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Evolution towards ergodic behavior of stationary fractal random processes with memory: application to the study of long-range correlations of nucleotide sequences in DNA

Marcel Ovidiu Vlad[a,1], Birgitt Schönfisch[b], Michael C. Mackey[c]

[a] *Centre for Nonlinear Dynamics in Physiology and Medicine, McGill University, 3655 Drummond Street, Montreal PQ, Canada H3G 1Y6*

[b] *Lehrstuhl für Biomathematik, Eberhard-Karls Universität Tübingen, Auf der Morgenstelle 10, D 72076 Tübingen, Germany*

[c] *Departments of Physiology, Physics and Mathematics, McGill University, 3655 Drummond Street, Montreal Quebec Canada H3G 1Y6*

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Abstract

The possible occurrence of ergodic behavior for large times is investigated in the case of stationary random processes with memory. It is shown that for finite times the time average of a state function is generally a random variable and thus two types of cumulants can be introduced: for the time average and for the statistical ensemble, respectively. In the limit of infinite time a transition from the random to the deterministic behavior of the time average may occur, resulting in an ergodic behavior. The conditions of occurrence of this transition are investigated by analyzing the scaling behavior of the cumulants of the time average. A general approach for the computation of these cumulants is developed; explicit computations are presented both for short and long memory in the particular case of separable stationary processes for which the cumulants of a statistical ensemble can be factorized into products of functions depending on binary time differences. In both cases the ergodic behavior emerges for large times provided that the cumulants of a statistical ensemble decrease to zero as the time differences increase to infinity. The analysis leads to the surprising conclusion that the scaling behavior of the cumulants of the time average is relatively insensitive to the type of memory considered: both for short and long memory the cumulants of the time average obey inverse power scaling laws. If the cumulants of a statistical ensemble tend towards asymptotic values different from zero for large time differences, then the time average is random even as the length of the total time interval tends to infinity and the ergodic behavior no longer holds. The theory is applied to the study of long range correlations of nucleotide sequences in DNA; in this case the length t of a sequence of nucleotides plays the role of the time variable. A proportionality relationship is established between the cumulants of the pyrimidine excess in a sequence of length t and the cumulants of the time (length) average of the probability of occurrence of a pyrimidine. It is shown that the statistical analysis of the DNA data presented in the literature

[1] Permanent address: Romanian Academy of Sciences, Centre for Mathematical Statistics, Casa Academiei, Calea 13 Septembrie No 13, OPB Bucuresti 5, 76100 Bucharest, Romania. Current address after 1 January 1996: Department of Chemistry, Stanford University, Stanford, CA 94305-5080, USA.

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is consistent with the occurrence of the ergodic behavior for large lengths. The implications of the approach to the analysis of the large time behavior of stochastic cellular automata and of fractional Brownian motion are also investigated.

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1. Introduction

Although initially formulated by physicists, the study of the problem of ergodic behavior has been almost entirely taken over by mathematicians. In physics the ergodic theory is mainly used for the analysis of the following problems: the fundamentation of statistical physics of equilibrium systems [1–4], the description of chaotic systems [5,6] and the description of the passage from the ergodic to the non-ergodic behavior in certain exotic systems, for instance in spin glasses [7]. Other subjects related to the ergodic theory, such as the ergodic behavior of stochastic processes with short or long memory [8,9], although well known to mathematicians, have been barely investigated by physicists [10,11].

The aim of this paper is to present a physically-oriented investigation of the possible occurrence of ergodic behavior for a class of stochastic processes with memory based on the use of a systematic cumulant expansion technique. Our approach originates from an investigation of the possible ergodic behavior in the particular case of stochastic cellular automata suggested by two of the present authors [12,13]. The initial approach [13] has led to incomplete results due to the nonsystematic expansion technique used. The systematic cumulant expansion presented in the following allows one to overcome the technical difficulties present in the initial approach [13] and makes it possible to establish under what circumstances the transition from the non-ergodic to the ergodic behavior is possible for large times. In developing this approach we have had two different possible applications in mind. The first application is the study of long-range correlations in the nucleotide sequences in DNA [14–16], and the second application is the investigation of the large time behavior of a general class of stochastic cellular automata with memory [13], which, even though are local in space, are non-local in time.

The structure of the paper is as follows. In Sections 2 and 3 we investigate the ergodic behavior of continuous and discrete systems, respectively. Section 4 deals with the application of the theory to the study of long-range correlations in DNA. Finally in Section 5 the implications of our approach for the study of stochastic cellular automata as well as of fractional Brownian motion are discussed.

2. Continuous systems

We start out by considering the case of random processes continuous in space and time; formally this case includes the random processes with discrete space and (or) time as particular cases.

The idea presented in Refs. [12,13] for investigating the ergodic behavior is limited to discrete random variables and discrete time. For a random system with a discrete state space with M states $u = 1, 2, \ldots, M$ the time average of a function $f(u)$ depending on the state u of the system is defined by:

$$\widetilde{f[u(t')]}(t) = \frac{1}{t} \sum_{t'=1}^{t} f[u(t')] . \tag{1}$$

In Refs. [12,13] it is suggested to express the time average (1) in the form

$$\widetilde{f[u(t')]}(t) = \frac{1}{t} \sum_{u=1}^{M} f(u) t_u, \qquad \sum_{u=1}^{M} t_u = t , \tag{2}$$

where t_u, $u = 1, \ldots, M$ are the total times spent by the system in the states $u = 1, \ldots, M$. By using Eq. (2) the evaluation of the time average $\widetilde{f[u(t')]}(t)$ reduces to the evaluation of the stochastic properties of the sojourn times $t_1, \ldots, t_M$. From Eq. (2) it follows that, as $t_1, \ldots, t_M$ are random, the time average $\widetilde{f[u(t')]}(t)$ is also random. The possible existence of ergodicity in the limit of large times $t \to \infty$ can be investigated by studying the stochastic properties of the sojourn times $t_1, \ldots, t_M$ in this limit.

If the system is characterized by a continuous state vector

$$\boldsymbol{x} = (x_1, x_2, \ldots) , \tag{3}$$

which is a random function of the continuous time t,

$$\boldsymbol{x} = \boldsymbol{x}(t) . \tag{4}$$

then the time average of a state function $f(\boldsymbol{x})$ for a time interval of length t is given by

$$\widetilde{f[\boldsymbol{x}(t')]}(t) = \frac{1}{t} \int_0^t f[\boldsymbol{x}(t')] \, \mathrm{d}t' . \tag{5}$$

In continuous time and space the sojourn times $t_1, \ldots, t_M$ are replaced by the density of states characterized by a state function between f and $f + \mathrm{d}f$:

$$\eta(f;t) = \int_0^t \delta(f - f[\boldsymbol{x}(t')] \, \mathrm{d}t', \qquad \int \eta(f;t) \, \mathrm{d}f = t . \tag{6}$$

By using definition (6) of the density of states $\eta(f;t)$ and the filtration property of the delta function it is easy to check that the time average (5) can be written in a form similar to Eq. (2):

$$\widetilde{f[\boldsymbol{x}(t')]}(t) = \frac{1}{t} \int_0^t f \eta(f;t) \, \mathrm{d}f . \tag{7}$$

We assume that the stochastic properties of the state vector $\boldsymbol{x}(t)$ are given by a stationary random process with memory whose behavior is characterized by a generalized characteristic functional of the Lax type [17]:

$$\mathscr{G}[Q(\boldsymbol{x}(t');t')] = \left\langle \exp\left(\mathrm{i} \int_0^\infty Q(\boldsymbol{x}(t');t')\,\mathrm{d}t' \right) \right\rangle, \tag{8}$$

where $Q(\boldsymbol{x}(t');t')$ is a suitable test function. The stationarity of the process can be expressed in the form

$$\mathscr{G}[Q(\boldsymbol{x}(t');t')] = \mathscr{G}[Q(\boldsymbol{x}(t'+\Delta t);t+\Delta t)], \tag{9}$$

where Δt is an arbitrary time difference.

According to their definitions, the state function at time $t, f(t) = f(\boldsymbol{x}(t))$ and the density of states $\eta(f;t)$ are also random functions; the characteristic functionals of these random functions can be expressed in terms of the characteristic functional $\mathscr{G}[Q(\boldsymbol{x}(t');t')]$ of the random vector $\boldsymbol{x}(t)$. We have:

$$\begin{aligned} G[\omega(t');t] &= \left\langle \exp\left(\mathrm{i} \int_0^t f(\boldsymbol{x}(t'))\omega(t')\,\mathrm{d}t' \right) \right\rangle \\ &= \mathscr{G}[Q(\boldsymbol{x}(t');t') = f(\boldsymbol{x}(t'))\omega(t')\,h(t-t')], \end{aligned} \tag{10}$$

$$\begin{aligned} \Xi[K(f',t');t] &= \left\langle \exp\left(\mathrm{i} \iint_0^t K(f',t')\eta(f';t')\,\mathrm{d}f'\,\mathrm{d}t' \right) \right\rangle \\ &= \mathscr{G}\left[Q(\boldsymbol{x}(t');t') = h(t-t') \int_{t'}^t \mathrm{d}t''\, K(f(\boldsymbol{x}(t'')),t') \right], \end{aligned} \tag{11}$$

where $\omega(t')$ and $K(f',t')$ are suitable test functions conjugate to $f(t) = f(\boldsymbol{x}(t))$ and $\eta(f';t')$, respectively, and $h(a)$ is the usual Heaviside step function.

