Evolution of probability densities in stochastic coupled map lattices

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This paper describes the statistical properties of coupled map lattices subjected to the influence of stochastic perturbations. The stochastic analog of the Perron-Frobenius operator is derived for various types of noise. When the local dynamics satisfy rather mild conditions, this equation is shown to possess either stable, steady state solutions (i.e., a stable invariant density) or density limit cycles. Convergence of the phase space densities to these limit cycle solutions explains the nonstationary behavior of statistical quantifiers at equilibrium. Numerical experiments performed on various lattices of tent, logistic, and shift maps with diffusivelike interelement couplings are examined in light of these theoretical results.

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I. INTRODUCTION

For the past decade or so, the dynamics of coupled map lattices (CML's) have been the focus of intense investigation (cf. [1] for a sample review). They are a natural extension of low-dimensional dynamical systems and provide spatially extended models that are particularly well suited for analytic and numerical investigations. They were originally introduced as alternatives to more complicated partial differential equations in an attempt to understand the generic properties of coupled chaotic units.

CML's have also been used to model various properties of reaction-diffusion systems [2], population dynamics [3], and genetic networks [4,5] and parallel image processing algorithms [6–8] to mention but a few applications. In all these models, however, the dynamics are deterministic. Though there are numerical investigations of the influence of stochastic perturbations on CML dynamics (see, for example, [9]), analytic insight into the statistical behavior of stochastic CML's is lacking.

The present paper is based on the study of the transfer operator for stochastically perturbed CML's and extends previous analytic descriptions of the Perron-Frobenius operator associated with deterministic lattices [10–12]. To draw an analogy with more familiar physical systems, the transfer operators discussed here describe the ensemble properties of stochastic CML's much as the Liouville equation describes the ensemble dynamics of ordinary differential equations (ODE's), the Fokker-Planck equation those of the Langevin equation, or the Perron-Frobenius operator those of deterministic maps. See Table I.

The approach presented here rests on the observation that the transfer operator induced by noisy CML's is a Markov operator defined by a stochastic kernel. We can therefore apply results available concerning such operators to the statistical description of noisy CML dynamics.

The types of models investigated here are introduced in Sec. II. Their behavior is discussed numerically in Sec. III. Section IV is a description of the thermodynamics of CML's and highlights the need to focus on the properties of the transfer operator to understand the behavior discussed in Sec. III. The main analytic results are presented in Sec. V. Implications for the proper construction of the thermodynamics of stochastically perturbed high-dimensional dynamical systems are discussed in Sec. VI.

TABLE I. Brief summary of the probabilistic descriptions associated with various types of discrete and continuous-time models.

Description of the model	Description of ensemble dynamics
Deterministic maps	Perron-Frobenius operator
Stochastic maps	Transfer operator
Deterministic ODE's	Generalized Liouville equation
Stochastic ODE's (white noise)	Fokker-Planck equation
Stochastic ODE's (nonwhite noise)	Kramers-Moyal equation
Differential delay equations	Hopf equation for the characteristic functional

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II. STOCHASTIC CML'S

A coupled map lattice is a mapping $\Phi:\mathbb{R}^N \mapsto \mathbb{R}^N$ governing the evolution of a vector $\mathbf{x}_t$ representing the state of the lattice at time t

$$\mathbf{x}_{t+1}=\Phi(\mathbf{x}_t)\ , \quad t=0,1,\ldots \ . \tag{1}$$

We consider cases where the phase space $\mathbf{X}$ of Φ is a restriction of $\mathbb{R}^N$ to the N-dimensional hypercube: $\mathbf{X}=[0,1]\times\cdots\times[0,1]$. In two spatial dimensions, the evolution of each site of a deterministic coupled map lattice with linear interelement coupling is given by

$$x_{t+1}^{(kl)}=\Phi^{(kl)}(\mathbf{x}_t)=(1-\varepsilon)S(x_t^{(kl)}) + \frac{\varepsilon}{p}\sum_{\substack{p \text{ nearest}\\ \text{neighbors}}} S(x_t^{(ij)})\ , \quad \varepsilon\in(0,1)\ , \tag{2}$$

where $S:[0,1]\mapsto[0,1]$ describes the local dynamics. When $p=4$, the coupling in (2) mimics a discrete version of the diffusion operator and when the p neighborhood encompasses the entire lattice, the coupling is known as mean field.

