = H_r \ln(r/r_0) \qquad \text{as} \qquad r \to \infty, \tag{31}$$

$$\nu(V) = H_V \ln(V/V_0) \qquad \text{as} \qquad V \to \infty. \tag{32}$$

As for r, Ω, $V \to \infty$ the average number of particles increases logarithmically with the size of the system it follows that for the statistical fractal model the distribution of particles is much rarer than for the geometrical fractal model.

4. CROSSOVER FROM GEOMETRICAL TO STOCHASTIC FRACTAL STATISTICS

The asymptotic laws (15)–(17) are valid only in the limit r, Ω, $V \to \infty$. However, in order to apply the theory to a given problem a set of properly normalized probability densities $\phi_0(r)$, $\phi_0(\Omega)$, $\phi_0(V)$ should be known for any values of r, Ω, V. Here we give a heuristical approach for the evaluation of the functions $\phi_0(r)$, $\phi_0(\Omega)$ and $\phi_0(V)$ for any values of r, Ω, V based on a suitable modification of the differential equation (29). Although without a profound theoretical basis such a procedure has the advantage of simplicity: it leads to properly defined probability densities ϕ_0 for any values of r, Ω, V by using only two parameters: the fractal exponent H_Ω and the fractal density $\mathcal{N}$ of particles in the limit r, Ω, $V \to 0$.

It is normal to expect that as $\Omega \to 0$ the average density of points also tends to zero

$$\nu(\Omega) \to 0 \qquad \text{as} \qquad \Omega \to 0. \tag{33}$$

Anyway even though the condition (33) is not satisfied ν, being an average number of particles, should be non-negative. On the contrary, as $\Omega \to 0$ equations (29)–(30) lead to an unphysical result

$$\nu(\Omega) \to -\infty \qquad \text{as} \qquad \Omega \to 0. \tag{34}$$

Our aim is to modify equation (29) in such a way so that in the limit $\Omega \to 0$, instead of the absurd result (34), to recover a behaviour which is typical for a geometrical fractal

$$\nu(\Omega) \to \mathcal{N}\Omega \quad \text{as} \quad \Omega \to 0, \tag{35}$$

which corresponds to

$$\partial\nu(\Omega)/\partial\Omega \to \mathcal{N} = \text{constant as } \Omega \to 0. \tag{36}$$

The simplest way of interpolating between the two asymptotic equations (29) and (36) is to add a positive constant K to the denominator of the rhs of equation (29):

$$\partial\nu(\Omega)/\partial\Omega = H_\Omega/(\Omega + K). \tag{37}$$

We assume that (37) is valid for any values of Ω between zero and infinity. From (37) we recover the asymptotic law (36) provided that K is given by

$$K = H_\Omega/\mathcal{N}. \tag{38}$$

By inserting equation (38) into equation (36) and integrating the resulting equation we obtain

$$\nu(\Omega) = H_\Omega \ln[1 + \mathcal{N}\Omega/H_\Omega], \tag{39}$$

which corresponds to

$$g(\Omega) = [H_\Omega/(\mathcal{N}\Omega + H_\Omega)]^{H_\Omega}, \tag{40}$$

and

$$\phi_0(\Omega)\,\mathrm{d}\Omega = [H_\Omega/(\mathcal{N}\Omega + H_\Omega)]^{H_\Omega+1}\mathcal{N}\,\mathrm{d}\Omega. \tag{41}$$

By using equations (3) and (4) we can also express ν and ϕ_0 in terms of r and V, respectively:

$$\nu(r) = \frac{H_r}{d_f}\ln\left[1 + \frac{\mathcal{N}d_f}{H_r}\cdot\frac{\pi^{d_f/2}r^{d_f}}{\Gamma(1 + d_f/2)}\right], \tag{42}$$

$$\nu(V) = \frac{nH_V}{d_f}\ln\left[1 + \frac{\mathcal{N}d_f}{nH_V}\cdot\frac{[\Gamma(1 + n/2)]^{d_f/n}}{\Gamma(1 + d_f/2)}(V)^{d_f/n}\right], \tag{43}$$

and

$$\phi_0(r)\,\mathrm{d}r = \frac{\mathcal{N}\pi^{d_f/2}d_f}{\Gamma(1 + d_f/2)}\left[\frac{H_r}{\dfrac{\mathcal{N}\pi^{d_f/2}d_f}{\Gamma(1 + d_f/2)}(r)^{d_f} + H_r}\right]^{1+H_r/d_f}\cdot(r)^{d_f-1}\,\mathrm{d}r, \tag{44}$$

$$\phi_0(V)\,\mathrm{d}V = \frac{d_f\mathcal{N}[\Gamma(1 + n/2)]^{d_f/n}}{n\Gamma(1 + d_f/2)}\left[\frac{H_V}{\dfrac{[\Gamma(1 + n/2)]^{d_f/n}\mathcal{N}d_f}{n\Gamma(1 + d_f/2)}(V)^{d_f/n} + H_V}\right]^{1+nH_V/d_f} \cdot(V)^{(d_f/n)-1}\,\mathrm{d}V. \tag{45}$$