If the cumulants $\langle\langle f(t_1') \dots f(t_m') \rangle\rangle$ and $\langle\langle \eta(f_1,t_1) \dots \eta(f_m,t_m) \rangle\rangle$, of the state function $f(\boldsymbol{x})$ at time $t, f(t) = f(\boldsymbol{x}(t))$, and of the density of states $\eta(f;t)$ exist and are finite, then the characteristic functionals $G[\omega(t');t]$ and $\Xi[K(f',t');t]$ can be expressed by the cumulant expansions [10]

$$\begin{aligned} G[\omega(t');t] = \exp\Bigg\{ \sum_{m=1}^\infty \frac{(\mathrm{i})^m}{m!} \int_0^t \cdots \int_0^t &\langle\langle f(t_1') \cdots f(t_m') \rangle\rangle\, \omega(t_1') \\ &\cdots \omega(t_m')\,\mathrm{d}t_1' \dots \mathrm{d}t_m' \Bigg\}, \end{aligned} \tag{12}$$

$$\Xi[K(f',t');t] = \exp\left(\sum_{m=1}^{\infty} \frac{(\mathrm{i})^m}{m!} \int\int_0^t \cdots \int\int_0^t \langle\langle \eta(f_1';t_1') \cdots \eta(f_m';t_m') \rangle\rangle \right.$$

$$\left. \times K(f_1',t_1') \cdots K(f_m',t_m'))\, \mathrm{d}f_1' \mathrm{d}t_1' \cdots \mathrm{d}f_m' \mathrm{d}t_m' \right). \tag{13}$$

Between the cumulants $\langle\langle f(t_m') \cdots f(t_m') \rangle\rangle$ of the state function $f(t) = f(x(t))$ and the cumulants $\langle\langle \eta(f_1;t_1) \ldots \eta(f_m,t_m) \rangle\rangle$ of the density of states $\eta(f;t)$ there is a relationship which can be derived by considering that the test function $K(f,t)$ has the product form

$$K(f,t) = fw(t). \tag{14}$$

In this case by using the definitions (10), (11) for $G[\omega(t');t]$ and $\Xi[K(f',t');t]$ we come to

$$\Xi[K = fw(t')] = G\left[\omega(t') = \int_{t'}^{t} w(t'')\, \mathrm{d}t'' \right]. \tag{15}$$

By inserting the cumulant expansions (12), (13) into Eq. (15) and identifying the coefficients of $w(t_1') \cdots w(t_m')$ from both terms we come to the identities

$$\int_0^{t_1} \cdots \int_0^{t_m} \langle\langle f(t_1') \cdots f(t_m') \rangle\rangle \, \mathrm{d}t_1' \cdots \mathrm{d}t_m' = \int \cdots \int f_1 \cdots f_m$$

$$\times \langle\langle \eta(f_1,t_1) \cdots \eta(f_m,t_m) \rangle\rangle \, \mathrm{d}f_1 \cdots \mathrm{d}f_m, \tag{16}$$

from which

$$\langle\langle f(t_1) \cdots f(t_m) \rangle\rangle$$

$$= \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \int \cdots \int f_1 \cdots f_m \langle\langle \eta(f_1,t_1) \ldots \eta(f_m,t_m) \rangle\rangle \, \mathrm{d}f_1 \cdots \mathrm{d}f_m, \tag{17}$$

and then the cumulants of the state function are completely determined by the cumulants of the density of states.

Since the density of states is random from Eq. (7) it turns out that the time average corresponding to a finite time interval of length t is also random. It follows that we can introduce the cumulants $\langle\langle \widetilde{f[x(t')]}^m \rangle\rangle(t)$ of the time average; these cumulants can be computed by defining a characteristic function for the time average $\widetilde{f[x(t')]}(t)$ and performing a cumulant expansion

$$\langle \exp(\mathrm{i}b\widetilde{f[x(t')]}) \rangle = \exp\left(\sum_1^{\infty} \frac{(\mathrm{i})^m}{m!} \langle\langle [\widetilde{f[x(t')]}]^m \rangle\rangle(t) b^m \right). \tag{18}$$

The cumulants of $f[x(x')]$ can be expressed in terms of the cumulants of the density of states. From Eqs. (7), (11) and (18) we note that

$$\langle \exp(\mathrm{i} b \overline{f[x(t')]}) \rangle = \Xi[K(t') = t^{-1}\delta(t-t') f b] . \tag{19}$$

By inserting Eqs. (13) and (17) into Eq. (18) and identifying the coefficients of the different powers of b we obtain:

$$\langle\langle [\overline{f[x(t')]}]^m \rangle\rangle(t) = t^{-m} \int \cdots \int f_1 \cdots f_m \langle\langle \eta(f_1,t) \cdots \eta(f_m,t) \rangle\rangle \, \mathrm{d}f_1 \cdots \mathrm{d}f_m , \tag{20}$$

from which, by taking Eqs. (16) into account we get:

$$\langle\langle [\overline{f[x(t')]}]^m \rangle\rangle(t) = t^{-m} \int_0^t \cdots \int_0^t \langle\langle f(t_1') \cdots f(t_m') \rangle\rangle \, \mathrm{d}t_1' \cdots \mathrm{d}t_m' . \tag{21}$$

We are interested in checking the possibility of existence of the ergodic behavior, i.e., we want to find out under what circumstances in the limit $t \to \infty$ the time average $\overline{f[x(t')]}(t)$ of the state function $f(x)$ is equal to the ensemble average of $f(x)$:

$$\lim_{t\to\infty} \overline{f[x(t')]}\,(t) = \langle f(x) \rangle , \tag{22}$$

Due to the stationarity of the random process describing the time evolution of the random state vector $x(t)$ the first cumulant of the state function $f(x)$ is time-independent,

$$\langle\langle f(x) \rangle\rangle = \langle f(x) \rangle = \text{independent of } t, \tag{23}$$

where we have used the property that the cumulant of first order of a random variable is equal to the average value of the random variable. From Eqs. (21) applied for $m = 1$ and from Eq. (23) we have

$$\langle \overline{f[x(t')]} \rangle(t) = \langle\langle \overline{f[x(t')]} \rangle\rangle(t) = t^{-1} \int_0^t \langle f(x) \rangle \, \mathrm{d}t' = \langle f(x) \rangle , \tag{24}$$

and therefore, in order that the ergodic property (22) be valid, we should have

$$\langle \overline{f[x(t')]} \rangle(\infty) = \overline{f[x(t')]}\;(\infty) , \tag{25}$$

that is, in the limit $t \to \infty$ the ensemble average of the time average should be equal to the time average itself; in other words the time average $\overline{f[x(t')]}(\infty)$ should be non-random. Since for a non-random quantity all cumulants of order bigger than one are equal to zero we should have:

$$\langle\langle [\overline{f[x(t')]}]^m \rangle\rangle(\infty) = \lim_{t\to\infty} \langle\langle [\overline{f[x(t')]}]^m \rangle\rangle(t) = 0, \qquad m \geqslant 2 . \tag{26}$$

In order to check whether the relationships (26) hold, we should evaluate the asymptotic behavior of the time integrals in Eqs. (21) in the limit $t \to \infty$. For studying the behavior of these integrals we note that the cumulants $\langle\langle f[x(t_1)] \cdots f[x(t_m)]\rangle\rangle$ of the state function $f[x(t)]$ fulfill the following properties:

(1) Since no restrictions are imposed on the times $t_1, \ldots, t_m$ the cumulant of the mth order of the state function is left unchanged by a permutation of the times $t_1, \ldots, t_m$.

(2) Due to the stationarity of the random process describing the evolution of the state vector x, and since the state function $f(x)$ depends on time only through $x = x(t)$ we have

$$\langle\langle f(t_1) \cdots f(t_m)\rangle\rangle = \langle\langle f(t_1 - \Delta t) \cdots f(t_m - \Delta t)\rangle\rangle, \quad f(t_u) = f[x(t_u)], \tag{27}$$

where Δt is an arbitrary time difference. Eqs. (27) are a consequence of the condition (9) of the time invariance of the random process describing the time evolution of the random vector $x = x(t)$.

By choosing the time difference Δt as the smallest of the times $t_1, \ldots, t_m$,

$$\Delta t = t_m^* = \min(t_1, \ldots, t_m), \tag{28}$$

we can express the cumulant of the mth order of the state function $f(x)$ as a function of $m - 1$ time differences

$$\langle\langle f(t_1) \cdots f(t_m)\rangle\rangle = C_m(t_1 - t_m^*, \ldots, t_m - t_m^*). \tag{29}$$

In Appendix A we show that by using Eq. (29) we can express the cumulants of the time average in the form

$$\langle\langle [\overline{f[x(t')]}]^m\rangle\rangle(t) = m t^{-m} \int_0^t d\theta \int_0^{t-\theta} d\tau_1 \cdots \int_0^{t-\theta} d\tau_{m-1}\, C_m(\tau_1, \ldots, \tau_{m-1}). \tag{30}$$

If moreover the mth cumulant $C_m(t_1 - t_m^*, \ldots, t_m - t_m^*)$ can be factorized into a product of $m - 1$ time differences

$$C_m(t_1 - t_m^*, \ldots, t_m - t_m^*) = A_m \prod_u \varphi(t_u - t_m^*), \quad C_1 = A_1 = \text{constant}, \tag{31}$$

then

$$\langle\langle [\overline{f[x(t')]}]^m\rangle\rangle(t) = A_m m t^{-m} \int_{\theta=0}^{t} d\theta \left(\int_0^{t-\theta} \varphi(\tau)\, d\tau \right)^{m-1}. \tag{32}$$

Now we apply Eqs. (30) and (32) to two different particular cases.