As mentioned in the Introduction, it is of interest to understand and clarify the influence of noisy perturbations on the evolution of these CML's. The perturbations considered here are random vectors of N numbers (for an N element CML), whose components are independent of one another, each being distributed according to a one-dimensional probability density g. The density g of the vector random variable $\xi=(\xi^{(1)},\ldots,\xi^{(N)})$ will therefore be constructed as the product of these identical components

$$g(\xi)=\prod_{i=1}^{N} g(\xi^{(i)})\ , \quad i=1,2,\ldots,N\ . \tag{3}$$

There are various ways in which a stochastic perturbation can influence the evolution of a coupled map lattice: the perturbation can be additive or multiplicative and it can be applied constantly or randomly. The influence of the noise on the dynamics depends on which of these is considered.

A. Additive and multiplicative perturbations

These are perturbations applied at each iteration step. When the stochastic perturbation is constantly applied, it can be either added to or multiplied by the original transformation Φ. In the former case, the evolution of a lattice site is given by a relation of the form

$$x_{t+1}^{(kl)}=\Phi^{(kl)}(\mathbf{x}_t)+\xi_t^{(kl)}\equiv\Phi_{\text{add}}^{(kl)}(\mathbf{x}_t) \tag{4}$$

and ξ is then referred to as *additive* noise. In the latter, we have

$$x_{t+1}^{(kl)}=\Phi^{(kl)}(\mathbf{x}_t)\times\xi_t^{(kl)}\equiv\Phi_{\text{mul}}^{(kl)}(\mathbf{x}_t) \tag{5}$$

and ξ is then referred to as *multiplicative* or *parametric noise.* In general, the effects of additive and multiplicative noise on CML's can be different since they model different perturbing mechanisms. The noise density (3) of the perturbations present in (4) and (5) is always defined so that the phase space of the perturbed transformations remains the N-dimensional hypercube $\mathbf{X}$ defined above. In other words, $\Phi_{\text{add}}: \mathbf{X}\mapsto\mathbf{X}$ and $\Phi_{\text{mul}}: \mathbf{X}\mapsto\mathbf{X}$. Before analytically discussing the dynamics of stochastic CML's, we numerically illustrate typical behaviors using several transformations that have been introduced in the literature.

III. NUMERICAL INVESTIGATION OF STOCHASTIC CML'S

Many deterministic CML's of the form (2) with chaotic local transformations S display "phases" that correspond to qualitatively different statistical behaviors. In some cases the lattices are statistically stable, while in others the lattices display statistical periodicity. This periodicity is the high-dimensional analog of the so-called noisy periodicity present in certain one-dimensional maps and discussed in [13,14] for the logistic and tent maps and in [15] for certain stochastically perturbed piecewise linear Keener maps. It has been described numerically [16] and analytically [17,12] in deterministic lattices and similar behavior has also been reported in cellular automata [18–20], where it is referred to as quasiperiodicity. A short discussion of the phenomenology is given in [21]. We now demonstrate its presence in stochastic CML's and verify that it provides a mechanism for collective behavior in these spatially extended models. Note that for all the systems described in the remainder of this section, periodic boundary conditions were implemented. In addition, the updating rule for the models presented here is, as usual, parallel (though the rigorous results presented in Sec. V are independent of this assumption).

The statistical descriptors used here to characterize the evolution of the various CML's are the temporal and spatial correlation functions, the Boltzmann-Gibbs entropy and the density of activity on the lattices. The rationale for choosing these rather than, say, the mutual information or the Lyapunov spectrum is that they provide sufficient information to unambiguously determine whether a given CML is in a statistically stable or statistically periodic phase. Before proceeding we therefore briefly review the definitions of these descriptors (for more detailed discussions, see [17]).

The density $\tilde{f}_t$ of activity across a lattice of N elements at time t is implicitly determined by

$$\langle x_t\rangle=\frac{1}{N}\sum_{k,l} x_t^{(kl)}\tilde{f}_t(x_t^{(kl)})\ , \tag{6}$$

where $\langle\ \rangle$ denotes the expectation of the quantity inside the angular brackets. Numerically, $\tilde{f}_t$ is approximated by the histogram of activity on a lattice at time t. It is investigated here because it reflects the behavior of the actual phase space (or ensemble) density f_t defined in Sec. IV, which is computationally costly. To briefly summarize the link between the numerically computed $\tilde{f}_t$ and f_t, we note without proof (cf. [17] for details) that if $\tilde{f}_t$ is eventually time invariant for almost all initial preparations, this invariance indicates that the CML generates an invariant measure, whereas if $\tilde{f}_t$ does not reach time invariant $\bar{f}_*$, then the corresponding ensemble density f_t does not reach equilibrium.