As expected for r, Ω, $V \to \infty$ equations (39), (42), (43) and (41), (44), (45) have the asymptotic behaviour given by equations (30), (32) and (15)–(17), respectively where the parameters Ω_0, r_0 and V_0 are given by

$$\Omega_0 = H_\Omega/\mathcal{N}, \tag{46}$$

$$r_0 = \frac{1}{\sqrt{\pi}} \cdot \left[\frac{H_r \Gamma(1 + d_f/2)}{d_f \mathcal{N}}\right]^{1/d_f}, \tag{47}$$

and

$$V_0 = \frac{1}{\Gamma(1 + n/2)} \left[\frac{n H_V}{\mathcal{N} d_f} \Gamma(1 + d_f/2)\right]^{n/d_f}, \tag{48}$$

and H_Ω, H_r and H_V are related to each other through the relationships (20). On the other hand, for r, Ω, $V \to 0$ these equations reduce to equations (9)–(11) characteristic to the geometrical fractal model. Note that in the limit H_r, H_Ω, $H_V \to \infty$ the geometrical fractal model is exact; in this limit (9)–(11) are fulfilled not only for small values of r, Ω, V but rather for any values in the range 0, ∞.

All positive moments corresponding to the probability densities (41), (44) and (45) can be computed exactly. After some algebra we get

$$\langle r^k \rangle = \infty, \quad k \geq H_r, \tag{49a}$$

$$= \frac{[\Gamma(1 + d_f/2)]^{k/d_f}}{\mathcal{N}^{k/d_f} \pi^{k/2}} \left(\frac{H_r}{d_f}\right)^{(k/d_f)+1} \frac{\Gamma(1 + k/d_f)\Gamma[(H_r - k)/d_f]}{\Gamma(1 + H_r/d_f)}, \quad 0 \leq k < H_r, \tag{49b}$$

$$\langle \Omega^k \rangle = \infty, \quad k \geq H_\Omega, \tag{50a}$$

$$= \mathcal{N}^{-k} (H_\Omega)^{k+1} \frac{\Gamma(1 + k)\Gamma(H_\Omega - k)}{\Gamma(1 + H_\Omega)}, \quad 0 \leq k < H_\Omega, \tag{50b}$$

$$\langle V^k \rangle = \infty, \quad k \geq H_V, \tag{51a}$$

$$= \frac{[\Gamma(1 + d_f/2)]^{kn/d_f}}{[\Gamma(1 + n/2)]^k \mathcal{N}^{kn/d_f}} \cdot \left(\frac{H_V n}{d_f}\right)^{(kn/d_f)+1} \frac{\Gamma(1 + kn/d_f)\Gamma[(H_V - k)n/d_f]}{\Gamma(1 + H_V n/d_f)}, \quad 0 \leq k < H_V. \tag{51b}$$

These equations are generalizations of equations (12)–(14) valid for geometrical fractals. As expected the statistical fractal features occur for small positive values of H_r, H_Ω, H_V. For small positive values of the fractal exponents only the first few moments are finite, a situation which is not far from a 'pure' statistical fractal. As the fractal exponents increase, more and more moments become finite. In the limit H_r, H_Ω, $H_V \to \infty$ all positive moments are finite and we recover the geometrical fractal model.

5. AN ALTERNATIVE APPROACH

The approaches presented in Sections 3 and 4 are not the only types of statistical fractal models conceivable. The catalogue of possible models is much broader; one can derive a succession of models the first of which correspond to the models presented in the preceding sections.

