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Industrial replacement, communication networks and fractal time statistics

Marcel Ovidiu Vlad[1], Michael C. Mackey

Centre for Nonlinear Dynamics in Physiology and Medicine, McGill University, 3655 Drummond Street, Montreal, Quebec Canada H3G 1Y6

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Abstract

Three models for the fractal time statistics of renewal processes are suggested. The first two models are related to the industrial replacement. A model assumes that the state of an industrial aggregate is described by a continuous positive variable X, which is a measure of its complexity. The failure probability exponentially decreases as the complexity of the aggregate increases. A renewal process is constructed by assuming that after the occurrence of a breakdown event the defective aggregate is replaced by a new aggregate whose complexity is a random variable selected from an exponential probability law. We show that the probability density of the lifetime of an aggregate has a long tail $\psi(t) \sim t^{-(1+H)}$ as $t \to \infty$ where the fractal exponent H is the ratio between the average complexity of an aggregate which leaves the system and the average complexity of a new aggregate. The asymptotic behavior of all moments of the number N of replacement events occurring in a large time interval may be evaluated analytically. For $1 > H > 0$ the mean and the dispersion of N behave as $\langle N(t)\rangle \sim t^{H}$ and $\langle \Delta N^2(t)\rangle \sim t^{2H}$ as $t \to \infty$ which outlines the intermittent character of the fluctuations.

A second model gives a discrete description of industrial replacement. The aggregates are assumed to be made up of variable numbers of basic units. Each basic unit has a probability α to be associated in an aggregate and a probability β of being in an active state. An aggregate can work if at least a basic unit is in an active state. The mechanism of replacement is the same as in the first model, the number of basic units from an aggregate playing the role of a complexity measure. The probability density of the lifetime has a long tail modulated by a periodic function in $\ln t$: $\psi(t) \sim t^{-(1+H)}\Xi(\ln t)$, where $H = \ln\alpha/\ln(1-\beta)$ and $\Xi(\ln t)$ is a periodic function of $\ln t$ with a period $-\ln(1-\beta)$.

A third model is related to the transmission errors in communication networks. A network is made up of a large number of communication channels; each channel has a probability α to be open and a probability β of transmitting a message. The number of open channels is a random variable which is kept constant as far as the transmission is possible; if a failure occurs, then the number of open channels is changed in a random way. We show that this model is

[1] Permanent address: Romanian Academy of Sciences, Centre for Mathematical Statistics, Casa Academiei, Calea 13 Septembrie No 13, OPB Bucuresti 5, 76100 Bucharest, Romania. Current address after 1 January 1996: Department of Chemistry, Stanford University, Stanford, CA 94305-5080, USA.

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approximately isomorphic with the second one. The probability density of the time between two successive errors has also an inverse power tail modulated by a periodic function in $\ln t$. The general implications of these models for the physics of fractal time are analysed.

1. Introduction

The term "fractal time" was introduced thirty years ago by Berger and Mandelbrot [1] in connection with the transmission errors in telephone networks, a problem which can be formulated by using the theory of renewal processes. However, most published papers deal with the application of fractal time statistics in physics and chemistry (see Shlesinger [2] and references therein). As far as we know no simple physical model for the occurrence of fractal time in renewal processes has been suggested yet.

Recently we have pointed out some relationships between the stochastic theory of branched chain processes and the fractal time statistics (Vlad [3,4]). These relationships have been analysed in an abstract way, without reference to a concrete physical model. In an attempt to illustrate our approach we have succeeded to come up with some models for the possible occurrence of fractal time statistics in the dynamics of renewal processes. The equations attached to these models are somewhat similar to the ones derived by using the branched chain approach. From the physical point of view, however, this similarity is rather superficial: no branched chain description is needed in the context of renewal processes. Although these models are not appropriate examples of the branched chain formalism, their analysis is however useful. From their study we can draw some general conclusions concerning the connections between the fractal time statistics and the behavior of complex systems.

The plan of the paper is as follows. In Section 2 we rephrase some aspects of the renewal theory in a language which is familiar to statistical physicists. In Section 3 a continuous description of the process of industrial replacement is suggested. A more detailed description of this process is given in Section 4, resulting in a discrete model. In Section 5 a discrete model for the statistics of transmission errors in communication networks is developed. Finally in Section 6 a comparison among the three models is performed and some general conclusions are drawn concerning the mechanism of occurrence of fractal time in complex physical and engineering systems.

2. Renewal theory as a continuous time random walk

Although the renewal theory has been already used in statistical physics (Moffat [5]), it is an approach less known to physicists. That is why we reformulate some basic aspects of this theory in terms of an approach commonly used in statistical physics, the continuous time random walk method (CTRW, Montroll and Weiss [6], Haus and Kehr [7]).

Although a part of the applied probability theory (Cox [8], Feller [9]) the study of renewal processes was initially developed in connection with the replacement of industrial aggregates and with population dynamics (Lotka [10] and references therein).

We consider an industrial aggregate which performs a certain task. Its lifetime t is a random variable which obeys a certain probability law

$$\psi(t)\,\mathrm{d}t, \quad \int_0^\infty \psi(t)\,\mathrm{d}t = 1\,. \tag{1}$$

After a breakdown event the defective aggregate is replaced by a new one. The new aggregate has a lifetime selected from the same probability law (1). We address the following question: given $\psi(t)$ which are the stochastic properties of the number N of replacement events which occur in a large time interval of length t?. We are interested mainly in the asymptotic behavior so that we neglect the complications related to the possibility that the stochastic properties of the lifetime of the first aggregate are different from the stochastic properties of the lifetimes of the following aggregates.