(1) For systems with short memory for which the mth cumulant $C_m(t_1 - t_m^*, \ldots, t_m - t_m^*)$ is the product of $m - 1$ exponentially decaying factors

$$C_m(t_1 - t_m^*, \ldots, t_m - t_m^*) = A_m \prod_u [\nu \exp[-\nu(t_u - t_m^*)]], \tag{33}$$

where, due to the symmetry of the mth cumulant C_m with respect to $t_1, \dots, t_m$, the frequency of fluctuations decay ν should be the same for all terms in Eq. (33). In Appendix B we show that a physical model of the shot-noise type leads to Eq. (33). Note that this situation includes as a particular case the stationary Gaussian and Markovian processes for which

$$A_m = 0, \qquad m \geqslant 3 . \tag{34}$$

By using the expression (33) for C_m Eq. (32) leads to

$$\langle\langle [\widetilde{f[x(t')]}]^m \rangle\rangle(t) = t^{-m} A_m \left(mt + \sum_{k=1}^{m-1} \frac{m!(-1)^k}{k!(m-1-k)!\nu k} [1 - \exp(-\nu k t)] \right), \tag{35}$$

and in the limit $t \to \infty$ we have

$$\langle\langle \widetilde{f[x(t')]} \rangle\rangle(t) = \langle f(x) \rangle, \qquad \langle f(x) \rangle = C_1 = A_1 , \tag{36}$$

$$\langle\langle [\widetilde{f[x(t')]}]^m \rangle\rangle(t) \sim t^{-(m-1)} m A_m, \qquad t \gg \nu^{-1}, \quad m \geqslant 2 , \tag{37}$$

and thus

$$\langle\langle [f[x(t')]]^m \rangle\rangle(\infty) = 0, \qquad m \geqslant 2 , \tag{38}$$

and therefore the ergodic property (22) holds. This situation includes the independent random processes as a particular case corresponding to

$$\nu \to \infty . \tag{39}$$

Eq. (33) becomes

$$C_m(t_1 - t_m^*, \dots, t_m - t_m^*) = A_m \prod_u \delta(t_u - t_m^*) \tag{40}$$

(see also Van Kampen [18]). From Eqs. (32) and (40) we obtain

$$\langle\langle \widetilde{f[x(t')]} \rangle\rangle(t) = A_1; \qquad \langle\langle [f[x(t')]]^m \rangle\rangle(t) = t^{m-1} m A_m, \quad m \geqslant 2 . \tag{41}$$

Eqs. (41) have the same form as the asymptotic laws (36), (37) derived for short memory. Eqs. (41) are exact for any time interval, short or long.

(2) The second case corresponds to random processes with long memory for which the statistical ensemble cumulants are given by a set of negative power laws:

$$C_m(t_1 - t_m^*, \dots, t_m - t_m^*) = B_m \left[\prod_u (t_u - t_m^*) \right]^{-H_m}, \quad C_1 = B_1 = \text{constant}, \; B_m \neq 0, \tag{42}$$

where H_m, $m = 1, 2, \ldots$ are nonnegative fractal exponents smaller than unity,

$$1 > H_m \geqslant 0, \qquad m = 2, 3, \ldots \tag{43}$$

In Appendix C we show that the scaling law (42) for the statistical ensemble cumulants can be generated by means of Shlesinger–Hughes stochastic renormalization [19, 20]. By computing the time integrals in Eq. (32) we come to

$$\langle\langle [\widetilde{f[x(t')]}]^m \rangle\rangle(t) = \frac{B_m m t^{-H_m(m-1)}}{[m(1 - H_m) + H_m](1 - H_m)^{m-1}} . \tag{44}$$

We note that for large time both the short and long memory lead to scaling conditions of the inverse power law type for the cumulants of the time average [see Eqs. (37), (44)]. The only effect of the long memory is that it leads to smaller scaling exponents than the short memory.

Concerning the ergodic behavior we distinguish two different subcases. (2.a.) If the fractal exponents H_m are positive, that is

$$1 > H_m > 0, \qquad m = 2, 3, \ldots , \tag{45}$$

then as $t \to \infty$ from Eq. (44) we come to

$$\langle\langle \widetilde{f[x(t')]} \rangle\rangle(\infty) = B_1 = \langle f(x) \rangle , \tag{46}$$

$$\langle\langle [\widetilde{f[x(t')]}]^m \rangle\rangle(\infty) = 0, \qquad m \geqslant 2 , \tag{47}$$

and therefore the ergodic property is valid.

(2) If the fractal exponents H_m are equal to zero

$$H_m = 0 , \tag{48}$$

the statistical ensemble cumulants are constant

$$C_m = B_m, \qquad m = 1, 2, \ldots \tag{49}$$

and in the limit $t \to \infty$ the cumulants of the time average are equal to the statistical ensemble cumulants

$$\langle\langle [\widetilde{f[x(t')]}]^m \rangle\rangle(\infty) = B_m, \qquad m \geqslant 1 . \tag{50}$$

Since the statistical ensemble cumulants $C_m = B_m \neq 0$ are different from zero [see Eq. (42)], it follows that the ergodic property no longer holds. Note that there is a less restrictive condition than the one given by Eqs. (48), (49) for which the ergodic property (22) is not valid. For the violation of ergodicity it is enough that residual correlations different from zero exist for large values of the time differences $t_1 - t_m^*$, $t_2 - t_m^*, \ldots ,$ in other words the statistical ensemble cumulants $C_m(t_1 - t_m^*, \ldots , t_m - t_m^*)$ tend to constant values different from zero as

$$t_1 - t_m^*, \ldots, t_m - t_m^* \to \infty,$$

$$\lim C_m(t_1 - t_m^*, \ldots, t_m - t_m^*) = C_m(\infty) \neq 0, \tag{51}$$

$$\forall t_u - t_m^* \to \infty .$$

By applying Eqs. (21) it is easy to check that

$$\langle\langle [\widetilde{f[x(t')]}]^m \rangle\rangle(\infty) = C_m(\infty) \neq 0, \tag{52}$$

and therefore the system has a non-ergodic behavior.

3. Discrete systems

Although most of the results derived in the preceding section also hold for discrete systems, there are however some differences. In this section we limit ourselves to the study of stationary stochastic systems in discrete time with a finite number M of states; these systems are directly connected to the problem of long-range correlations in DNA sequences and of the large time behavior of stochastic cellular automata.

For a discrete system state vector $\boldsymbol{x}$ is replaced by a discrete label u which can take a finite number M of values, $u = 1, 2, \ldots, M$, and the time t is a positive and integer number. Under these circumstances the time average of a state function $f(u)$ is given by Eqs. (1) and (2), where the individual sojourn times $t_u = t_u(t)$ corresponding to a total time interval of length t, are the discrete analogs of the density of states $\eta(f; t)$. The sojourn times can be expressed by the equation

$$t_u(t) = \sum_{t'=1}^{t} \delta_{(u)[u(t')]}, \qquad \sum_{u=1}^{M} t_u(t) = t, \tag{53}$$

which is the discrete analog of Eq. (6). By following the same steps as in the case of continuous systems we can relate the statistical ensemble cumulants of the state function $f(u)$, $\langle\langle f(t_1) \ldots f(t_m) \rangle\rangle$, with $f(u) = f[u(t)]$, to the cumulants $\langle\langle t_{u_1}(t_1) \cdots t_{u_m}(t_m) \rangle\rangle$ of the sojourn times:

$$\sum_{t_1'=1}^{t_1} \cdots \sum_{t_m'=1}^{t_m} \langle\langle f(t_1') \cdots f(t_m') \rangle\rangle$$

$$= \sum_{u_1=1}^{M} \cdots \sum_{u_m=1}^{M} f(u_1) \cdots f(u_m) \langle\langle t_{u_1}(t_1) \ldots t_{u_m}(t_m) \rangle\rangle, \tag{54}$$

$$\langle\langle f(t_1) \cdots f(t_m) \rangle\rangle = \Delta_{t_1} \cdots \Delta_{t_m}$$

$$\times \sum_{u_1=1}^{M} \cdots \sum_{u_m=1}^{M} f(u_1) \cdots f(u_m) \langle\langle t_{u_1}(t_1) \ldots t_{u_m}(t_m) \rangle\rangle, \tag{55}$$

where $\Delta_t \cdots$

$$\Delta_t \varphi(t) = \varphi(t) - \varphi(t-1), \tag{56}$$

are difference operators. We can also express the cumulants of the time average $\langle\langle [\overline{f[u(t')]}]^m \rangle\rangle(t)$, in terms of the cumulants of the sojourn times, $\langle\langle t_{u_1}(t_1) \cdots t_{u_m}(t_m) \rangle\rangle$ and of the statistical ensemble cumulants $\langle\langle f(t_1) \cdots f(t_m) \rangle\rangle$:

$$\langle\langle [\overline{f[u(t')]}]^m \rangle\rangle(t) = t^{-m} \sum_{u_1=1}^{M} \cdots \sum_{u_m=1}^{M} f(u_1) \cdots f(u_m) \langle\langle t_{u_1}(t) \ldots t_{u_m}(t) \rangle\rangle, \tag{57}$$

$$\langle\langle [\overline{f[u(t')]}]^m \rangle\rangle(t) = t^{-m} \sum_{t_1'=1}^{t} \cdots \sum_{t_m'=1}^{t} \langle\langle f[u(t_1')] \cdots f[u(t_m')] \rangle\rangle. \tag{58}$$

Eqs. (54)–(58) are the discrete analogs of Eqs. (16), (17), (20) and (21), respectively; their detailed derivation is left to the reader.

By using the condition of time homogeneity due to the stationarity of the random process, which describes the time evolution of the random variable u, we can express the statistical ensemble cumulants by Eq. (29) where now the times $t_1, \ldots, t_m$ are discrete variables. The discrete analog of Eq. (30) is:

$$\langle\langle [\overline{f[u(t')]}]^m \rangle\rangle(t) = mt^{-m} \sum_{\theta=1}^{t} \sum_{\tau_1=0}^{t-\theta} \cdots \sum_{\tau_{m-1}=0}^{t-\theta} C_m(\tau_1, \ldots, \tau_{m-1}), \tag{59}$$

from which, by assuming that the cumulants C_m can be factorized in the product form (31), we obtain:

$$\langle\langle [\overline{f[u(t')]}]^m \rangle\rangle(t) = A_m m t^{-m} \sum_{\theta=1}^{t} \left(\sum_{\tau=0}^{t-\theta} \varphi(\tau) \right)^{m-1}. \tag{60}$$

A random process with short memory is described by a set of cumulants of the product from (31) where the function $\varphi(\tau)$ is exponentially decreasing:

$$\varphi(\tau) = (1-\lambda)\lambda^\tau, \qquad 1 > \lambda > 0. \tag{61}$$

In this case the evaluation of the sums in Eq. (60) reduces to the repeated summation of several geometrical progressions; to save space we give here only the scaling law for the cumulants of the time average valid for large times:

$$\langle\langle \overline{f[u(t')]} \rangle\rangle(t) = \langle f(x) \rangle, \qquad t \gg [\ln(1/\lambda)]^{-1}. \tag{62}$$

$$\langle\langle [\overline{f[u(t')]}]^m \rangle\rangle(t) \sim \text{const.}\ \ t^{-(m-1)}, \qquad t \gg [\ln(1/\lambda)]^{-1}, \quad m \geqslant 2. \tag{63}$$

We notice that in the limit $t \to \infty$ the cumulants of the time average of order bigger than one vanish and thus the behavior of the system is ergodic.