The starting point of such an approach is the work of Vlad [9]. By using a stochastic renormalization approach [9] Vlad has derived a class of very broad probability densities whose tails are given by logarithmic rather than inverse power laws. We do not enter into the details of the derivation of these probability densities; we confine ourselves to assume that the probability density $\phi_0(\Omega)$ of the empty fractal hypervolume has the asymptotic behavior:

$$\phi_0(\Omega)\,d\Omega \sim \frac{d\Omega}{\Omega \ln(\Omega/\Omega_0)\ln^2(\Omega/\Omega_0)\ldots\ln^{m-1}(\Omega/\Omega_0)[\ln^m(\Omega/\Omega_0)]^{H_\Omega+1}}$$
$$\sim \frac{d[\ln^m(\Omega/\Omega_0)]}{[\ln^m(\Omega/\Omega_0)]^{H_\Omega+1}}, \qquad m = 0, 1, 2, \ldots, H_\Omega > 0, \qquad \Omega \to \infty, \tag{52}$$

where

$$\ln^m(\Omega/\Omega_0) = \underbrace{\ln\ldots\ln}_{m \text{ times}}(\Omega/\Omega_0), \qquad \ln^0(\Omega/\Omega_0) = \Omega/\Omega_0, \tag{53}$$

is the mth iterated logarithm of Ω/Ω_0 and Ω_0 is a cutoff value of the fractal hypervolume. For $m = 0$ equation (52) reduces to (16) characteristic for the 'pure' statistical fractal model. The models with $m \geq 1$ differ from the model corresponding to $m = 0$ in at least one respect: passing in equation (52) from the fractal hypervolume to radius r or to the Euclidean hypervolume V we come to two asymptotic laws similar to (15) and (17):

$$\phi_0(r)\,dr \sim d[\ln^m(r/r_0)]/[\ln^m(r/r_0)]^{H_\Omega+1}, \qquad m = 1, 2, \ldots, r \to \infty, \tag{54}$$

$$\phi_0(V)\,dV \sim d[\ln^m(V/V_0)]/[\ln^m(V/V_0)]^{H_\Omega+1}, \qquad m = 1, 2, \ldots, V \to \infty. \tag{55}$$

However, unlike for $m = 0$ in equations (52)–(54) the fractal exponent is the same. It is easy to check that this is indeed the case by inserting equations (18) and (19) into equation (52) and keeping the dominant terms as r, Ω, $V \to \infty$. That is why in this section we shall drop the subscript Ω of H_Ω.

We notice that equations (22)–(26) remain valid in the logarithmic fractal case; the asymptotic expression for the probability $g(\Omega)$ is however different. From equation (52) we get

$$g(\Omega) = \int_\Omega^\infty \phi_0(\Omega)\,d\Omega = H^{-1}[\ln^m(\Omega/\Omega_0)]^{-H} \qquad \text{as} \qquad \Omega \to \infty. \tag{56}$$

By combining (23), (52) and (56) we get a differential equation for $\nu(\Omega)$ similar to (29):

$$\frac{\partial \nu}{\partial \Omega} = H\frac{d\ln^{m+1}(\Omega/\Omega_0)}{d\Omega}, \qquad \Omega \to \infty, \tag{57}$$

from which by integration we obtain

$$\nu(\Omega) \sim H\ln^{m+1}(\Omega/\Omega_0), \qquad \Omega \to \infty. \tag{58}$$

From equation (58) we note that for $m \geq 1$ the average number of particles increases much more slowly than in the 'pure' statistical fractal case $m = 0$.

Now we derive a model leading to properly defined probability densities of the empty space for any values of r, Ω, V between zero and infinity. By following the procedure used in Section 4 we rewrite (57) in the form

$$\frac{\partial \nu}{\partial \Omega} = \frac{H}{\Omega\ln(\Omega/\Omega_0)\ldots\ln^m(\Omega/\Omega_0)} \qquad \text{as} \qquad \Omega \to \infty, \tag{59}$$

and add to the Ω-factors the positive constants $K_0, K_1, \ldots, K_m$:

$$\frac{\partial \nu}{\partial \Omega} = \frac{H}{(\Omega + K_0)\left[K_1 + \ln\left(\frac{K_0+\Omega}{\Omega_0}\right)\right]\left[K_2 + \ln\left[K_1 + \ln\left(\frac{K_0+\Omega}{\Omega_0}\right)\right]\right]\ldots}. \tag{60}$$

We choose the constants $K_0, K_1, \ldots, K_m$ in such a way so that for $\Omega \to 0$ equation (60) reduces to (36) characteristic for a geometrical fractal. We obtain

$$\Omega_0 = K_0 = H/\mathcal{N}, \qquad K_1 = K_2 = \ldots = K_m = 1. \tag{61}$$