For a physicist this problem is equivalent to a directed CTRW in one dimension. In terms of $\psi(t)$ we introduce the probability

$$l(t) = \int_t^\infty \psi(t)\,\mathrm{d}t\,, \tag{2}$$

that in the time interval from $t = 0$ to $t = t$ no replacements have occurred. The probability $P_N(t)$ that from $t = 0$ to $t = t$, N replacement events have taken place is a multiple convolution product of $\psi(t)$ and $l(t)$:

$$P_N(t) = [\psi(t) \otimes]^N \otimes l(t)\,, \tag{3}$$

where $\otimes$ denotes the temporal convolution product. By introducing the Laplace transforms

$$\bar{P}_N(s) = \mathscr{L} P_N(t) = \int_0^\infty \exp(-\,\mathrm{st}) P_N(t)\,\mathrm{d}t, \qquad \text{etc.}, \tag{4}$$

where s is the Laplace variable, and $\mathscr{L}$ and the overbar denote the Laplace transformation, we have

$$\bar{P}_N(s) = s^{-1}(1 - \bar{\psi}(s))\bar{\psi}^N(s)\,. \tag{5}$$

Eq. (5) can be used to evaluate, at least in principle, all moments of the random variable N. We introduce the generating function

$$G(z,t) = \sum z^N P_N(t), \quad |z| \leqslant 1; \qquad \bar{G}(z,s) = \mathscr{L} G(z,t)\,. \tag{6}$$

Eq. (5) leads to

$$\bar{G}(z,s) = s^{-1}[1 - \bar{\psi}(s)][1 - z\bar{\psi}(s)]^{-1}. \tag{7}$$

The Laplace transforms of the factorial moments of N may be computed from Eq. (7) by taking the derivatives of $\bar{G}(z,s)$ for $z = 1$. The behavior of the moments in the time domain depends on the function $\psi(t)$. In the renewal theory one usually assumes that all moments of the lifetime exist and are finite. As a result $\bar{\psi}(s)$ is analytic in s around $s = 0$:

$$\bar{\psi}(s) = 1 + \sum_{k=1}^{\infty} (-1)^k (k!)^{-1} \langle \tau^k \rangle s^k, \tag{8}$$

where

$$\langle \tau^k \rangle = \int_0^{\infty} \tau^k \psi(\tau)\, \mathrm{d}\tau, \tag{9}$$

are the moments of the lifetime. By using Eqs. (7), (8) we get the following asymptotic expressions for the mean and dispersion of N as $t \to \infty$:

$$\langle N(t) \rangle \cong t/\langle \tau \rangle \quad \text{as } t \to \infty, \tag{10}$$

$$\langle \Delta N^2(t) \rangle \cong [\langle \Delta\tau^2 \rangle \langle \tau \rangle^{-3}]t \quad \text{as } t \to \infty, \tag{11}$$

where

$$\langle \Delta\tau^2 \rangle = \langle \tau^2 \rangle - \langle \tau \rangle^2, \tag{12}$$

is the dispersion of the lifetime. In this case the relative fluctuation of the number of replacement events tends to zero as $t \to \infty$:

$$\frac{\langle \Delta N^2(t) \rangle^{1/2}}{\langle N(t) \rangle} \cong \left(\frac{\langle \Delta\tau^2 \rangle}{\langle \tau \rangle} \right)^{1/2} t^{-1/2} \to 0 \quad \text{as } t \to \infty. \tag{13}$$

In this limit the fluctuations of N are negligible and the mean number of replacement events increases linearly in time.

The same analysis applies to the transmission errors in communication networks. If the probability density of the time between two failures has finite moments then as $t \to \infty$ the number of errors increases linearly in time and the relative fluctuation becomes negligible.

By making an analogy with the CTRW theory (Haus and Kehr [7]) it follows that a fractal time behavior may occur if the moments $\langle \tau^k \rangle$, $k = 1,2, \ldots$ are infinite. In the following sections we shall present some models which may lead to this situation.

3. A continuous approach to industrial replacement

We assume that each aggregate is characterised by a continuous and positive variable X which is a measure of its complexity. During the lifetime of an aggregate two types of breakdown events may occur: "non-lethal" and "lethal" ones. If a "non-lethal" breakdown occurs then the aggregate is mended in a time scale which is much smaller than the characteristic time t in which the replacement events are counted; if a "lethal" breakdown occurs the aggregate cannot be longer mended and it is replaced by a new one.

We denote by $\mathscr{E}(X)$ the probability that an aggregate of complexity X survives a breakdown event (i.e. that a breakdown event is "non lethal"). We introduce the rate $\rho\Delta X$ of the occurrence of a lethal defect for an aggregate with a complexity between X and $X + \Delta X$. We can write the probability balance equation

$$\mathscr{E}(X + \Delta X) = \mathscr{E}(X)[1 - \rho\Delta X], \tag{14}$$

from which for $\Delta X \to 0$ we get the differential equation

$$\partial\mathscr{E}(X)/\partial X = -\rho\mathscr{E}(X). \tag{15}$$

We assume that the "lethal" breakdown events are entirely due to chance, i.e. that the rate $\rho\Delta X$ is independent of the complexity. We choose a complexity scale for which a complexity equal to zero corresponds to a probability of lethal breakdown equal to unity