The other statistical quantifiers of the motion discussed here are the Boltzmann-Gibbs entropy of the density $\bar{f}_t$

$$H(\bar{f}_t)=-\int_0^1 \bar{f}_t(x)\ln\bar{f}_t(x)dx \qquad (7)$$

and the linear autocorrelation function of a trajectory $\{y_i\}_{i=1}^{Q}$, which may be either the temporal evolution of a single site or the activity along one spatial direction

$$\rho(i)=\frac{c_i}{c_0}\ ,\quad c_i=\frac{1}{Q}\sum_{j=1}^{Q-i}(y_j-\langle y\rangle)(y_{j+i}-\langle y\rangle)\ . \qquad (8)$$

Given these preliminary definitions, it is now possible to describe the evolution of various stochastic systems whose deterministic counterparts have been investigated [12,18].

A. Tent map and logistic map lattices

These CML's are of the form (2), with $p=4$ in dimensionality 2, and the local map S is either the generalized tent map

$$S(x)=\min_{x\in[0,1]}\{ax,a(1-x)\}\ ,\quad a\in(1,2] \qquad (9)$$

or the logistic map

$$S(x)=ax(1-x)\ ,\quad a\in(1,4]\ . \qquad (10)$$

The tent map is discussed here because much information is available about the statistical behavior of the single map [14,22]. The transformation (9) is chaotic and it can be either statistically stable or statistically periodic. More precisely [14],

$$2^{1/2^{n+1}}<a\le 2^{1/2^n} \Longleftrightarrow \text{statistical cycling of period } 2^n\ ,\quad n=0,1,2,\ldots\ . \qquad (11)$$

This "band-splitting" behavior survives diffusive coupling, modulo a dependence of the location of the band splitting on the strength of the diffusive coupling [23].

The logistic map (10) has been the subject of intense scrutiny and, despite its apparent simplicity, is known to display a wide range of dynamical behaviors [13,24,25]. It possesses stable periodic orbits that bifurcate via the period-doubling scenario as the parameter a increases from 3 to the critical value $a_c\simeq 3.5699$. For $a_c<a<4$ the chaotic trajectories correspond either to statistically stable regimes (described by invariant measures) or to banded chaotic windows (described by metastable measures). At $a=4$ and for a set of a values of positive measure, it is statistically stable.

1. The tent map lattice

The behavior of the CML (2), with S given by the tent map (9), perturbed as in (4) remains qualitatively similar to that of the unperturbed system. As the local slope a is decreased from 2 to 1, for a given ε and a given noise amplitude, the period of the banded chaotic behavior dou-

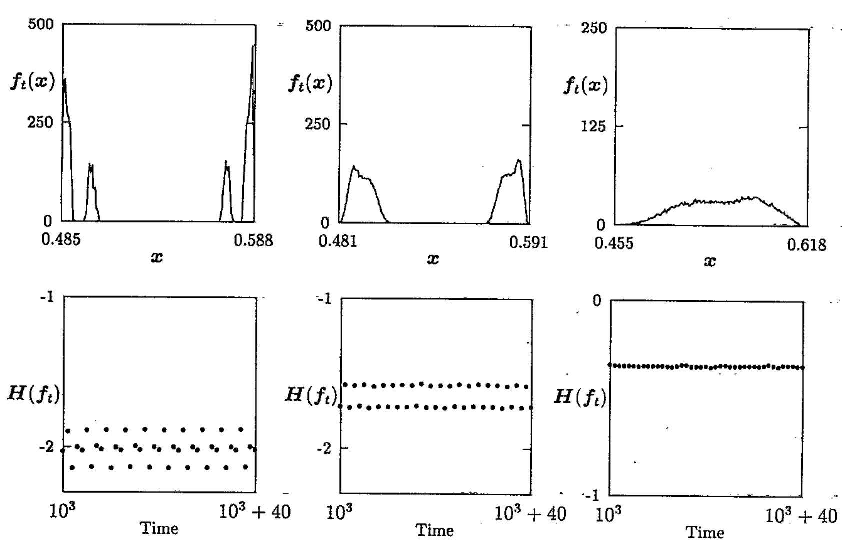

FIG. 1. Noise-induced band merging in a lattice of 200×200 tent maps coupled diffusively, with $a=1.175$ and $\varepsilon=0.1$. The evolution of the system is described by (4) with Φ given by (2) and S by (9). Top row: left, the function $\bar{f}_t(x)$ defined in (6), which is a histogram of the values of all the sites on a lattice at time $t=10^3$ in the absence of noise; center, same histogram when the lattice is perturbed by additive noise uniformly supported on [0,0.003]; right, additive noise supported on [0,0.03] [the three noise densities are given by (17), where $b=0$ and c increases from left to right]. The subintervals of [0,1] displayed in these figures are chosen so that the support of $\bar{f}_t(x)$ occupies the entire panel. Each histogram was produced with 200 bins on $[a-a^2/2-c, a/2+c]$ (which is the subinterval of [0,1] on which the activity of a single tent map with additive noise on $[0,c]$ is supported). Bottom row: from left to right, the panels display the temporal behavior of the Boltzmann-Gibbs entropy (7) associated with the histograms displayed in the top row. As expected, the lattices are statistically stabilized as the noise amplitude increases.