In the limit $\lambda \to 0$ Eq. (61) becomes

$$\varphi(\tau) = \delta_{0\tau}, \tag{64}$$

and the random process describing the time evolution of the label u is independent, that is, at each time step the label u is randomly selected from a constant probability distribution

$$P(u), \qquad \sum_{u=1}^{M} P(u) = 1 . \tag{65}$$

In Appendix D we show that in this case all cumulants of the sojourn time and of the time average can be computed exactly for any time interval, short or long. We have

$$\begin{aligned}\langle\langle t_{u_1}(t) \ldots t_{u_m}(t)\rangle\rangle = t\sum_n \frac{(-1)^n}{n} \sum_{\beta_1} \cdots \sum_{\beta_n} \delta_{m(\Sigma\beta_q)} \sum_{v_1=1}^{M} \cdots \sum_{v_n=1}^{M} P(v_1) \\ \cdots P(v_n) \frac{m!}{\prod_q \beta_q!} \delta_{u_1 v_1} \cdots \delta_{u_{\beta_1} v_1} \cdots \delta_{(u_{\beta_1+} \ldots {}_{+\beta_{n-1}+1}) v_n} \\ \times \delta_{(u_{\beta_1+} \ldots {}_{+\beta_n}) v_n} ,\end{aligned} \tag{66}$$

$$\begin{aligned}\langle\langle [\overline{f[u(t')]}]^m \rangle\rangle(t) = t^{-(m-1)} \sum_n \frac{(-1)^{n-1}}{n} \sum_{\beta_1} \\ \cdots \sum_{\beta_n} \delta_{m(\Sigma\beta_q)} \frac{m!}{\prod_q \beta_q!} \prod_{q=1}^{n} \langle f^{\beta_q}(u)\rangle ,\end{aligned} \tag{67}$$

where the moments of the state function $\langle f^{\beta_q}(u)\rangle$ are computed in terms of the one-time probability distribution $P(u)$ [Eq. (65)] of the label u:

$$\langle f^{\beta_q}(u)\rangle = \sum_{u=1}^{M} P(u) f^{\beta_q}(u) . \tag{68}$$

Eqs. (67) are the discrete time analogues of the relationships (41) derived in Section 2 for independent processes in continuous space and time. Note that both equations display the same scaling law for the cumulants of the time average, $\sim t^{-(m-1)}$, which is valid for any time interval, short or long.

By analogy with the continuous time case we consider a random process with long memory for which the statistical ensemble cumulants of the state variable $f(u)$ have the product form (42) for $t_u \neq t_m^*$. In the case of continuous time systems the divergence generated by the equality $t_u = t_m^*$ disappears due to the integration in Eqs. (30) and (32). In the case of discrete time, however, the divergence is not removed by summation and thus we should assume that for $t_u = t_m^*$ Eq. (29) is not valid anymore and that in this case the corresponding cumulant is given by a finite constant. By evaluating the cumulants of the time average in the limit of large times we get a scaling law similar to Eq. (44)

$$\langle\!\langle [\widetilde{f[u(t')]}]^m \rangle\!\rangle(t) \sim \text{const.} \quad t^{-H_m(m-1)}, \qquad t \gg 0\,. \tag{69}$$

Just as in the case of continuous systems for positive fractal exponents $1 > H_m > 0$ [Eq. (45)] the system is ergodic as $t \to \infty$ whereas for fractal exponents equal to zero residual correlations different from zero exist for large time differences and the ergodic behavior does not hold anymore. By analogy with the case of continuous systems it is easy to show that for the violation of ergodicity it is enough that the condition (51) holds and then

$$\langle\!\langle [\widetilde{f[u(t')]}]^m \rangle\!\rangle(\infty) = C_m(\infty) \neq 0\,. \tag{70}$$

4. Long range correlations of nucleotide sequences in DNA

Recently extensive studies of the statistical correlations of the nucleotide sequences in the DNA molecule have been reported [14–16]. Special attention has been paid to the identification of different self-similar features of the coding and non-coding regions of DNA (exons and introns, respectively) with the purpose of establishing statistical tests for the location of these regions.

For the statistical study of correlations the notion of DNA walk has been introduced [14–16]. Given a strand of DNA made up of t nucleotides for each of the t sites $t' = 1, \ldots, t$ we attach a spin-like variable $s_{t'}$, $t' = 1, \ldots, t$ where

$$s_{t'} = +1 \quad \text{if the site is occupied by a pyrimidine,} \tag{71}$$

$$s_{t'} = -1 \quad \text{if the site is occupied by a purine.} \tag{72}$$

For the set of t sites we introduce the overall variable

$$y(t) = \sum_{t'=1}^{t} s_{t'}, \tag{73}$$

which is a random function of the total number of nucleotides t. Concerning the physical interpretation of the random variable $y(t)$ from Eq. (73) we notice that it is the difference between the number $n_+(t)$ of pyrimidines in a chain of t nucleotides

$$n_+(t) = \sum_{t'=1}^{t} \delta_{(+1)s_{t'}}, \tag{74}$$

and the corresponding number of purines

$$n_-(t) = \sum_{t'=1}^{t} \delta_{(-1)s_{t'}}. \tag{75}$$

We have

$$n_+(t) + n_-(t) = t\,, \tag{76}$$

$$n_+(t) - n_-(t) = y(t)\,. \tag{77}$$

According to Eq. (77) we can use the name of pyrimidine excess for the function $y(t)$.

The methodology of identifying the long-range correlations in the nucleotide sequences suggested in the literature consists in the experimental study of the stochastic properties of the pyrimidine excess $y(t)$ as a function of the total number of nucleotides t. These stochastic properties can be characterized in terms of the cumulants of the pyrimidine excess corresponding to different total numbers of nucleotides $t_1, \ldots, t_m$

$$\langle\langle y(t_1) \ldots y(t_m) \rangle\rangle\,, \qquad m = 1, 2, \ldots\,, \tag{78}$$

which are defined by means of a cumulant expansion of the characteristic functional of the pyrimidine excess

$$\left\langle \exp\left(i \sum_{t'=1}^{t} y(t')\xi(t') \right) \right\rangle = \exp\left(\sum_{m=1}^{\infty} \frac{(i)^m}{m!} \sum_{t'_1=1}^{t} \cdots \sum_{t'_m=1}^{t} \xi(t'_1) \cdots \xi(t'_m) \langle\langle y(t'_1) \cdots y(t'_m) \rangle\rangle \right). \tag{79}$$

The data reported in the literature [14–16] concern only the equal-site-number cumulants of the first and second order, $\langle\langle y(t) \rangle\rangle$ and $\langle\langle y^2(t) \rangle\rangle$. It has been shown that for a large class of DNA sequences these two cumulants obey the scaling laws

$$\langle\langle y(t) \rangle\rangle \sim \text{const}\ t, \qquad t \gg 1\,, \tag{80}$$

$$\langle\langle y^2(t) \rangle\rangle \sim \text{const}\ t^{\sigma_2}, \quad t \gg 1\,, \tag{81}$$

where σ_2 is a positive fractal exponent. In many cases the proportionality constant in Eq. (80) is equal to zero, a situation which corresponds to an average ratio of pyrimidines to purines of one to one; otherwise for const. > 0 there is a pyrimidine excess and for const. < 0 there is a purine excess. Concerning the value of the fractal exponent σ_2 attached to the second cumulant of the pyrimidine excess there is a controversy in the literature. Peng et al. [14] claim that σ_2 is very close to unity for the coding regions of DNA (exons) whereas for the non-coding regions (introns) it is bigger than the unity $\sigma_2 = 1.2$–1.3 and that this difference may serve as a basis for identifying the exons by means of a statistical analysis. On the other hand Voss et al. [15] claim that the difference of the σ_2-values for exons and introns is an artifact generated by the statistical approach used by Peng et al. for the analysis of experimental data. By using an alternative statistical approach Voss et al. obtain fractal exponents bigger than unity and smaller than two both for exons and introns and draw the conclusion that the value of the exponent σ_2 cannot be used for identifying the exons.

In the following we try to establish a connection between the experimental scaling laws (80), (81) for the first two cumulants of the pyrimidine excess and the possible existence of an ergodic behavior for long sequences of nucleotides. To accomplish this we establish a connection between the DNA walk and the discrete formalism introduced in Section 3. First of all we notice that the total number t of nucleotides plays the same role as the time variable t and that the value of the label u is given by the spin variable s; the state space is made up of only two states $s = +1$ $(u = 1)$ and $s = -1$ $(u = 2)$ and $M = 2$.

The time average (or rather the length average) of the spin variable s corresponding to a nucleotide sequence of total length t is given by

$$\widetilde{s(t')}(t) = (1/t) \sum_{t'=1}^{t} s(t') = y(t)/t\,, \tag{82}$$

and thus the pyrimidine excess is the product of the total length t of the sequence and of the time average of the spin variable

$$y(t) = t\,\widetilde{s(t')}(t). \tag{83}$$

Now we introduce the one-length characteristic function of the pyrimidine excess, which is a particular case of the characteristic functional (79) corresponding to

$$\xi(t') = \delta_{tt'}\zeta(t)\,, \tag{84}$$

that is

$$\langle \exp(\mathrm{i}y(t)\zeta(t))\rangle = \exp\left(\sum_{m=1}^{\infty} \frac{(\mathrm{i})^m}{m!}\zeta^m(t)\langle\langle y^m(t)\rangle\rangle\right), \tag{85}$$

and the characteristic function of the time average $\widetilde{s(t')}(t)$ of the spin variable $s(t')$,

$$\langle \exp(\mathrm{i}\,\widetilde{s(t')}(t)b(t))\rangle = \exp\left(\sum_{m=1}^{\infty} \frac{(\mathrm{i})^m}{m!}\langle\langle[\widetilde{s(t')}]^m\rangle\rangle(t)b^m(t)\right). \tag{86}$$

We insert Eq. (82) into Eq. (86) and compare the result with Eq. (85); we come to

$$\langle\langle y^m(t)\rangle\rangle = t^m\langle\langle\widetilde{[s(t')]^m}\rangle\rangle(t), \qquad m \geqslant 1\,. \tag{87}$$

By using Eq. (87) we can apply the general relationships derived in Section 3 for the study of ergodic behavior for discrete systems to the case of DNA walks. If the spin variable s has a short memory, then from Eqs. (58) and (87) we get the following expressions for the cumulants of the pyrimidine excess:

$$\langle\langle y(t)\rangle\rangle \cong t\langle s\rangle = t[P(s = +1) - P(s = -1)], \qquad t \gg [\ln(1/\lambda)]^{-1}, \tag{88}$$

$$\langle\langle y^m(t)\rangle\rangle \cong \mathrm{const}'\ t,\ t \gg [\ln(1/\lambda)]^{-1}, \qquad m \geqslant 2\,, \tag{89}$$

equations which are consistent with the results of Peng et al. for the exonic regions [14]. Note that the proportionality constant in Eq. (88) is also consistent with the results from Refs. [14] mentioned before, that is, the proportionality constant is equal to zero if the probability of occurrence of a pyrimidine $P(+1)$ is equal to the probability $P(-1)$ of occurrence of a purine, $P(+1) = P(-1) = \frac{1}{2}$, and it is $\gtrless 0$ if $P(+1) \gtrless P(-1)$, respectively.