By inserting (61) into (60) and integrating the resulting equation we get the following expression for the average number of particles:

$$\nu(\Omega) = H \underbrace{\ln(1 + \ln(1 + \ln(1 \ldots + \ln(1 + \mathcal{N}\Omega/H) \ldots)))}_{(m+1)\text{ times}}. \tag{62}$$

Now it is useful to introduce a set of special functions

$$\mathcal{L}_m(x) = \underbrace{\ln(1 + \ln(1 + \ln(1 \ldots + \ln(1 + x) \ldots)))}_{m\text{ logarithms}}, \qquad x \geqslant 0, \qquad m = 1, 2, \ldots \tag{63a}$$

$$\mathcal{L}_0(x) = x, \tag{63b}$$

which are successive functional iterates of the logarithmic function $\ln(1 + x)$. $\mathcal{L}_m(x)$ fulfill the forward and backward functional equations

$$\mathcal{L}_m(x) = \ln[1 + \mathcal{L}_m(x)] \qquad \text{and} \qquad \mathcal{L}_m(x) = \mathcal{L}_{m-1}[\ln(1 + x)]. \tag{64}$$

In equations (64) both iterative processes start from the initial condition (63b). For non-negative x all functions $\mathcal{L}_m(x)$ are non-negative, which ensures the non-negativity of the average number of particles $\nu(\Omega)$ for any values of the fractal hypervolume. The derivative of $\mathcal{L}_m(x)$ may be easily evaluated from the first of the two functional equations (64). We get

$$\frac{d\mathcal{L}_m(x)}{dx} = \frac{d\mathcal{L}_{m-1}(x)}{dx} \cdot \frac{1}{1 + \mathcal{L}_{m-1}(x)} = \prod_{m'=0}^{m-1} \frac{1}{1 + \mathcal{L}_{m'}(x)}. \tag{65}$$

In terms of $\mathcal{L}_m(x)$ the average number of particles is given by

$$\nu(\Omega) = H\mathcal{L}_{m+1}(\mathcal{N}\Omega/H). \tag{66}$$

Similarly, by combining equations (26) and (66) and making use of equation (65) we get a closed expression for the probability density $\phi_0(\Omega)$ of the empty space

$$\phi_0(\Omega)\,d\Omega = \frac{\mathcal{N}\,d\Omega}{\left\{\prod_{m'=0}^{m-1}[1 + \mathcal{L}_{m'}(\mathcal{N}\Omega/H)]\right\}[1 + \mathcal{L}_m(\mathcal{N}\Omega/H)]^{H+1}}$$

$$= \frac{H\,d\mathcal{L}_m(\mathcal{N}\Omega/H)}{[1 + \mathcal{L}_m(\mathcal{N}\Omega/H)]^{H+1}}. \tag{67}$$

Just as in equation (66) for the average number of particles the expression (67) for the probability density of the empty space is properly defined for any values of Ω between zero and infinity. Due to the non-negativity of $\mathcal{L}_m(x)$, $\phi_0(\Omega)$ is also non-negative. On the other hand, we can prove from (67) that $\phi_0(\Omega)\,d\Omega$ is normalized to unity. We have

$$\int_0^\infty \phi_0(\Omega)\,d\Omega = \int_0^\infty H\,d\mathcal{L}_m(x)/[1 + \mathcal{L}_m(x)]^{H+1} = 1. \tag{68}$$

The fact that the probability density (67) is much broader than the one corresponding to the inverse power statistical fractals (equation (16)) is also displayed by the behaviour of the corresponding moments. For equation (67) with $m \geqslant 1$ not only all positive moments of the fractal hypervolume are infinite, but the positive moments of the first $m-1$ iterate logarithms of Ω/Ω_0, $\ln^{m'}(\Omega/\Omega_0)$, $m' = 0, 1, \ldots, m-1$ are also infinite for any positive value of the fractal exponent. Only the divergence or convergence of the moments of the mth iterate logarithm of Ω/Ω_0, $\ln^m(\Omega/\Omega_0)$ depend on the value of the fractal exponent H. The moments of $\ln^m(\Omega/\Omega_0)$ cannot be computed exactly; however the moments of $\mathscr{L}_m(\mathscr{N}\Omega/H)$, which in the limit $\Omega \to \infty$ have the same behaviour

$$\mathscr{L}_m(\mathscr{N}\Omega/H) \to \ln^m(\Omega/\Omega_0) \quad \text{as} \quad \Omega \to \infty,\ m = 0, 1, 2, \ldots, \tag{69}$$