bles successively from period 1 to period 2 to period 4 and so on. As the period increases, the separation between different bands in the deterministic system diminishes without bound away from 0. As expected, in the stochastic system, the doubling stops when the amplitude of the noise becomes larger than this band separation. Figure 1 illustrates this noise-induced band merging, when two bands merge into a single one, under the action of additive perturbations. Numerical investigations of this and other CML's indicate that in certain regions of parameter space, the systems are more sensitive to the action of parametric noise than to that of additive noise. This is striking, for example, in the case of the lattices of tent maps when the parameters are such that the deterministic lattice is in the period-two regime. In this case the transition between period 2 and period 1 (stability) behavior is induced for additive perturbations that have an amplitude of about $\frac{1}{10}$ or multiplicative perturbations that have an amplitude of about $\frac{1}{100}$ (i.e., in this case the variable ξ is uniformly supported on [0.995,1.005]). High sensitivity to multiplicative noise is also observed in lattices of logistic maps and lattices of piecewise linear maps introduced in Sec. III B. Preliminary investigations of a CML introduced in [12], where S is smooth (sigmoidal) and the interelement coupling is nonlinear, show that this system is also more "sensitive" to multiplicative than to additive noise. We should point out, however, that this apparent greater susceptibility to parametric noise is not observed everywhere in the parameter space.

A surprising effect of the perturbation of a system by noise is the resulting "statistical hysteresis" shown in Fig. 2, which displays a bifurcation diagram for the Boltzmann-Gibbs entropy of the collapsed density of various lattices subjected to noise. As explained in Sec. IV, the abrupt changes in the behavior of H as the control parameters are varied are signatures of phase transitions in the corresponding lattice. When H asymptotically reaches a periodic cycle, reflected by the cycling of ensemble densities, the statistical quantifiers of the motion are also time periodic. This cycling is similar to the "periodicity on average" described in various cellular automata [21]. Similar behavior has also been reported in deterministic CML's by Chaté and Manneville (Sec. 3.2.2 of [18]). Note that the bifurcation diagram of Fig. 2 depends, in certain regions of parameter space, on whether the parameter is being increased or decreased. This indicates that the ensemble properties of the CML depend on the initial ensemble density. We will return in Sec. V to an explanation of this phenomenon, which is expected in large classes of stochastic CML's (including the systems whose behavior is displayed in Fig. 2) on the basis of the spectral properties of the transfer operator.

Pattern formation in the tent map lattice is illustrated in Fig. 3. As expected, as the noise amplitude is increased, patterns gradually disappear. Numerical observations indicate that pattern formation coincides with statistical cycling in this system (this is clearly illustrated by the top panels of Figs. 1 and 3). To understand this connection, note that when the lattice cycles statistically, the density of activity is supported on disjoint supports (as in Fig. 1, for example). The diffusivelike effect of the coupling in Eq. (2) ensures that nearby sites tend to belong to the same "band" (or support of the distribution) and if this coupling is not strong enough to force all sites into the same band, the state of the lattice at time t consists of clusters of sites that belong to the different bands [strictly speaking, the discretization of the diffusion operator yields a coupling of the form (2), where the p neighborhood includes only the nearest neighbors, but when more neighbors are taken into account, the coupling still mimics the effect of diffusion in a system with a large diffusion coefficient]. If there is only one support of the collapsed density $\bar{f}_t$ (cf. the rightmost panel of Figs. 1 and 3), no discernible patterns occur since the entire attracting interval is occupied with nonzero probability, irrespective of the coupling strength.