Similarly for long memory we get

$$\langle\langle y(t) \rangle\rangle \cong t[P(+1) - P(-1)], \qquad t \gg 1, \tag{90}$$

$$\langle\langle y^m(t) \rangle\rangle \cong \text{const}'\, t^{\sigma_m}, \qquad t \gg 1, \tag{91}$$

where the fractal exponents σ_m are related to the fractal exponents H_m introduced in Section 3 by the relationship

$$\sigma_m = m(1 - H_m) + H_m, \qquad m \geqslant 2. \tag{92}$$

As $1 > H_m \geqslant 0$ [see Eq. (43)] we have

$$m \geqslant \sigma_m > 1. \tag{93}$$

The behavior of the average pyrimidine excess is the same as in the case of short memory. The cumulants of order bigger than one attached to the pyrimidine excess obey a power law with a fractal exponent depending linearly on the fractal exponent of statistical ensemble cumulants of the spin variable. This situation corresponds to the results obtained by Peng et al. [14] for introns and with the results of Voss et al. [15] both for introns and for exons.

Concerning the possible existence of the ergodic behavior for the spin variable s in the limit of large DNA lengths, $t \to \infty$, a complete discussion is not possible because the literature [14–16] reports only results concerning the cumulants of the first and the second order of the pyrimidine excess. Note that the violation of the ergodic behavior as $t \to \infty$ may occur only for long memory with fractal exponents equal to zero, $H_m = 0$, that is, for scaling exponents σ_m equal to

$$\sigma_m = m. \tag{94}$$

As far as we know the results presented in the literature give $\sigma_2 < 1.4$ [14–16], which is compatible with ergodicity. Although the violation of ergodicity due to the equality $\sigma_m = m$ for the cumulants of the pyrimidine excess of order bigger than two, $m \geqslant 3$, cannot be completely ruled out, it seems to be rather improbable; indeed usually the cumulants of the second order give the most useful information about the fluctuations of a set of random variables. A definite answer to this question can be given only if information concerning the cumulants of superior order of the pyrimidine excess is extracted from the experimental data.

5. Ergodic behavior for stochastic cellular automata and generalized fractional Brownian motion

In this section we discuss briefly two additional applications of our approach for studying the ergodic behavior, in the case of a class of stochastic cellular automata and of generalized fractional Brownian motion. The stochastic cellular automata considered here [12, 13] are made up of a finite number of cells arranged on a regular d_s-dimensional lattice. To each cell we attach a finite number of states and for each cell we define a local neighborhood and a stochastic local function which assigns an elementary state of the cell for each occupation of the neighborhood. The evolution of the cellular automaton is followed step by step, on a discrete time scale $t = 1, 2, \ldots$. Depending on the way a cell is evaluated, we distinguish synchronous cellular automata for which all cells are evaluated at the same steps; otherwise, if the evaluation of the cells is sequential and only one group of cells is evaluated in a given step, then the automaton is asynchronous.

In Ref. [13] a general approach for investigating the ergodic behavior has been suggested which is independent of the detailed definition of the neighborhood, the stochastic local function and of the evaluation rule (synchronous or asynchronous). The main assumption made is that, at least in principle, it is possible to identify all possible states in which the whole lattice may exist. The total number of configurations M, although possibly very large is always finite for a finite lattice for which the cells may exist in a finite number of states. The finite number of states makes possible the replacement of the local description of a cellular automaton by a global description by attaching a numerical label $u = 1, 2, \ldots, M$ to each global state of the lattice. Since the evaluation of the cells takes place step by step according to a given stochastic local function, in this global description the whole automaton performs a Markovian random walk among the M states of the lattice $u = 1, \ldots, M$. The Markovian nature of the random walk is due to the action of the local function which is localized in time and establishes a correspondence between the states of the cells corresponding to two successive time steps. This Markovian behavior is independent of the detailed structure of the automaton defined by the type of neighborhood chosen, the type of local function or the synchronous or the asynchronous evaluation of the cells.

By assuming that all M states of the automaton are connected [21] and using a Lippman–Schwinger expansion technique in Ref. [13], it has been shown that the cumulants of first and second order of the sojourn times t_u attached to the different states of the automaton $u = 1, \ldots, M$ have the following large time behavior:

$$\langle\langle t_u \rangle\rangle(t) = p^{st}(u)t, \qquad t \gg 0\,, \tag{95}$$

$$\langle\langle t_{u_1} t_{u_2} \rangle\rangle(t) = \text{const.}\ t\,, \qquad t \gg 0\,, \tag{96}$$

where $P^{st}(u)$ is the stationary probability of occurrence of the state u. Eqs. (95), (96) are consistent with the ergodic behavior of the cellular automaton as $t \to \infty$. The asymptotic behavior of the cumulants of the sojourn times t_u, $u = 1, \ldots, M$ of order bigger than two could not be evaluated even though exact analytical expressions for all positive moments of the sojourn times have been derived [13].

In addition to this Markovian type of stochastic cellular automata in Ref. [13] a non-Markovian type has been also considered for which the local function, although local in space, is non-local in time and defined for a succession of many time steps. By assuming that a stationary random behavior emerges in the long run, such stochastic cellular automata with long memory can be described by an infinite chain of probability densities

$$P^{(t)}(u_1,1; \dots ; u_t; t), \quad \sum_{u_1} \cdots \sum_{u_t} P^{(t)}(u_1,1; \dots ; u_t; t) = 1,$$
$$t = 1,2, \dots . \tag{97}$$

A general approach for computing the moments and the cumulants of the sojourn times has been suggested in Ref. [13] by assuming that the joint probabilities (97) are known. By assuming the stationarity of the random process described by Eqs. (97) only the asymptotic behavior of the cumulants of the first and the second order has been evaluated [13]. The resulting expressions are

$$\langle\langle t_u \rangle\rangle(t) = P^{st}(u)t, \qquad t \gg 0\,, \tag{98}$$

$$\langle\langle t_{u_1} t_{u_2} \rangle\rangle(t) = t\,P^{st}(u_1)[\delta_{u_1u_2} + P^{st}(u_2)] + \sum_{\varepsilon=1}^{t} (t-\varepsilon)[\phi^{(2)}_{u_1u_2}(0,\varepsilon) + \phi^{(2)}_{u_1u_2}(\varepsilon,0)], \qquad t \gg 0\,, \tag{99}$$

where

$$\phi^{(m)}_{u_1 \cdots u_m}(t_1, \dots, t_m) = P^{(t)}(u_1,t_1; \dots ; u_m, t_m) - P(u_1) \cdots P(u_m)\,, \tag{100}$$

is a measure of the correlations of the states $u_1, \dots, u_m$ which plays a similar role to the statistical ensemble cumulants $\langle\langle f(t_1) \cdots f(t_m) \rangle\rangle$ in our approach. A straightforward analysis shows that if

$$\phi^{(2)}_{u_1u_2}(t_1,t_2) \to 0 \quad \text{as } |t_1 - t_2| \to \infty\,, \tag{101}$$

then

$$\langle\langle [\widetilde{f[u(t')]}]^2 \rangle\rangle(\infty) = \lim_{t\to\infty} t^{-2} \sum_u \sum f(u_1) f(u_2) \langle\langle t_{u_1} t_{u_2} \rangle\rangle(t) = 0\,, \tag{102}$$

a condition which is compatible with the ergodic behavior. On the other hand if residual correlations different from zero exist for large time differences

$$\phi^{(2)}_{u_1u_2}(t_1,t_2) \to \phi^{(2)}_{u_1u_2}(\infty) \neq 0 \quad \text{as } |t_1 - t_2| \to \infty\,, \tag{103}$$

then

$$\langle\langle [\widetilde{f[u(t')]}]^2 \rangle\rangle(\infty) = \sum_{u_1}\sum_{u_2} f(u_1) f(u_2)\, \phi^{(2)}_{u_1 u_2}(\infty), \tag{104}$$

and thus the second cumulant of the time average is generally different from zero and the ergodic property does not hold anymore.

By describing the long memory correlations among the different states of the cellular automaton, not in terms of the joint probability densities (97) and of the functions (100), but in terms of the statistical ensemble cumulants $\langle\langle f(t_1) \cdots f(t_m)\rangle\rangle$ the analysis presented in Ref. [13] for the asymptotic behavior of the cumulants of the first and second order of the time average can be easily extended to cumulants of any order. It is easy to see that by using a cumulant description the cellular automaton problem is isomorphic with the general discrete problem discussed in Section 3. In particular the asymptotic expressions (98), (99) for a Markovian cellular automaton are equivalent to the first two of the Markovian equations (62), (63). In addition to Eqs. (62), (63) for the cumulants of order one and two, our systematic expansion technique also provides information concerning the cumulants of the time average of order bigger than two which all tend to zero as $t \to \infty$ and thus the ergodic property holds.