$$\mathscr{E}(0) = 1. \tag{16}$$

By integrating Eq. (15) we come to

$$\mathscr{E}(X) = \exp(-\rho X). \tag{17}$$

From Eq. (17) it follows that for infinite complexity the rate of lethal breakdown is equal to zero. The function

$$\eta_b(X)\,dX = -[\partial\mathscr{E}(X)/\partial X]\,dX = \rho\exp(-\rho X)\,dX, \tag{18}$$

is the probability density of the complexity of the aggregates which leave the system as a result of a "lethal" breakdown. The corresponding mean complexity $\langle X\rangle_b$ is equal to

$$\langle X\rangle_b = \int X\eta_b(X)\,dX = 1/\rho, \tag{19}$$

and thus the probability $\mathscr{E}(X)$ can be expressed as

$$\mathscr{E}(X) = \exp(-X/\langle X\rangle_b). \tag{20}$$

The probability $\phi(m)$ that an aggregate is replaced after m breakdown events is equal to:

$$\phi(m) = \int \eta_n(X)[1 - \mathscr{E}(X)]^{m-1}\mathscr{E}(X)\,dX \tag{21}$$

where

$$\eta_n(X)\,dX, \qquad \int \eta_n(X)\,dX = 1\,, \tag{22}$$

is the probability density of the complexity of a new aggregate. Eq. (21) expresses the fact that the first $m-1$ breakdown events should be "non-lethal", which corresponds to a probability $[1-\mathscr{E}(X)]^{m-1}$ and that the last one should be lethal, which corresponds to a probability $\mathscr{E}(X)$. Finally an average with respect to the possible values of the complexity measure of a new aggregate is done. To elucidate the conditions in which Eq. (21) may lead to a statistical fractal we evaluate the average number of breakdown events at the moment of replacement:

$$\langle m\rangle = \sum m\phi(m) = \int \eta_n(X)\langle m\rangle_X\,dX\,, \tag{23}$$

where

$$\langle m\rangle_X = \sum m[1-\mathscr{E}(X)]^{m-1}\mathscr{E}(X) = \exp[X/\langle X\rangle_b]\,, \tag{24}$$

is the average number of breakdown events for an aggregate of complexity X. The average number of breakdowns exponentially increases with the increase of the complexity. A statistical fractal may be realised by means of a compensation mechanism: although possible, the very complex aggregates should be very rare. The simplest choice of the function $\eta_n(X)\,dX$ which fulfills this condition corresponds to an exponential law similar to Eq. (18):

$$\eta_n(X)\,dX = (\langle X\rangle_n)^{-1}\exp[-X/\langle X\rangle_n]\,dX\,, \tag{25}$$

where the corresponding mean value $\langle X\rangle_n$ is generally different from $\langle X\rangle_b$. At first sight it would seem that the choice of the probability law (25) is somewhat arbitrary; however in the following section we shall see that it may be justified as a consequence of the way in which the aggregates may be generated.

Now the function $\phi(m)$ can be evaluated exactly. By inserting Eqs. (20) and (25) into Eq. (21) we get:

$$\phi(m) = HB(m, H+1) = H(m-1)!/[(H+1)\ldots(H+m)]\,, \tag{26}$$

where H is the ratio between the average complexity $\langle X\rangle_b$ of an aggregate at the moment of replacement and the average complexity $\langle X\rangle_n$ of a new aggregate;

$$H = \langle X\rangle_b/\langle X\rangle_n\,, \tag{27}$$

and $B(p,q)$ is the complete beta function

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \qquad p,q > 0\,. \tag{28}$$

The factorial moments corresponding to the probability $\phi(m)$ given by Eq. (26) are equal to

$$\langle m(m-1)\cdots(m-l+1)\rangle$$

$$= [(l-1)!]^2 lH \sum_{k=0}^{l-1} (-1)^k$$

$$\times [k!(l-k-1)!(H-l-k)]^{-1}, \quad \text{for } l < H, \tag{29.a}$$

$$= \infty, \qquad \text{for } l \geqslant H. \tag{29.b}$$

Thus the behavior of the number of breakdown events is determined by the ratio between the two average complexities. If $(l+1)\langle X\rangle_{\mathrm{n}} \geqslant \langle X\rangle_{\mathrm{b}} > l\langle X\rangle_{\mathrm{n}}$ only the first l moments are finite. In particular, for $\langle X\rangle_{\mathrm{n}} \geqslant \langle X\rangle_{\mathrm{b}} > 0$, i.e. for $1 > H > 0$ all moments are infinite. The physical significance of this fact is clear. From Eq. (21) it turns out that the factorial moments of m can be expressed as

$$\langle m(m-1)\cdots(m-l+1)\rangle$$

$$= l! \int \eta_{\mathrm{n}}(X)[\mathscr{E}(X)]^{l-1}[1-\mathscr{E}(X)]^{-l}\,\mathrm{d}X. \tag{30}$$

We see that as $\mathscr{E}(X) \to 1$ the moments are infinite. If $\langle X\rangle_{\mathrm{b}} > \langle X\rangle_{\mathrm{n}}$, $\mathscr{E}(X)$ increases towards its asymptotic value 1 and the very large values of the complexity have only a small contribution to the moments; in this case only the superior moments are infinite. On the contrary , if $\langle X\rangle_{\mathrm{n}} > \langle X\rangle_{\mathrm{b}}$, $\mathscr{E}(X)$ increases faster as $\eta_{\mathrm{n}}(X)$ decreases and the contribution of aggregates with very large complexities is significant; in this case all moments of m are infinite and we expect that $\phi(m)$ has the properties of a statistical fractal. Indeed, by evaluating the asymptotic behavior of $\phi(m)$ by means of the Stirling formula, we come to

$$\phi(m) \cong H\Gamma(1+H)m^{-(1+H)} \quad \text{as } m \to \infty, \tag{31}$$

where

$$\Gamma(p) = \int_0^\infty x^{p-1}\exp(-x)\,\mathrm{d}x, \quad p > 0, \tag{32}$$

is the complete gamma function.