As expected [26], the transients leading to the various states of equilibria can be extremely long. The length of those transients is determined by the specificities of the

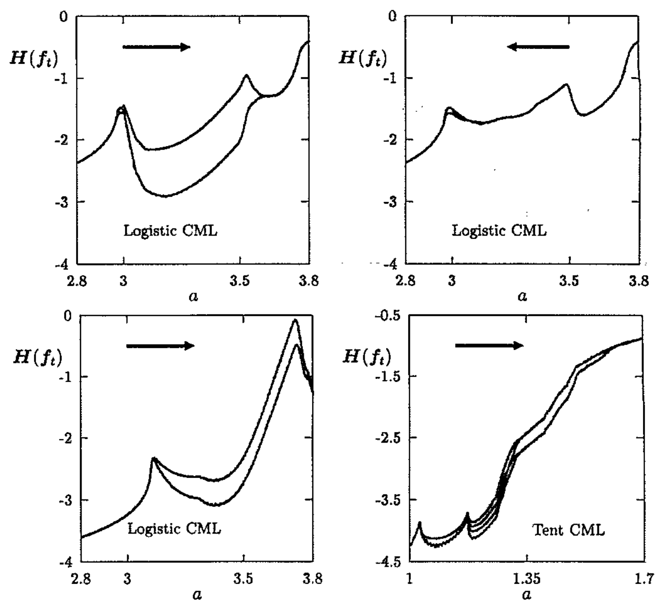

FIG. 2. "Adiabatic" bifurcation diagrams for the Boltzmann-Gibbs entropy of the distribution of activity in lattices of diffusively coupled tent and logistic maps. For each value of a, the CML's were iterated 10^3 times, with an initial condition given by the last one of the 10^3 iterations corresponding to the previous a value (the increment is $\Delta a = 0.002$ and the arrows indicate the direction in which the parameter is changed). Top row: hysteresis in a lattice of logistic maps, with $\varepsilon = 0.1$, and additive noise supported uniformly on [0,0.05] (for $a = 2.8$, the initial condition was spatially isotropic and uniform on [0.5,1]). The bifurcation diagram for H depends on whether a is increased (left panel) or decreased (right panel), a feature that is shown in Sec. V to reflect the asymptotic periodicity of the transfer operator (cf. also Definition 1). Bottom left: same parameters as above, with multiplicative noise supported uniformly on [0.95,1]. Bottom right: diagram for a diffusively coupled tent map lattice with $\varepsilon = 0.45$ (as in Fig. 3) and additive noise supported on [0,0.01]. The hysteresis observed in the top row is also observed in tent map lattices and when the noise is multiplicative.

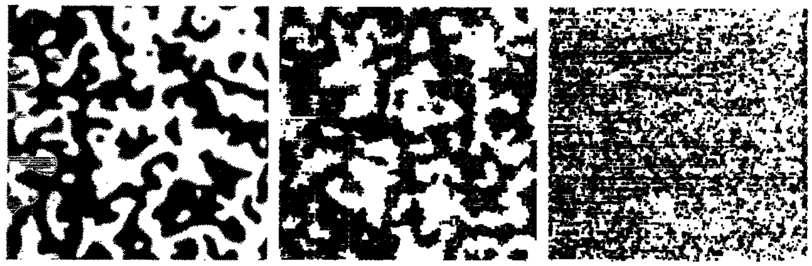

FIG. 3. Three snapshots displaying the state of a 200×200 lattice of diffusively coupled tent maps at time $t=10^3$, with $a=1.175$ and $\varepsilon=0.45$. The evolution of the system is described by Eq. (4) with Φ given by (2) (with $p=4$) and S given by (9). The noise density is given by (17) and the interval on which it is supported (called [b,c] in (17)) is changed from panel to panel. From left to right the support of the additive noise is widened as in Fig. 1 ($c=0,0.003,0.03$). For each panel the initial preparation of the lattice was featureless: the initial value for every site was a random variable distributed uniformly on [0.5,1]. The value of each $x_{10^3}^{(kl)}$ is represented by a gray pixel. Our coding scheme contains 256 gray levels that cover the interval within which the $x^{(kl)}$'s are contained: black corresponds to $x_{10^3}^{(kl)}=0.484-c$ [note that $0.484=a(1-a/2)$], while white corresponds to $x_{10^3}^{(kl)}=0.5875+c$ $(0.5875=a/2)$.

local map, the interelement coupling, and the size of the lattice.

2. *The logistic map lattice*

The effects of the noise on the dynamics of the logistic map lattice depend to a large extent on whether the unperturbed CML is periodic in time and/or space or chaotic. A comprehensive overview of the dynamics of system (2) with S given by (10) is presented in [18]. When $a<a_\infty \simeq 3.5699\ldots$, the CML can possess stable spatiotemporal orbits (i.e., such as the "coherent structures" discussed in [27,28]). In this scenario, each lattice site then eventually settles onto a periodic cycle and the lattices reach frozen spatial structures consisting of domains that contain different phases in the cycle. When noise is present, the system can become either statistically periodic or statistically stable, depending on the strength of the perturbation. In the case of additive noise, when the perturbation is small compared to the amplitude of the periodic cycle, the solutions are statistically periodic and are reminiscent of the banded chaotic trajectories. If the amplitude of the perturbation is increased, possibly prexisting spatiotemporal structures are gradually lost and eventually the CML's are seen to be spatiotemporally chaotic in the sense that correlations in time and space decay exponentially.