Similarly for long memory the general approach presented in Section 3 is consistent with the results from Ref. [13]. In particular, for both approaches the ergodic property is not valid if residual correlations different from zero exist for large time differences. The main difference between the two methods is that in Ref. [13] the correlations are expressed in terms of the functions (100) whereas in the approach presented here they are given by the statistical ensemble cumulants $\langle\langle f(t_1) \cdots f(t_m)\rangle\rangle$. Another difference is that our approach allows us to compute the asymptotic behavior of all cumulants of the time average in contrast with the non-systematic approach from Ref. [13] which leads to asymptotic expressions for large times only for the cumulants of first and second order.

Even though more complete than the study presented in Ref. [13], the analysis presented in Section 3 does not exhaust the study of the ergodic behavior of stochastic cellular automata. One question remains unanswered for systems with long memory; unlike in the case of Markovian processes [13], our approach does not show under what circumstances a stationary random process eventually emerges for stochastic cellular automata with long memory. Note however that, although our physically oriented approach does not provide an answer to this question, similar problems have been extensively studied in the mathematical literature [9]. Further research should include a comparison between our approach and these mathematical studies.

The fractional Brownian motion (FBM, [22, 23]) is a simple example of a fractal random process introduced by Mandelbrot in the late sixties [22]. Due to its simplicity, the FBM is a popular means for describing a variety of statistical fractals corresponding to various phenomena such as floods, the fluctuations of the stock market or of the heartbeat, the random topography of certain surfaces, anomalous diffusion, etc. [22–24]. In this section we investigate a simple multidimensional generalization of the FBM model. We start out from an M-dimensional random vector $\boldsymbol{x}$ with zero average value

$$\langle \boldsymbol{x}(t)\rangle = \boldsymbol{0}\,, \tag{105}$$

and assume that the corresponding random process in Gaussian and stationary. It follows that all stochastic properties of the random vector $\boldsymbol{x} = \boldsymbol{x}(t)$ are characterized by its cumulants of the second order which can be arranged in a correlation matrix

$$\boldsymbol{C}(t_1, t_2) = [C_{u_1 u_2}(t_1, t_2)]_{u_1, u_2 = 1, \dots, m}\,, \qquad m = 1, 2, \dots\,, \tag{106}$$

where the matrix elements $C_{u_1 u_2}$ obey the condition of temporal invariance

$$\begin{aligned} C_{u_1 u_2}(t_1, t_2) &= \langle\langle x_{u_1}(t_1) x_{u_2}(t_2)\rangle\rangle = \langle\langle x_{u_1}(t_1 - t_2) x_{u_2}(0)\rangle\rangle, \quad \text{for } t_1 \geqslant t_2\,, \\ &= \langle\langle x_{u_1}(0)\, x_{u_2}(t_2 - t_1)\rangle\rangle, \quad \text{for } t_2 \geqslant t_1\,. \end{aligned} \tag{107}$$

We assume that these correlation functions obey scaling laws of the negative power law type

$$\begin{aligned} C_{u_1 u_2}(t_1, t_2) &= (1 - H_{u_1 u_2}) A_{u_1 u_2} (t_1 - t_2)^{-H_{u_1 u_2}}, \quad 1 > H_{u_1 u_2} \geqslant 0, \quad t_1 \geqslant t_2\,, \\ &= (1 - H_{u_2 u_1}) A_{u_2 u_1} (t_2 - t_1)^{-H_{u_2 u_1}}, \quad 1 > H_{u_2 u_1} \geqslant 0, \quad t_2 \geqslant t_1\,, \end{aligned} \tag{108}$$

where $H_{u_1 u_2}$, $H_{u_2 u_1}$ are fractal exponents between zero and unity and $A_{u_1 u_2}$ are the elements of a matrix with non-negative diagonal elements. Since we do not assume microscopic reversibility, the matrix $\boldsymbol{H} = [H_{uu'}]$ of the fractal exponents as well as the matrix $\boldsymbol{A} = [A_{uu'}]$ are generally non-symmetric.

Our aim is to characterize the stochastic properties of the time integral [22–24]:

$$\boldsymbol{y}(t) = \int_0^t \boldsymbol{x}(t')\,\mathrm{d}t'\,. \tag{109}$$

In Appendix E we show that the random behavior of the vector $\boldsymbol{y}(t)$ is described by a non-stationary random process with long memory with zero average value and with the characteristic functional

$$\begin{aligned} \mathscr{F}[\boldsymbol{K}(t')] &= \left\langle \exp\left(\mathrm{i} \int_0^\infty \boldsymbol{K}(t') \cdot \boldsymbol{y}(t')\,\mathrm{d}t'\right)\right\rangle \\ &= \exp\left(-\frac{1}{2} \sum_{u_1} \sum_{u_2} \int_0^\infty \int_0^\infty \langle\langle y_{u_1}(t_1) y_{u_2}(t_2)\rangle\rangle\, K_{u_1}(t_1) K_{u_2}(t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2\right), \end{aligned} \tag{110}$$

where $\boldsymbol{K}(t') = [K_u(t')]$ is a suitable vectorial test function conjugate to the random vector $\boldsymbol{y}(t)$,

$$\langle\langle y_{u_1}(t_1) y_{u_2}(t_2)\rangle\rangle = \mathscr{I}_{u_1u_2}(t_1) + \mathscr{I}_{u_2u_1}(t_2) \\ - \mathscr{I}_{u_1u_2}(t_1 - t_2)h(t_1 - t_2) - \mathscr{I}_{u_2u_1}(t_2 - t_1)h(t_2 - t_1), \tag{111}$$

is the correlation matrix of the random vector $\boldsymbol{y}(t)$, $h(\tau)$ is the usual Heaviside function and the functions $\mathscr{I}_{u_1u_2}(t)$ are given by:

$$\mathscr{I}_{u_1u_2}(t) = \int_0^t (t-\tau)C_{u_1u_2}(\tau)\,\mathrm{d}\tau = A_{u_1u_2}(2 - H_{u_1u_2})^{-1}t^{2-H_{u_1u_2}}. \tag{112}$$

The m-point joint probability density of the random vector $\boldsymbol{y}(t)$

$$P^{(m)}(\boldsymbol{y}_1, t_1; \dots; \boldsymbol{y}_m, t_m)\mathrm{d}\boldsymbol{y}_1 \dots \mathrm{d}\boldsymbol{y}_m, \qquad \int \cdots \int P^{(m)}\mathrm{d}\boldsymbol{y}_1 \dots \mathrm{d}\boldsymbol{y}_m = 1, \tag{113}$$

is given by a multivariate Gaussian law (see Appendix E)

$$P^{(m)}(\boldsymbol{y}_1, t_1; \dots; \boldsymbol{y}_m, t_m) = [\det \mathscr{B}]^{1/2}(2\pi)^{-Mm/2}\exp(-\tfrac{1}{2}\boldsymbol{Y}\mathscr{B}^{-1}\boldsymbol{Y}^+). \tag{114}$$

$$\boldsymbol{Y} = [\boldsymbol{y}_u]_{u=1,\dots,m}; \quad \mathscr{B} = [\langle\langle \boldsymbol{y}^+(t_{u_1})\boldsymbol{y}(t_{u_2})\rangle\rangle]_{u_1u_2=1,\dots,m}. \tag{115}$$

The generalized fractional Brownian motion considered here is a multivariable analogue of the one-dimensional FBM model, which is recovered as a particular case for $M = 1$. Other important particular situations include the multidimensional classical Brownian motion which results in the limit

$$\forall H_{uu'} \to 1, \tag{116}$$

and a random process with infinite memory for which

$$\forall H_{uu'} = 0. \tag{117}$$

Concerning the ergodic behavior of the random vector $\boldsymbol{x} = \boldsymbol{x}(t)$ we notice that for the generalized FBM model considered here there is a proportionality relationship between the time average

$$\widetilde{\boldsymbol{x}(t')}(t) = t^{-1}\int_0^t \boldsymbol{x}(t')\mathrm{d}t', \tag{118}$$

and the random vector $\boldsymbol{y}(t)$ which is similar to the relationship (83) derived in Section 4 in the case of DNA walks

$$\boldsymbol{y}(t) = t\widetilde{\boldsymbol{x}(t')}(t). \tag{119}$$

Since the random process describing the behavior of the random vector $y(t)$ is Gaussian and with zero average values, the possible existence of the ergodic behavior can be investigated by computing the cumulants of the time average of the second order which are the only cumulants different from zero. By using the method developed in Section 2 we obtain

$$\langle\langle \widetilde{x_{u_1}(t')}\ \widetilde{x_{u_2}(t')} \rangle\rangle(t) = A_{u_1 u_2}(2 - H_{u_1 u_2})^{-1} t^{-H_{u_1 u_2}} + A_{u_2 u_1}(2 - H_{u_2 u_1})^{-1} t^{-H_{u_2 u_1}} . \tag{120}$$

Note that for $H_{uu'} \neq 0$ all these cumulants tend to zero as $t \to \infty$ and thus the ergodic property holds. If at least one fractal exponent $H_{uu'} \neq 0$ and $A_{uu'} \neq 0$ then at least one of the cumulants (120) tends towards a value different from zero as $t \to \infty$ and the ergodic property does not hold anymore.

6. Conclusions

The investigation of the conditions of existence of the ergodic behavior is of importance both from the theoretical and practical point of view. The ergodicity is an important feature of a stochastic process; if it exists it simplifies the evaluation of the average values of the state functions depending on the stochastic variables. Although the investigation of ergodicity of stochastic processes is an active field of mathematical research little attention has been paid to this problem by the physicists. In this paper we have suggested a physically oriented approach for the study of the conditions of existence of ergodic behavior for a stationary fractal random process; our method is based on the observation that for a finite time interval the time average of a state function is random and that the transition from the non-ergodic to the ergodic behavior corresponds to the passage from the random to the deterministic behavior of the time averages. For investigating the transition from the random to the deterministic behavior a systematic approach has been suggested based on the evaluation of the cumulants of the time average. By using this technique a number of examples of biological and physical interest have been investigated namely the DNA walks describing the long range correlations of nucleotide sequences, the stochastic cellular automata with long memory and a multivariable generalization of Mandelbrot's fractional Brownian motion. The analysis of the examples presented in this paper shows that for stationary fractal random processes with long memory a violation of the ergodic behavior may occur provided that long-range correlations different from zero exist for large time differences.