Our aim is to determine the probability density $\psi(t)\,\mathrm{d}t$ of the time between two replacement events and the stochastic properties of the number N of replacement events. By assuming that the time between two breakdown events is a random variable selected from a probability density with finite moments, $\psi_0(t)\,\mathrm{d}t$, we obtain:

$$\psi(t) = \sum \phi(m)[\psi_0(t)\otimes]^m. \tag{33}$$

By applying the Laplace transform the sum in Eq. (33) may be evaluated exactly. By combining Eqs. (26), (28) and (33) we get:

$$\bar{\psi}(s) = H \int_0^1 y^H [g(s) + y)]^{-1} \, dy , \tag{34}$$

where

$$g(s) = [1 - \bar{\psi}_0(s)]/\bar{\psi}_0(s) . \tag{35}$$

By applying a method suggested by Shlesinger and Hughes [11] we replace the fraction $(g + y)^{-1}$ by its inverse Mellin transform and perform the integration over y. The resulting equation is an expression for $\bar{\psi}(s)$ depending on an integral in the complex plane over the Mellin variable; by taking into account the corresponding poles and evaluating the complex integral we obtain

$$\bar{\psi}(s) = 1 - [g(s)]^H \pi H/[\sin(\pi H)] + \mathcal{M}(g(s)) , \tag{36}$$

where $\mathcal{M}(g)$ is an analytic function in g:

$$\mathcal{M}(g) = \sum_{k=1}^{\infty} (-1)^{k+1} H(k - H)^{-1} g^k . \tag{37}$$

Eq. (36) cannot be used to evaluate the probability density $\psi(t)$ for arbitrary time; however, it may be used to evaluate the asymptotic behavior as $t \to \infty$. Since all moments of $\psi_0(t)$ exist and are finite, $\bar{\psi}_0(s)$ may be represented as a Taylor series in s similar to Eq. (8):

$$\bar{\psi}_0(s) = 1 + \sum_{k=1}^{\infty} (-1)^k (k!)^{-1} \langle \tau^k \rangle_0 s^k \cong 1 - s\langle \tau \rangle_0, \qquad s \to 0 , \tag{38}$$

where

$$\langle \tau^k \rangle_0 = \int_0^{\infty} \tau^k \psi_0(\tau) \, d\tau , \tag{39}$$

are the moments of the time interval between two breakdown events. From Eqs. (36), (37) and (39) we note that for $1 > H > 0$ the nonanalytic term of Eq. (36) has the main contribution to $\bar{\psi}(s)$. We have:

$$\bar{\psi}(s) \cong 1 - s^H (\langle \tau \rangle_0)^H \pi H/[\sin(\pi H)], \quad \text{as } s \to 0, \quad 1 > H > 0 , \tag{40}$$

which corresponds to the following large-time behavior:

$$\psi(t) \cong (\langle \tau \rangle_0)^H H \Gamma(1 + H) t^{-(H+1)} \quad \text{as } t \to \infty, \quad 1 > H > 0 . \tag{41}$$

An exact expression of $\psi(t)$ for arbitrary time may be derived if the time between two breakdown events is a random variable selected from an exponential law

$$\psi_0(t) = \nu \exp(-\nu t), \tag{42}$$

where

$$\nu = \langle \tau \rangle_0^{-1} \tag{43}$$

is the frequency of occurrence of a breakdown event. We have:

$$[\psi_0(t) \otimes]^m = \nu(\nu t)^{m-1}[(m-1)!]^{-1} \exp(-\nu t), \tag{44}$$

and the application of Eq. (33) leads to:

$$\psi(t) = \nu^{-H} t^{-(1+H)} \gamma(H+1, \nu t), \tag{45}$$

where

$$\gamma(p, x) = \int_0^x y^{p-1} \exp(-y)\, \mathrm{d}y, \quad p, x > 0, \tag{46}$$

is the incomplete gamma function. The Laplace transform corresponding to Eq. (45) is

$$\bar{\psi}(s) = 1 - (s/\nu)^H H B[H, 1-H, \nu/(\nu+s)], \tag{47}$$

where

$$B(p, q, x) = \int_0^x y^{p-1}(1-y)^{q-1}\, \mathrm{d}y, \qquad x \leqslant 1, \quad p, q > 0, \tag{48}$$

is the incomplete beta function. Eqs. (45) and (37) are consistent with the general asymptotic results derived before. For $s \to 0$ and $t \to \infty$ Eqs. (45) and (47) reduce to Eqs. (41) and (40), respectively.