When $a>a_\infty$ and the unperturbed CML is chaotic, the influence of additive noise is exactly as for the coupled tent map lattices discussed above and the resulting system is again either statistically periodic or statistically stable. It is of interest to note that the evolution of the statistical quantifiers of the motion in the deterministic case has been conjectured to be quasiperiodic or chaotic (see, for example, the discussion surrounding Figs. 19 and 20 of [18]). We do not observe such statistical evolutions when the transients are long enough. The linearity of the Perron-Frobenius operator for the deterministic lattices [17] implies that statistics in these CML's asymptotically reach either stable steady states or periodic cycles. The results of Sec. V (and more specifically the linearity of the transfer operators for stochastic CML's) imply the same conclusion in the presence of noise.

The influence of parametric noise on the evolution of the logistic map lattices is qualitatively the same as the influence of additive perturbations, but, as with tent map lattices, in some regions of parameter space it is observed that the amplitude of parametric noise needed to statistically stabilize the lattices is much smaller than the additive noise amplitude required to produce the same effect. We now discuss the influence of noise on a CML that possesses chaotic regimes that cannot be described by probability densities in the absence of noise.

B. Keener map lattices

Here the local transformation is a piecewise linear map with constant slope on [0,1], which was considered by Keener [29]:

$$S(x)=(ax+b)\bmod 1, \quad a,b\in(0,1)\ , \quad x\in[0,1]\ . \tag{12}$$

There exists a range of values for the parameters a and b such that the trajectories are chaotic in the sense that they are attracted to a subset of [0,1] of zero Lebesgue measure (a Cantor set) [29]. Numerically, this is reflected by the fact that if the histogram along a trajectory is constructed, the number of histogram peaks will increase as the bin size decreases.

For this type of system the Perron-Frobenius operator does not possess a fixed point in the space of probability densities. In fact, it asymptotically transforms almost all initial probability densities into generalized functions. A rigorous treatment of such operators is possible and studying the nonequilibrium statistical properties of the corresponding CML's involves the reformulation of the problem in terms of the evolution of measures. However, this picture is simplified by the presence of noise in the map (12) because under the influence of noise, the map induces a transfer operator acting on well defined densities [30]. Furthermore, the ensemble densities asymptotically reach a limit cycle in density space (as in Fig. 1) that

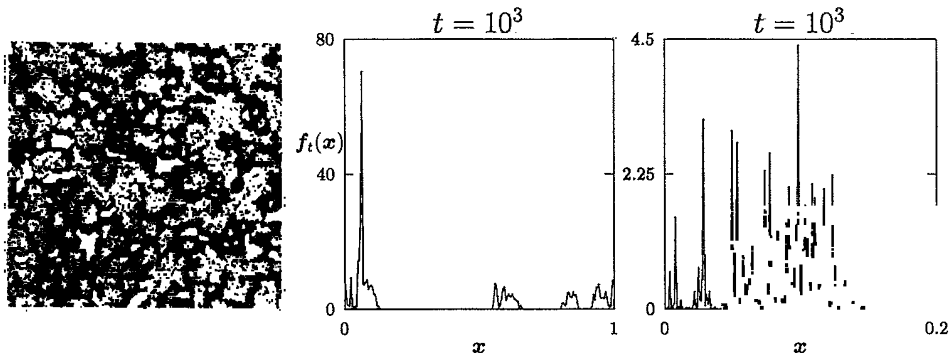

FIG. 4. Left: Gray scale representation of the state of a 200×200 lattice of diffusively coupled Keener maps without noise, at time $t=10^3$ with $a=0.5$, $b=0.571$, and $\varepsilon=0.45$. The 256-level gray scale is such that if $x^{(kl)}_{10^3}=0$ it is represented by a black pixel, while if $x^{(kl)}_{10^3}=1$ it is represented by a white pixel. Center panel: 200-bin histogram of the state of the lattice displayed on the left panel. The fractal nature of the support of this distribution is suggested by the right panel, which was obtained from a larger lattice (10^6 sites) and larger number of bins on [0,1] (10^3).

reflects the underlying asymptotic periodicity of the transfer operator (a property rigorously discussed in Sec. IV).