Our approach outlines a fundamental difference between two different types of fractal random processes which are often confounded in the literature. The type of fractal random processes considered here is characterized by finite moments and by probability densities with short tails. These processes display self-similar features of the fractal time type; the statistical ensemble cumulants of the process have long tails for large time differences of the inverse power law type. For the second kind of fractal processes, which is not considered here, the probability densities are of the Lévy type being characterized by infinite moments and by long tails for large values of the random variables. The problem of ergodicity is properly defined only for the fractal random processes belonging to the first type. For the second type of fractal random processes, due to the wild fluctuations of the random variables, the ensemble average cumulants do not exist and the notion of ergodicity has no physical meaning.

Further research should concentrate on the topic of establishing a connection between the physically oriented approach presented here and the mathematical analysis of the same problem presented in the literature of probability theory [8, 9].

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Appendix A

For proving Eq. (30) we use the integral identity

$$\int_0^t \cdots \int_0^t F(t_1, \ldots, t_m) \mathrm{d}t_1 \cdots \mathrm{d}t_m = m! \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{m-1}} \mathrm{d}t_m F(t_1, \ldots, t_m), \tag{A.1}$$

which is valid for any function $F(t_1, \ldots, t_m)$ which is symmetric with respect to any permutation of the integration variables $t_1, \ldots, t_m$. We denote by l the subscript of the smallest of the time variables $t_1, \ldots, t_m$:

$$t_l = t_m^* = \min(t_1, \ldots, t_m), \tag{A.2}$$

and introduce the integration variables

$$\tau_u = t_u - t_m^*, \quad u = 1, \ldots, l-1, \qquad \tau_{v-1} = t_v - t_m^*, \qquad v = l+1, \ldots, m. \tag{A.3}$$

By combining Eqs. (21) and (29), taking into account that $C_m(\tau_1, \ldots, \tau_{m-1})$ is symmetrical with respect to $t_1, \ldots, t_m$ and using the integral identity (A.1) we obtain

$$\langle\langle [\widetilde{f[x(t')]}]^m \rangle\rangle(t) = \frac{1}{t^m} m! \int_0^t \mathrm{d}a\, J(t-a), \tag{A.4}$$

$$J(t)=\int_0^t \mathrm{d}\tau_1 \int_0^{\tau_1} \mathrm{d}\tau_2 \cdots \int_0^{\tau_{m-2}} \mathrm{d}\tau_{m-1}\, C_m(\tau_1, \ldots, \tau_{m-1}) . \tag{A.5}$$

By using again the integral identity (A.1), now for the function $J(t)$, for which the integrand $C_m(\tau_1, \ldots, \tau_{m-1})$ is symmetric with respect to $\tau_1, \ldots, \tau_{m-1}$, we have

$$J(t)=\frac{1}{(m-1)!}\int_0^t \mathrm{d}\tau_1 \cdots \int_0^t \mathrm{d}\tau_{m-1}\, C_m(\tau_1, \ldots, \tau_{m-1}) . \tag{A.6}$$

Inserting Eq. (A.6) into Eq. (A.4) we come to Eq. (30).

Appendix B

Following Refs. [11, 18, 25] we consider the shot noise generated by a Poissonian distribution of point events occuring with a constant frequency Ω, each event having an exponentially decreasing effect characterized by the contribution

$$c\,\varphi(t-t') , \tag{B.1}$$

$$\begin{aligned} \varphi(t-t') &= \nu \exp[-\nu(t-t')], && \text{for } t \geqslant t' , \\ &= 0, && \text{for } t < t' , \end{aligned} \tag{B.2}$$

where t' is the time of occurrence of the event, t is the current time, ν is the frequency of the decay of the effect and c is a random amplitude factor selected from a given probability density

$$p(c)\mathrm{d}c, \qquad \int p(c)\mathrm{d}c = 1 . \tag{B.3}$$

A realization of the random state function f at time t is given by

$$f(t) = \sum_{u=1}^{N} c_u\, \varphi(t-t'_u) , \tag{B.4}$$

where N is the total number of events, $t'_1, \ldots, t'_N$ are the times of occurrence of these events and $c_1, \ldots, c_N$ are the realizations of the amplitude factors corresponding to the different events.

Considering a large time interval of length T the probability $P(N;T)$ of occurrence of N events is given by the Poissonian law

$$P(N;T) = [\Omega T]^N (N!)^{-1} \exp(-\Omega T) . \tag{B.5}$$

We introduce the characteristic functional

$$G_T[\omega(t');t] = \left\langle \exp\left(\mathrm{i}\int f(t')\omega(t')\mathrm{d}t'\right)\right\rangle, \quad t \leqslant T, \tag{B.6}$$

of the random function $f(t)$ corresponding to the time interval of length T. The characteristic functional $G[\omega(t');t]$ given by Eq. (10) corresponds to the limit $T \to \infty$:

$$G[\omega(t');t] = \lim_{T\to\infty} G_T[\omega(t');t]. \tag{B.7}$$

The average in Eq. (B.6) can be easily evaluated by taking into account the distributions of all possible values of N, $t'_1, \dots, t'_N$ and $c_1, \dots, c_N$. We have

$$\begin{aligned} G_T[\omega(t');t] &= \sum_N \frac{(\Omega T)^N}{N!}\exp(-\Omega T)\int \mathrm{d}c_1 \cdots \int \mathrm{d}c_N \int_0^T \frac{\mathrm{d}t'_1}{T} \cdots \int_0^T \frac{\mathrm{d}t'_N}{T} \\ &\quad \times \exp\left(\mathrm{i}\sum_{n=1}^{N}\int c_n \varphi(t-t'_n)\omega(t)\mathrm{d}t\right) p(c_1)\cdots p(c_N) \\ &= \exp\left\{-\int_0^T \mathrm{d}t' \int p(c)\mathrm{d}c\left[1-\exp\left(\mathrm{i}c\int_{t'}^{t}\varphi(t-t')\omega(t)\mathrm{d}t\right)\right]\right\}. \end{aligned} \tag{B.8}$$

By passing to the limit $T \to \infty$ we get an explicit expression for the characteristic functional $G[\omega(t');t]$ which has the same form as Eq. (B.8) with the difference that the upper integration limit for t' is $t' = \infty$.

From Eq. (12) it follows that the cumulants $\langle\langle f(t_1)\cdots f(t_m)\rangle\rangle$ of the state function $f(t)$ can be evaluated by computing the functional derivatives

$$\langle\langle f(t_1)\cdots f(t_m)\rangle\rangle = (-\mathrm{i})^m \frac{\delta^m \ln G[\omega(t')=0;t]}{[\delta\omega(t_1)\cdots\delta\omega(t_m)]}. \tag{B.9}$$

By evaluating these derivatives we obtain (for a similar computation see Ref. [25]):

$$\langle\langle f(t_1)\cdots f(t_m)\rangle\rangle = \Omega\langle c^m\rangle \int_0^{t_m^*} \varphi(t_1-t')\cdots\varphi(t_m-t')\mathrm{d}t', \tag{B.10}$$

as $\forall t_u \to \infty$, $u = 1, \dots, m$ with $\tau_u = t_u - t_m^* = \text{constant}$ $u = 1, \dots, m$. By inserting Eq. (B.2) into Eq. (B.10), computing the integral over t' and evaluating the limit we have

$$\langle\langle f(t_1)\cdots f(t_m)\rangle\rangle = \frac{1}{m}\Omega\langle c^m\rangle \prod_{u \neq l}^{m} [\nu\exp[-\nu(t_u - t_m^*)]], \tag{B.11}$$

which has the same form as Eq. (33) where the factor A_m is given by

$$A_m = \Omega\langle c^m\rangle/m\,, \tag{B.12}$$

and where

$$\langle c^m\rangle = \int c^m p(c)\,\mathrm{d}c\,, \tag{B.13}$$

are the moments of the amplitude factor c.

Appendix C

We start out from a stationary random process with short memory characterized by the statistical ensemble cumulants $C_m(\tau_1, \ldots, \tau_{m-1})$ which have short tails for large values of the time differences $\tau_1, \ldots, \tau_{m-1}$; we assume that as $\tau_u \to \infty$ these tails decrease exponentially or even faster towards the asymptotic value zero. Following Shlesinger and Hughes [19] and Vlad [20] we apply to these cumulants a succession of scale transformations of the renormalization group type, each time difference $\tau_1, \ldots, \tau_{m-1}$ being treated independently

$$\begin{aligned} \tilde{C}_m(\tau_1, \ldots, \tau_{m-1}) = \sum_{q_1=0}^{\infty} \cdots \sum_{q_{m-1}=0}^{\infty} \prod_{u=1}^{m-1} [(1-\lambda_u)(\lambda_u)^{q_u}] \\ \times C_m(\tau_1(b_1)^{-q_1}, \ldots, \tau_{m-1}(b_{m-1})^{-q_{m-1}}), \\ b_u \geqslant 1,\ 1 \geqslant \lambda_u \geqslant 0,\ u = 1, \ldots, m-1\,, \end{aligned} \tag{C.1}$$

where $\tilde{C}_m(\tau_1, \ldots, \tau_{m-1})$ are the renormalized cumulants, $b_1, \ldots, b_{m-1} \geqslant 1$ are characteristic multiplicative scaling factors attached to the different time differences $\tau_1, \ldots, \tau_{m-1}$ and $1 \geqslant \lambda_u \geqslant 0, u = 1, \ldots, m-1$ are the probabilities that a scaling step takes place for each of the time differences $\tau_1, \ldots, \tau_{m-1}$, respectively. Eq. (C.1) has a structure typical for a stochastic renormalization group equation [19–20], which generates negative power law tails in $\tau_1, \ldots, \tau_{m-1}$ for the renormalized cumulants $\tilde{C}_m(\tau_1, \ldots, \tau_{m-1})$, characterized by the fractal exponents

$$H_u = \ln(1/\lambda_u)/\ln b_u, \qquad u = 1, \ldots, m-1\,, \tag{C.2}$$

modulated by logarithmic oscillations in $\ln\tau_1, \ldots, \ln\tau_{m-1}$ with the periods $\ln b_1, \ldots, \ln b_{m-1}$, respectively. In order to get rid of the logarithmic oscillations in $\ln\tau_1, \ldots, \ln\tau_{m-1}$ we introduce the limit

$$b_u \searrow 1, \quad \lambda_u \nearrow 1, \qquad H_u = \text{constant}, \quad u = 1, \ldots, m-1\,. \tag{C.3}$$

This limit has been introduced for one variable systems in Ref. [26]; it leads to the vanishing of the logarithmic oscillations even though the long tails of the negative power law type are left unchanged.