may be expressed in a closed form in terms of gamma functions. From equation (67) we have

$$\langle [\mathscr{L}_m(\mathscr{N}\Omega/H)^k] \rangle = H\int_0^\infty (\mathscr{L}_m)^k \, d\mathscr{L}_m/(1 + \mathscr{L}_m)^{H+1}, \tag{70}$$

from which after some arrangements we get

$$\langle [\mathscr{L}_m(\mathscr{N}\Omega/H)]^k \rangle = \Gamma(k+1)\Gamma(H-k)/\Gamma(H) \quad \text{for} \quad k < H, \tag{71a}$$

$$= \infty \quad \text{for} \quad k \geq H. \tag{71b}$$

Equations (71a) and (71b) are similar to (50a) and (50b) derived in the case $m = 0$.

The relationships of the generalized model characterized by equations (66)–(67) with the other models discussed in this paper are straightforward. As expected for $\Omega \to \infty$ (66) and (67) lead to (58) and (52) which were the starting point of our considerations. In the other limit $\Omega \to 0$ we recover the geometrical fractal statistics; moreover, as $H \to \infty$ the geometrical fractal statistics is exact for any value of the fractal hypervolume Ω. This can be shown by using a property of the functions $\mathscr{L}_m$

$$H\mathscr{L}_m(\mathscr{N}\Omega/H) \to \mathscr{N}\Omega \quad \text{as } H \to \infty \text{ for } m = 0, 1, 2, \ldots \tag{72}$$

which can be proved by complete induction.

The approaching to the geometrical fractal model as $H \to \infty$ is illustrated by the relations (71a) and (71b) for the moments: for small values of H only the first few moments $\langle (\mathscr{L}_m)^k \rangle$ are finite; as H increases more and more moments become finite and in the limit $H \to \infty$ all moments are finite and we come to the geometrical fractal model.

The relationships between the two generalized models are also clear. For $m = 0$ (66) and (67) reduce to (39) and (41) and thus the generalized model from Section 4 is a particular case of the model considered here, at least when the fractal hypervolume Ω is assumed to be the basic random variable. It might seem that the use of r and V as random variables leads to a contradiction. Indeed the transformation of the fractal exponents obeys different laws for $m = 0$ and $m \geq 1$ (compare equations (15) and (17) with (54) and (55)); however, this contradiction is only apparent. From equations (18) and (19) and (66) and (67) we can express both v and ϕ_0 in terms of r or V for any $m \geq 0$. From the resulting equations we recover both equations (15) and (17) ($m = 0$; r, $V \to \infty$) and equations (54) and (55) ($m \geq 1$; r, $V \to \infty$) as particular cases. The detailed computation is left to the reader as an exercise.

6. DISCUSSION

In order to elucidate the meaning of the generalized models we define the specific fractal hypervolume

$$\tilde{\omega}(\Omega) = \partial\Omega/\partial v. \tag{73}$$

$\tilde{\omega}(\Omega)$ is the average fractal hypervolume necessary for the occurrence of a particle corresponding to a total fractal hypervolume Ω; in a way it is a notion similar to the partial molar quantities used in the thermodynamics of multicomponent systems [10]. By making an analogy with the partial molar volume from thermodynamics [10] we can give a clearer physical interpretation of $\tilde{\omega}(\Omega)$: it is the increase in the fractal hypervolume resulting by adding a supplementary particle to a system which already has the fractal hypervolume Ω.

By combining equations (39) and (73) we can compute the expression of the specific fractal hypervolume valid for the first generalized model ($m = 0$)

$$\tilde{\omega}(\Omega) = 1/(\partial v/\partial \Omega) = \frac{1}{\mathcal{N}} + \frac{\Omega}{H_\Omega}. \tag{74}$$

It follows that for $m = 0$ the specific fractal hypervolume is made up of two additive terms: a constant contribution equal to the reciprocal value of the average density of particles in the limit $\Omega \to 0$ and a variable contribution which increases linearly with the total fractal hypervolume available. As $\Omega \to 0$, $\tilde{\omega}(\Omega)$ is practically constant and we recover the geometrical fractal statistics; in the other limit $\Omega \to \infty$ the contribution of the Ω/H_Ω term outweighs the contribution of the constant term $1/\mathcal{N}$ and we get the statistical fractal statistics. We see now that the introduction of the constant K into equation (29) corrects an unphysical feature of the statistical fractal model, namely the fact that for $\Omega \to 0$, $\tilde{\omega}(\Omega)$ tends to zero.