Now we can proceed to evaluate the stochastic properties of the number of replacement events occurring in a large time interval of length t. From Eqs. (6), (7) we have

$$\begin{aligned}\mathscr{L}\langle N(N-1)\cdots(N-l+1)\rangle &= \partial^l \bar{G}(z,s)/\partial z^l|_{z=1} \\ &= l!\,[\bar{\psi}(s)/(1-\bar{\psi}(s))]^l s^{-1},\end{aligned} \tag{49}$$

from which, by using the asymptotic expression (40) and going back to the time variable, we obtain:

$$\begin{aligned}&\langle N(N-1)\cdots(N-l+1)\rangle \\ &\quad \cong l!\{[\sin(\pi H)]/\pi H\}^l [\Gamma(1+lH)]^{-1} [\langle \tau \rangle_0^{-1} t]^{lH} \\ &\qquad \text{as } t \to \infty, \quad 1 > H > 0.\end{aligned} \tag{50}$$

In particular we get the following expressions for the average value and the dispersion of $N(t)$,

$$\langle N(t)\rangle \cong \frac{\sin(\pi H)}{\pi H^2 \Gamma(H)}[\langle\tau\rangle_0^{-1} t]^H, \quad \text{as } t\to\infty, \quad 1 > H > 0, \tag{51}$$

$$\langle \Delta N^2(t)\rangle \cong \left(\frac{\sin(\pi H)}{\pi H^2 \Gamma(H)}\right)^2 \left(\frac{\sqrt{\pi}\,\Gamma(H+1)}{2^{2H-1}\Gamma(H+\frac{1}{2})} - 1\right)[\langle\tau\rangle_0^{-1} t]^{2H},$$
$$1 > H > 0, \quad t\to\infty. \tag{52}$$

Thus the increase in time of the mean number of replacement events is slower than the linear increase given by Eq. (10). The relative fluctuation

$$\frac{\langle \Delta N^2(t)\rangle^{1/2}}{\langle N(t)\rangle} \cong \left(\frac{\sqrt{\pi}\,\Gamma(H+1)}{2^{2H-1}\Gamma(H+\frac{1}{2})} - 1\right)^{1/2}, \quad 1 > H > 0, \quad \text{as } t\to\infty, \tag{53}$$

tends towards a constant value rather than decreasing to zero. This illustrates the intermittent behavior of the fluctuations of the number of replacement events. The fractal exponent of the time increase of the dispersion is two times bigger than the fractal exponent corresponding to the average value. It follows that the fluctuations of the number of the replacement events are not negligible. These results are valid for any value of the exponent H between 0 and 1 ($1 > H > 0$). In this range the difference $\sqrt{\pi}\,\Gamma(H+1) - 2^{2H-1}\Gamma(H+\frac{1}{2})$ is always positive. As $H \to 1$ this difference tends to zero and the termen in t^{2H} in the expression of dispersion vanishes; for $H = 1$ the dispersion increases linearly in time. For $H \geqslant 1$ the asymptotic expression (40) is no longer valid. For $2 \geqslant H > 1$ the average value increases linearly in time and the dispersion cannot be evaluated in a simple way. For $H > 2$ the moments $\langle\tau\rangle_0$ and $\langle\Delta\tau^2\rangle_0$ of $\psi_0(t)$ exist and are finite and Eqs. (10), (11) derived in Section 2 are valid.

4. A discrete approach to industrial replacement

In this section we give a more detailed description of an industrial aggregate. We assume that each aggregate is made up of the same type of basic units. The number q of basic units from an aggregate is a discrete analogue of the complexity measure X used in Section 3. We assume that each basic unit has the same probability α to be associated into an aggregate and the same probability β of being in an active state. It follows that the probability $\eta_n(q)$ that an aggregate is made up of q basic units is the product between the probability $\alpha \cdots \alpha = \alpha^{q-1}$ that $q-1$ association events take place and the probability $1-\alpha$ that no further association events occur:

$$\eta_n(q) = (1-\alpha)\alpha^{q-1}. \tag{54}$$

This is an exponential law in q similar to the exponential law (25) used in the preceding section.

An aggregate made up of q units is in a passive state if all q basic units do not work, i.e. if q possible lethal breakdown events can occur. The corresponding probability is equal to

$$\mathscr{E}(q) = (1-\beta)^q . \tag{55}$$

It follows that in the case of a discrete description Eq. (21) for the probability $\phi(m)$ that an aggregate is replaced after m breakdown events is replaced by:

$$\begin{aligned}\phi(m) &= \sum_{q=1}^{\infty} \eta_n(q)\,[1-\mathscr{E}(q)]^{m-1}\mathscr{E}(q)\\ &= (1-\alpha)\sum_{q=1}^{\infty} \alpha^{q-1}[1-(1-\beta)^q]^{m-1}(1-\beta)^q .\end{aligned} \tag{56}$$

The factorial moments corresponding to Eq. (56) are:

$$\begin{aligned}&\langle m(m-1)\cdots(m-l+1)\rangle\\ &= l[(l-1)!]^2(1-\alpha)\sum_{k=0}^{l-1}(-1)^{l-1-k}[k!(l-1-k)!]^{-1}[(1-\beta)^{k+1}-\alpha]^{-1},\\ &\qquad \text{for } l < H ,\end{aligned} \tag{57}$$

$$= \infty \quad \text{for } l \geqslant H , \tag{58}$$

where the exponent H is equal to

$$H = \ln\alpha/\ln(1-\beta) . \tag{59}$$

The asymptotic behavior of $\phi(m)$ as $m \to \infty$ may be evaluated by computing in Eq. (56) the sum over q by means of the Poisson summation formula (Titchmarsh [12]) and by keeping the dominant terms as $m \to \infty$. We obtain:

$$\phi(m) = m^{-(H+1)}\Xi(\ln m) \quad \text{as } m \to \infty , \tag{60}$$

where $\Xi(\ln m)$ is a periodic function of $\ln m$ with period $-\ln(1-\beta)$,