By using the technique developed in Ref. [26] in the limit (C.3) we can derive a partial differential equation for the renormalized cumulants $\tilde{C}_m(\tau_1, \dots, \tau_{m-1})$:

$$\prod_{u=1}^{m-1}\left(\frac{\tau_u}{H_m}\frac{\partial}{\partial \tau_u} + 1\right)\tilde{C}_m(\tau_1, \dots, \tau_{m-1}) = C_m(\tau_1, \dots, \tau_{m-1}), \quad m = 2, 3, \dots \tag{C.4}$$

In Eqs. (C.4) we have taken into account that, due to the symmetry of the nonrenormalized and the renormalized cumulants with respect to any permutation of the time differences $\tau_1, \dots, \tau_{m-1}$, the fractal exponents H_u should be independent of the label u of the time differences; however for cumulants of different orders the corresponding exponents may have different values, $H_2, \dots, H_m, \dots$:

$$H = H_m, \qquad m = 1, 2, \dots \tag{C.5}$$

For solving Eqs. (C.4) we consider the boundary conditions

$$\tilde{C}_m(\tau_1, \dots, \tau_{m-1}) \to C_m(\tau_1, \dots, \tau_{m-1}) \qquad \text{as } H_m \to \infty, \quad m = 2, 3, \dots . \tag{C.6}$$

which express the fact that for very large fractal exponents, $H_m \to \infty$, the numbers of scaling steps tend to zero and thus the renormalization transformation does not lead to a change of the expressions of the cumulants. The corresponding solutions of Eqs. (C.4) are:

$$\tilde{C}_m(\tau_1, \dots, \tau_{m-1}) = \left(\prod_u \tau_u\right)^{-H_m} (H_m)^{m-1} \int_0^{\tau_1} \cdots \int_0^{\tau_{m-1}} \left(\prod_u a_u\right)^{H_m - 1} \times C_m(a_1, \dots, a_{m-1})\, \mathrm{d}a_1, \dots, \mathrm{d}a_{m-1} \, . \tag{C.7}$$

Note that the tails of the renormalized cumulants $\tilde{C}_m(\tau_1, \dots, \tau_{m-1})$ given by Eqs. (C.7) have the same form as Eqs. (42) where the constants B_m are given by

$$B_1 = C_1, \qquad B_m = (H_m)^{m-1} \int_0^{\infty} \cdots \int_0^{\infty} \left(\prod_u a_u\right)^{H_m - 1} \times C_m(a_1, \dots, a_{m-1})\, \mathrm{d}a_1, \dots, \mathrm{d}a_{m-1}, \quad m = 2, 3, \dots . \tag{C.8}$$

The convergence of the integrals in Eqs. (C.8) is ensured by the fact that as $a_1, \dots, a_{m-1} \to \infty$ the tails of the non-renormalized cumulants decrease exponentially or faster to zero.

Appendix D

For computing the cumulants of the sojourn times $t_1, \dots, t_M$ for independent and discrete random processes with M states we introduce the multitime characteristic function

$$\Xi(k_1, \dots, k_M; t) = \left\langle \exp\left(\mathrm{i} \sum_{u=1}^{M} k_u t_u \right) \right\rangle . \tag{D.1}$$

which is the discrete analogue of the characteristic functional $\Xi[K(f', t'); t]$ of the density of states $\eta(f, t)$ defined by Eq. (11). For independent random processes this characteristic function can be easily evaluated by computing the average in Eq. (D.1) over all possible values of the intermediate states of $u_1, \dots, u_t$ of the system:

$$\begin{aligned} \Xi(k_1, \dots, k_M; t) &= \sum_{u_1=1}^{M} \cdots \sum_{u_t=1}^{M} P(u_1) \cdots P(u_t) \exp\left(\mathrm{i} \sum_{u=1}^{M} k_u \sum_{t'=1}^{t} \delta_{(u)(u_{t'})} \right) \\ &= \left(\sum_{u=1}^{M} P(u) \exp(\mathrm{i} k_u) \right)^t = \exp\left[t \ln\left(1 + \sum_{u=1}^{M} P(u) [\exp(\mathrm{i} k_u) - 1] \right) \right] . \end{aligned} \tag{D.2}$$

We expand the characteristic function $\Xi(k_1, \dots, k_M; t)$ in a cumulant series

$$\Xi(k_1, \dots, k_M; t) = \exp\left(\sum_{m=1}^{\infty} \frac{(\mathrm{i})^m}{m!} \sum_{u_1=1}^{M} \cdots \sum_{u_m=1}^{M} k_{u_1} \cdots k_{u_m} \langle\langle t_{u_1} \cdots t_{u_m} \rangle\rangle (t) \right), \tag{D.3}$$

and perform a similar expansion in Eq. (D.2) by expressing the logarithm in a Taylor series. By comparing the resulting equation a with Eq. (D.3) we get Eqs. (67).

Appendix E

In this section we derive the main stochastic properties of the random vector $\boldsymbol{y}(t)$ defined by Eq. (109). We notice that as the random vector $\boldsymbol{x}(t)$ is Gaussian and with average zero its characteristic functional is given by:

$$G[\boldsymbol{W}(t'); t] = \left\langle \exp\left(\mathrm{i} \int \boldsymbol{W}(t') \cdot \boldsymbol{x}(t') \, \mathrm{d}t' \right) \right\rangle \tag{E.1}$$

$$= \exp\left(-\frac{1}{2} \int_0^t \int_0^t \boldsymbol{W}^+(t_1) \boldsymbol{C}(t_1, t_2) \boldsymbol{W}(t_2) \, \mathrm{d}t_1 \, \mathrm{d}t_2 \right) . \tag{E.2}$$

As, according to Eq. (109), $\boldsymbol{y}(t)$ is a linear functional of $\boldsymbol{x}(t)$, the random vector $\boldsymbol{y}$ is also Gaussian; its mean value can be evaluated by direct averaging of Eq. (109), resulting in $\langle \boldsymbol{y}(t) \rangle = \boldsymbol{0}$. It follows that the characteristic functional $\mathscr{F}[\boldsymbol{K}(t')]$ of the random vector $\boldsymbol{y}(t)$ has the form (110). For computing the correlation functions $\langle\langle y_{u_1}(t_1) y_{u_2}(t_2) \rangle\rangle$ we insert Eq. (109) into Eq. (110) and express the resulting equation in terms of the characteristic functional $G[W(t'); \infty]$. We obtain

$$\mathscr{F}[K(t')] = \exp\left(-\frac{1}{2}\sum\sum\iint\limits_{0}^{\infty}\int\limits_{0}^{t_1}\int\limits_{0}^{t_2} C_{u_1u_2}(t_1', t_2')\,\mathrm{d}t_1'\,\mathrm{d}t_2'\,K_{u_1}(t_1)K_{u_2}(t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2\right). \tag{E.3}$$

By comparing Eq. (110) with Eq. (E.3) we come to

$$\langle\langle y_{u_1}(t_1)y_{u_2}(t_2)\rangle\rangle = \int\limits_{0}^{t_1}\int\limits_{0}^{t_2} C_{u_1u_2}(t_1', t_2')\,\mathrm{d}t_1'\,\mathrm{d}t_2'\,. \tag{E.4}$$

Taking into account the condition of stationarity (107) of the random vector $x(t)$ after some changes of variables and rearrangements of the integrals Eq. (E.4) can be rewritten in the form (111).

The multiple Fourier transform of the joint probability density $P^{(m)}(y_1, t_1; \dots; y_m, t_m)$:

$$\bar{P}^{(m)}(k_1, t_1; \dots; k_m, t_m) = \int\cdots\int\exp\left(\mathrm{i}\sum_{u=1}^{m} k_u \cdot y_u\right)P^{(m)}(y_1, t_1; \dots; y_m, t_m)\,\mathrm{d}y_1\cdots\mathrm{d}y_m, \tag{E.5}$$

can be expressed in terms of the characteristic functional $\mathscr{F}[K(t')]$:

$$\bar{P}^{(m)}(k_1, t_1; \dots; k_m, t_m) = \mathscr{F}\left(K(t') = \sum_{u=1}^{m} k_u\,\delta(t' - t_u)\right). \tag{E.6}$$

The joint probability density $P^{(m)}(y_1, t_1; \dots; y_m, t_m)$ can be computed from Eq. (E.6) by means of an inverse Fourier transformation

$$P^{(m)}(y_1, t_1; \dots; y_m, t_m) = (2\pi)^{-mM}\int\cdots\int\mathscr{F}\left(K(t') = \sum_{u=1}^{m} k_u\,\delta(t' - t_u)\right)\mathrm{d}k_1\cdots\mathrm{d}k_m\,. \tag{E.7}$$

By inserting Eq. (110) into Eq. (E.7) we get a multidimensional Gaussian integral over $k_1, \dots, k_m$. By computing this multidimensional integral after lengthy manipulation we obtain Eq. (114).

Note added in proof

The different definitions for FBM suggested in the literature are not equivalent to each other, even though they all lead to fractal time scaling. According to its original definition [22, 23] FBM is a linear integral transformation of the usual Brownian motion. Other definitions relate the FBM to the white noise by means of different linear functional transformations, such as the fractional integral (Maccone, Nuovo

Cimento B61(1981)229; B65(1981)259; Cherayil and Biswas, J. Chem. Phys. 99(1993)9230). Studies of DNA statistics [14–16] have suggested a definition of the FBM as the first integral of a non-Markovian Gaussian noise with long memory. The generalization suggested in Section 5 is based on this last definition. Recently Sebastian (J. Phys. A Math. Gen. 28(1995)4305 has pointed out some differences between two definitions of FBM. Further research would extend Sebastian's analysis to the model discussed in Section 5.

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