Figure 4 shows that the fractal nature of the attractor of the single map survives linear coupling. The effects of adding noise in the system are shown in Fig. 5 and it is clear that the activity of the lattice is no longer supported on a Cantor set (in $\mathbb{R}^N$). In addition, the temporal evolution of densities is reminiscent of the statistical cycling described above for the tent and logistic map lattices. Figure 5 provides a clear illustration of period-3 statistical cycling (also described as quasiperiodicity). It has been proposed (Sec. 3.2.2 of [18]) that deterministic CML's do not display such behavior. Figure 5 demonstrates that this observation does not hold for noisy CML's. In addition, the fact that we have not found period-3 cycling in the absence of noise in the tent and quadratic map lattices is probably due to the absence of statistical period 3 in the local transformations (9) and (10) for the parameters used here in the simulations. It is likely that when the local map does possess period-3 statistical cycling, the CML will display the same behavior (at least for small enough values of the coupling). An example of noisy period 3 in the logistic map is given by Lorenz [13]. Lukin and Shestopalov illustrate the same behavior in a map relevant to the modeling of resonant electromagnetic cavities with nonlinearly reflecting boundaries [31].

Note also that the stochastic perturbation of system (2) with (12) yields a (mathematically) simpler system than in the absence of noise. As mentioned above, the Perron-Frobenius operator associated with the deterministic map

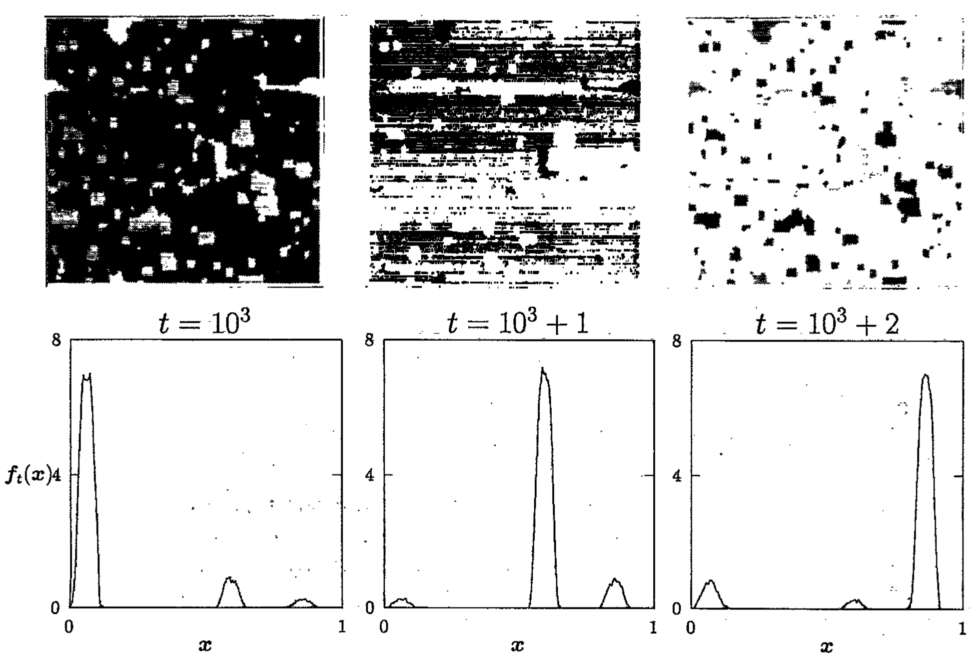

FIG. 5. Noise-induced statistical cycling in a lattice of 200×200 "Keener maps," with $a=0.5$, $b=0.571$, $\varepsilon=0.1$, and additive noise uniformly supported on [0,0.05]. The top panels display three successive iterations and the bottom panels display the corresponding histograms (produced with 200 bins). The gray scale for the top row is the same as in Fig. 4.

acts on (singular) measures, whereas the transfer operator associated with the stochastic map acts on densities. The results describing the dynamical properties of operators acting on densities are more numerous than those associated with measure evolving operators and, consequently, analytic investigations of the stochastic CML's are more straightforward than those of their deterministic counterparts (for a discussion of the evolution of measures under the action of linear operators, refer to Chap. 12 of [32]).

Before proceeding, we briefly summarize our numerical experiments. Deterministic CML's undergo the following, changes when subjected to stochastic perturbations.

(i) Not surprisingly, the addition (multiplication) of noise to (by) a system generating a deterministic periodic cycle yields a statistically periodic system if the noise amplitude is small and a statistically stable one if the noise is large enough to "wash out" the underlying periodicity. Similarly, the addition (multiplication) of noise to (by) a system generating a chaotic trajectory associated with a probability density yields a system that is either statistically periodic or statistically stable at equilibrium. Again, temporal periodicities in this case survive only small perturbations and disappear when the noise is sufficiently strong.