$$\begin{aligned}\Xi(b) &= \frac{1-\alpha}{-\alpha\ln(1-\beta)}\\ &\times\left\{\Gamma(1+H) + 2\sum_{k=1}^{\infty}\left[F^{+}\left(1+H,\frac{2\pi k}{-\ln(1-\beta)}\right)\cos\left(\frac{2\pi k b}{-\ln(1-\beta)}\right)\right.\right.\\ &\left.\left.+ F^{-}\left(1+H,\frac{2\pi k}{-\ln(1-\beta)}\right)\sin\left(\frac{2\pi k b}{-\ln(1-\beta)}\right)\right]\right\},\end{aligned} \tag{61}$$

and $F^{\pm}(a,c)$ are the real and the imaginary parts of the complete gamma function of the complex argument, respectively

$$F^{\pm}(a,c) = [\mathrm{Re}, \mathrm{Im}]\,\Gamma(p = a + c\mathrm{i}) . \tag{62}$$

From these equations we note that the main difference between the two models is that the discrete description of the complexity of an aggregate leads to the logarithmic oscillations described by Eqs. (60), (61), a phenomenon which is missing in the continuous case.

The waiting time distribution of the replacement time may be derived by using the same method as before. By inserting Eq. (56) into Eq. (33) and applying the Laplace transform we come to:

$$\bar{\psi}(s) = (1-\alpha) \sum_{q=1}^{\infty} \alpha^{q-1}(1-\beta)^q [g(s) + (1-\beta)^q]^{-1}, \tag{63}$$

where the function $g(s)$ is given by Eq. (35). By applying the Shlesinger–Hughes method (1981) we obtain

$$\bar{\psi}(s) = 1 - [g(s)]^H K[\ln(g(s))] + \mathcal{N}[g(s)], \tag{64}$$

where $K[\ln g]$ is a periodic function of $\ln g$ with a period $-\ln(1-\beta)$

$$K[\ln g] = \frac{\pi(1-\alpha)}{-\alpha\ln(1-\beta)}\left\{\frac{1}{\sin(\pi H)}\right.$$
$$+2\sum_{k=1}^{\infty}\left[\left[\exp\left(\frac{2\pi^2 k}{\ln(1-\beta)}\right)\sin\left(\pi H + \frac{2\pi k \ln g}{-\ln(1-\beta)}\right)\right.\right.$$
$$\left. + \exp\left(-\frac{2\pi^2 k}{\ln(1-\beta)}\right)\sin\left(\pi H + \frac{2\pi k\ln g}{\ln(1-\beta)}\right)\right]$$
$$\left.\left.\left[\cosh\left(\frac{4\pi^2 k}{\ln(1-\beta)}\right) - \cos(2\pi H)\right]\right]\right\}, \tag{65}$$

and $\mathcal{N}(g)$ is an analytic function of g:

$$\mathcal{N}(g) = (1-\alpha)\sum_{k=1}^{\infty}(-1)^k g^k[(1-\beta)^k - \alpha]^{-1}. \tag{66}$$

We expect that the logarithmic oscillations in the Laplace variable lead to similar logarithmic oscillations in the time domain. We have failed to evaluate the logarithmic oscillations of $\psi(t)$ in the general case. By neglecting the oscillatory terms in $\ln s$ and by using Eqs. (35) and (38) we get the following asymptotic expression for $\bar{\psi}(s)$ as $s \to 0$ and $1 > H > 0$:

$$\bar{\psi}(s) = 1 - \frac{\pi(1-\alpha)}{-\alpha\ln(1-\beta)}\frac{[\langle\tau\rangle_0 s]^H}{\sin(\pi H)}, \quad \text{as } s > 0, \quad 1 > H > 0. \tag{67}$$

By using this equation we get the following expressions for the asymptotic behavior of the probability density $\psi(t)$ and for the moments of the number of replacements:

$$\psi(t) = \frac{\pi H(1-\alpha)}{-\alpha\ln(1-\beta)} \frac{(\langle\tau\rangle_0)^H t^{-(H+1)}}{\Gamma(1-H)\sin(\pi H)}, \quad \text{as } t\to\infty, \quad 1 > H > 0, \tag{68}$$

$$\langle N(N-1)\cdots(N-l+1)\rangle = l!\left(\frac{-\alpha\ln(1-\beta)}{(1-\alpha)\pi}\right)^l \frac{[\langle\tau\rangle_0^{-1}t]^{lH}}{\Gamma(1+lH)}[\sin(\pi H)]^l,$$
$$\text{as } t\to\infty, \quad 1 > H > 0, \tag{69}$$

$$\langle N(t)\rangle = \frac{-[\alpha\ln\alpha]\sin(\pi H)}{(1-\alpha)\pi H^2\Gamma(H)}[\langle\tau\rangle_0^{-1}t]^H, \quad \text{as } t\to\infty, \quad 1 > H > 0, \tag{70}$$

$$\langle\Delta N^2(t)\rangle = \left(\frac{-[\alpha\ln\alpha]\sin(\pi H)}{(1-\alpha)\pi H^2\Gamma(H)}\right)^2\left(\frac{\sqrt{\pi}\Gamma(H+1)}{2^{2H-1}\Gamma(H+\frac{1}{2})} - 1\right)[\langle\tau\rangle_0^{-1}t]^{2H},$$
$$\text{as } t\to\infty, \quad 1 > H > 0, \tag{71}$$

$$\frac{\langle\Delta N^2(t)\rangle^{1/2}}{\langle N(t)\rangle} = \left(\frac{\sqrt{\pi}\Gamma(H+1)}{2^{2H-1}\Gamma(H+\frac{1}{2})} - 1\right)^{1/2}, \quad \text{as } t\to\infty, \quad 1 > H > 0. \tag{71a}$$

We note that for $1 > H > 0$ the intermittent behavior of the fluctuation of the number of replacement events is still present.