The same type of approach holds for $m \geq 0$. In this case the expression of the specific fractal hypervolume is given by:

$$\begin{aligned}\tilde{\omega}(\Omega) &= \left(\frac{1}{\mathcal{N}} + \frac{\Omega}{H}\right)\prod_{m'=1}^{m}\left[1 + \mathcal{L}_{m'}\left(\frac{\mathcal{N}\Omega}{H}\right)\right] \\ &= \left(\frac{1}{\mathcal{N}} + \frac{\Omega}{H}\right)\left[1 + \ln\left(1 + \frac{\mathcal{N}\Omega}{H}\right)\right]\ldots\end{aligned} \tag{75}$$

The first multiplicative term in equation (75) is the same as in the case $m = 0$ and thus the choice $\Omega_0 = K_0 = H/\mathcal{N}$ for $m \geq 1$ is equivalent to the choice $K = H_\Omega/\mathcal{N}$ for $m = 0$. Equation (75) also shows why the constants $K_1, \ldots, K_m$ in (60) should be equal to unity. For arbitrary positive $K_1, K_2, \ldots, K_m$ (75) becomes

$$\tilde{\omega}(\Omega) = \left(\frac{1}{\mathcal{N}} + \frac{\Omega}{H}\right)\left[K_1 + \ln\left(1 + \frac{\mathcal{N}\Omega}{H}\right)\right]\left[K_2 + \ln\left[K_1 + \ln\left(1 + \frac{\mathcal{N}\Omega}{H}\right)\right]\right]\ldots \tag{76}$$

Only for $K_1 = \ldots = K_m = 1$ all multiplicative terms containing logarithms collapse to unity for $\Omega \to 0$ and $\tilde{\omega}(\Omega)$ tends towards the value $1/\mathcal{N}$ typical for a geometrical fractal structure.

In the other limit $\Omega \to \infty$ the values of $K_1, \ldots, K_m$ are irrelevant and both (75) and (76) lead to

$$\tilde{\omega}(\Omega) = \frac{\Omega}{H}\prod_{m'=1}^{m}[\ln^{m'}(\mathcal{N}\Omega/H)], \quad \text{as} \quad \Omega \to \infty. \tag{77}$$

An open problem is the relationship between the generalized logarithmic model characterized by equations (66) and (67) and the stochastic renormalization procedure of Vlad [9]. Vlad also suggests a closed expression for a probability density of a positive random variable having a long tail of the type (52) which is properly defined for any values of the random variable. The expression of Vlad [9] is different from the relationship (67) derived here: it is a multiple integral expression depending on the incomplete gamma function. Although appropriate from the theoretical point of view the expression derived by Vlad [9] is not useful for concrete calculations; on the contrary equation (67), although without a deep theoretical significance, is more useful for applications. It might be possible that equation (67) could also be obtained as a result of a renormalization approach; till now we have failed to find whether this is possible.

Although rather different the geometric and statistical fractal models can be recovered as particular cases of the same type of generalized approaches. This is due to the fact that, in spite of their different behaviour all models presented in this paper share a common feature, that is, the translational invariance. For all models the average number of particles enclosed in a hypersphere of a given radius is independent of the position of the hypersphere in space; it depends only on the parameters of the models. The way in which we have matched the statistical and geometrical fractal models is rather formal. Further developments include the elaboration of a more realistic model for the transition from the geometrical to statistical fractal approaches, based on a clustering mechanism.

Another possibility of generalization is to make a connection between the probabilistic approach presented here and the multifractal analysis [11, 12]. Such a description would correspond to tails similar to the ones given by equations (15)–(17) where the exponents H_r, H_Ω, H_V are no longer constants but depend on a certain parameter (or class of parameters). For the elucidation of the implications of the multifractal structure on the behaviour of the average number of points further investigations are necessary.

7. CONCLUSIONS

The main new contribution presented in this paper is a feature of statistical fractal description of space distribution of independent particles; for a statistical fractal the average number of particles increases logarithmically with the size of the system. This feature is general in the sense that is independent of the details of a specific model. In order to illustrate the new approach a generalized model has been elaborated; although for small systems it reduces to a geometrical fractal its large volume behaviour obeys the statistical fractal laws. The generalized model is amenable to a complete analytical treatment: closed expressions for the average number of particles enclosed in a hypersphere of a given radius and for the probability densities of the empty space are available. Moreover, the generalized approach can be extended for the case when the statistical fractal distributions are replaced by very broad probability laws with iterated logarithmic tails.

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