If $\psi_0(t)$ is an exponential function given by Eq. (42) then the logarithmic oscillations in the time domain can be computed exactly. By applying the Poisson summation method we get an exact expression for $\psi(t)$:

$$\psi(t) = (1-\alpha)\sum_{q=1}^{\infty}\alpha^{q-1}(1-\beta)^q \nu\exp[-\nu t(1-\beta)^q]$$
$$= \tfrac{1}{2}(1-\alpha)\nu^*\exp(-\nu^* t) + (\nu^*)^{-H}t^{-(H+1)}\Xi^*[\nu^* t, \ln(\nu^* t)], \tag{72}$$

where $\nu^* = \nu(1-\beta)$,

$$\Xi^*(a,b) = \frac{1-\alpha}{-\alpha\ln(1-\beta)}\Bigg\{\gamma(1+H,a)$$
$$+ 2\sum_{k=1}^{\infty}\Bigg[\mathscr{F}^+\left(H+1, \frac{2\pi k}{-\ln(1-\beta)}, a\right)\cos\left(\frac{2\pi kb}{-\ln(1-\beta)}\right)$$
$$+ \mathscr{F}^-\left(H+1, \frac{2\pi k}{-\ln(1-\beta)}, a\right)\sin\left(\frac{2\pi kb}{-\ln(1-\beta)}\right)\Bigg]\Bigg\}, \tag{73}$$

$\gamma(1+H,a)$ is the incomplete gamma function (Eq. (46)) and $\mathscr{F}^+(1+H,c,a)$ are the real and imaginary parts of the incomplete gamma function of complex argument

$$\mathscr{F}^{\pm}(1+H,c,a) = [\mathrm{Re},\mathrm{Im}]\gamma(p = 1+H+ci, a). \tag{74}$$

For $t \to \infty$ Eq. (72) becomes:

$$\psi(t) \cong (v^*)^{-H} t^{-(H+1)} \Xi[\ln(v^* t)], \quad \text{as } t \to \infty, \tag{75}$$

where $\Xi(b)$ is the periodic function given by Eq. (61).

5. Communication networks

Now we discuss the problem for which the name of "fractal time" was initially suggested (Berger and Mandelbrot [1]). We consider a communication network made up of a large number of channels. Each channel has the same probability α to be open. Not all open channels can lead to a successful transmission. The probability β of a successful transmission is also the same for all open channels. The network works according to the following rules: (a) one tries to transmit a message by using all open channels; (b) the number q of open channels is a random variable which is kept constant until a failure of transmitting a message occurs. This model is approximately isomorphic with the discrete model presented in Section 4; however, some minor differences exist. An aggregate is made up of at least one basic unit; that is why in Section 4 the number of basic units varies between 1 and ∞. For a communication network the number of open channels can take any value from 0 to ∞. In Section 4 an aggregate is replaced after at least one breakdown event and thus the number m of breakdown events varies between 1 and ∞. For a communication network, however, there is the possibility that two successive attempts of transmitting a message result in a failure; it follows that the number m of transmitted messages between two failures varies from 0 to ∞. These distinct features do not lead to a difference in the asymptotic behavior of the model; they only generate different expressions for certain coefficients.

The probability $\eta_n(q)$ that q channels are open is equal to:

$$\eta_n(q) = (1-\alpha)\alpha^q . \tag{76}$$

The probability $\mathscr{E}(q)$ that for a network with q open channels a failure occurs is given by Eq. (55) where q can now take the value zero. The probability $\phi(m)$ that m messages are transmitted between two failures is given by an equation similar to Eq. (56):

$$\begin{aligned}\phi(m) &= \sum_{q=0}^{\infty} \eta_n(q)[1-\mathscr{E}(q)]^m \mathscr{E}(q) \\ &= (1-\alpha) \sum_{q=0}^{\infty} \alpha^q [1-(1-\beta)^q]^m (1-\beta)^q .\end{aligned} \tag{77}$$

The asymptotic behavior of $\phi(m)$ is given by:

$$\phi(m) = m^{-(H+1)} \alpha\, \Xi(\ln m), \quad \text{as } m \to \infty, \tag{78}$$

where the periodic function $\Xi(b)$ is given by Eq. (61).

In this case $\psi_0(t)$ is the probability density of the time necessary for transmitting a message and $\psi(t)$ is the probability density of the time interval between two failure events. If $\psi_0(t)$ is given by an exponential distribution, $\psi(t)$ is equal to:

$$\begin{aligned}\psi(t) &= (1-\alpha)\sum v\alpha^q(1-\beta)^q \exp[-vt(1-\beta)^q] \\ &= \tfrac{1}{2}(1-\alpha)\, v\exp(-vt) + v^{-H}t^{-(H+1)}\alpha\Xi^*[vt, \ln(vt)]\,, \end{aligned} \tag{79}$$

where $\Xi^*(a,b)$ is given by Eq. (73). The corresponding asymptotic expression is

$$\psi(t) = \alpha v^{-H}t^{-(H+1)}\Xi[\ln(vt)], \quad \text{as } t\to\infty\,. \tag{79'}$$

In the general case, when the function $\psi_0(t)$ is not necessary given by an exponential law, only the non-periodic components of $\psi(t)$ and of the moments of the number N of failures can be evaluated:

$$\psi(t) \cong \frac{\pi H(1-\alpha)}{-\ln(1-\beta)}\frac{(\langle\tau\rangle_0)^H t^{-(H+1)}}{\Gamma(1-H)\sin(\pi H)}, \quad \text{as } t\to\infty, \quad 1>H>0\,, \tag{80}$$

$$\langle N(N-1)\cdots(N-l+1)\rangle = l!\left(\frac{-\ln(1-\beta)}{(1-\alpha)\pi}\right)^l \frac{[\langle\tau\rangle_0^{-1}t]^H}{\Gamma(1+lH)}[\sin(\pi H)]^l,$$

$$\text{as } t\to\infty, \quad 1>H>0\,, \tag{81}$$

$$\langle N(t)\rangle \cong \frac{[-\ln\alpha]\sin(\pi H)}{(1-\alpha)\pi H^2\Gamma(H)}[\langle\tau\rangle_0^{-1}t]^H, \quad \text{as } t\to\infty, \quad 1>H>0\,, \tag{82}$$

$$\langle\Delta N^2(t)\rangle \cong \left(\frac{[-\ln\alpha]\sin(\pi H)}{(1-\alpha)\pi H^2\Gamma(H)}\right)^2\left(\frac{\sqrt{\pi}\,\Gamma(H+1)}{2^{2H-1}\Gamma(H+\frac{1}{2})}-1\right)[\langle\tau\rangle_0^{-1}t]^{2H}$$

$$\text{as } t\to\infty, \quad 1>H>0\,. \tag{83}$$

Eq. (71a) for the relative fluctuation remains valid.

6. Discussion

The above models illustrate a general feature related to the fractal time statistics, the existence of two opposite factors which compensate each other. The equilibration of these factors lead to the absence of a characteristic time scale of the system, a situation which generates a statistical fractal. For the models of industrial replacement by increasing the complexity of an aggregate the probability of lethal breakdown falls off to zero and the average lifetime increases to infinity; however, the contribution of very complex aggregates decreases to zero as the complexity tends to infinity. For a communication network, as the number of open channels increases to infinity, the average time between two failure events increases to infinity. On the contrary, the probability that a large number of channels are open decreases exponentially to zero as the number of channels increases to infinity.

A controversial feature is the occurrence of logarithmic oscillations. Their origin is somewhat mysterious: for some models presented in the literature they exist, whereas for others they are missing. For the models presented in this paper the logarithmic oscillations are due to the discrete nature of the complexity measure. If a continuous complexity measure is used, the logarithmic oscillations disappear.

To show that this is indeed the case we shall try to recover the continuous model presented in Section 3 as a particular case of the discrete model. Eq. (63) for $\bar{\psi}(s)$ can be rewritten in a self-similar form

$$\bar{\psi}[(1-\beta)g(s)] = (1-\alpha)/[g(s)+1] + \alpha\bar{\psi}[g(s)]\,. \tag{84}$$

We consider the limit $\alpha \nearrow 1$, $\beta \searrow 0$ with the restriction that the fractal exponent H is constant,

$$\alpha \nearrow 1, \quad \beta \searrow 0, \quad H = \ln\alpha/\ln(1-\beta) = \text{constant}\,. \tag{85}$$

From Eq. (59) we have

$$\alpha = (1-\beta)^H \cong 1 - \beta H, \quad \text{as } \beta \to 0\,. \tag{86}$$

By inserting Eq. (86) into Eq. (84) in the limit $\beta \to 0$ we get a differential equation in $\bar{\psi}(s)$

$$g(s)\frac{\partial\bar{\psi}(s)}{\partial s} = H\frac{\partial g(s)}{\partial s}\left(\bar{\psi}(s) - \frac{1}{g(s)+1}\right). \tag{87}$$

The initial condition of Eq. (87)

$$\bar{\psi}(s=0) = 1\,, \tag{88}$$

may be derived from the normalization condition of $\psi(t)$, $\int_0^\infty \psi(t)\,dt = 1$. By integrating Eq. (87) we recover Eq. (34) which has been derived by using the continuous model. In order to clarify the physical significance of the limit (85), we shall write Eqs. (54), (55) in a form similar to Eqs. (25) and (20) respectively

$$\eta_n(q) = (\alpha^{-1} - 1)\exp[-q\ln(1/\alpha)]\,, \tag{89}$$

$$\mathscr{E}(q) = \exp\{-q[-\ln(1-\beta)]\}\,. \tag{90}$$

From these equations we can establish the following correspondence relations between the discrete and continuous models:

$$\langle X\rangle_n \leftrightarrow [\ln(1/\alpha)]^{-1}\,, \tag{91}$$

$$\langle X\rangle_b \leftrightarrow [-\ln(1-\beta)]^{-1}\,. \tag{92}$$

Thus the limit (85) conserves the ratio of the average complexities of a new aggregate and of an aggregate at the moment of replacement. In the limit (85) all sums over q become integrals with respect to the continuous complexity measure X.

It might be possible that the models presented here would be applied to other physical or engineering systems for which a fractal time statistics exists. From the practical point of view we assume that the possibility of application of the first two models to realistic systems with replacement is rather uncertain. Indeed, in practice it is usually very difficult to realize aggregates with very large size. We believe however, that the third model is more than a simple academic exercise. The telephone networks are finite but sufficiently large so that the existence of a cutoff value for the number of open channels has little influence on the self-similar behavior of the time distributions.

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