(ii) Perhaps less intuitive is the observation that stochastic perturbations (no matter how minute) of CML's can result in fundamental qualitative changes in both the topology of the attracting sets and the statistical evolution of the model. Section III B illustrates this by using a CML of Keener maps that possesses a Cantor set as an attractor in the absence of noise and cycles statistically (with period 3) in the presence of noise.

(iii) CML's that display quasiperiodicity when they are subjected to stochastic perturbations are also likely to display statistical hysteresis displayed in Fig. 2. In this case, the asymptotic value of the statistical quantifiers of the motion are seen to depend on the initial preparation of the system (this property is explained analytically in Sec. V).

In general, dynamical systems need not possess stable thermodynamic equilibrium states and it is therefore interesting to note that perturbations of CML's by noise yields systems that seem to often eventually reach equilibrium conditions, even if the notion of equilibrium must be extended to include statistical periodic cycles.

In Sec. V we examine these numerical observations analytically using the theory of Markov operators defined by stochastic kernels. As illustrated in the next section, these operators arise naturally in the thermodynamic description of stochastic CML's.

IV. THERMODYNAMICS OF CML'S

To clarify the notion of a thermodynamic state for CML's (stochastic or deterministic), suppose that the dynamics of a physical system are modeled by a (deterministic or stochastic) coupled map lattice denoted by $\mathcal{T}: \mathbf{X} \mapsto \mathbf{X}$ (many examples of such systems can be found in [33]). Suppose further that some observable $\mathcal{O}(\mathbf{x}_t)$, which depends on the state $\mathbf{x}_t$ of $\mathcal{T}$, is being measured at time t (the observable $\mathcal{O}$ is arbitrary, though it must be a bounded measurable function). The expectation value of this observable, denoted $E(\mathcal{O}_t)$, is the mean value of $\mathcal{O}(\mathbf{x}_t)$ when the measurement is repeated a large (ideally infinite) number of times. Mathematically it is given by

$$E(\mathcal{O}_t)=\int_{\mathbf{X}} f_t(\mathbf{y})\mathcal{O}(\mathbf{y})d\mathbf{y} , \qquad (13)$$

where $f_t(\mathbf{x})$ is the density of the variable $\mathbf{x}_t$, i.e., the probability $p(\mathbf{x}_t')$ of finding $\mathbf{x}_t$ between $\mathbf{x}_t'$ and $\mathbf{x}_t'+\delta\mathbf{x}_t'$ is

$$p(\mathbf{x}_t')=\int_{\mathbf{x}_t'}^{\mathbf{x}_t'+\delta\mathbf{x}_t'} f_t(\mathbf{y})d\mathbf{y} .$$

All thermodynamic functions that characterize the ensemble properties of a system are observables whose expectation values are defined by (13) since $\mathcal{O}$ was arbitrary. Therefore, the thermodynamic state of the CML $\mathcal{T}$ at time t is completely characterized by the density function f_t. Hence a complete description of the thermodynamics of $\mathcal{T}$ must focus on the behavior and properties of f_t. To this end, we introduce the transfer operator associated with $\mathcal{T}$, denoted $\mathcal{P}_{\mathcal{T}}$, which governs the time evolution of f_t (again, when $\mathcal{T}$ is a deterministic nonsingular discrete time map, $\mathcal{P}_{\mathcal{T}}$ is known as the Perron-Frobenius operator):

$$f_{t+1}(\mathbf{x})=\mathcal{P}_{\mathcal{T}} f_t(\mathbf{x}) , \quad t=0,1,\ldots . \qquad (14)$$

Of particular interest is the behavior of the sequence of densities $\{f_t\}$, which is intimately linked to the equilibrium and nonequilibrium properties of the CML. For example, $\mathcal{T}$ is ergodic if and only if the sequence is *weak Cesàro convergent* to the invariant density $f_*(\mathbf{x})$,

$$\lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1}\int_{\mathbf{X}} f_k(\mathbf{x})q(\mathbf{x})d\mathbf{x}=\int_{\mathbf{X}} f_*(\mathbf{x})q(\mathbf{x})d\mathbf{x} \quad \text{for all } q\in L^{\infty}(\mathbf{X}) ,$$

and all initial probability densities $f_0(\mathbf{x})$. A stronger (but familiar) property—mixing—is equivalent to the *weak convergence* of the sequence to f_*:

$$\lim_{t\to\infty}\int_{\mathbf{X}} f_t(\mathbf{x})q(\mathbf{x})d\mathbf{x}=\int_{\mathbf{X}} f_*(\mathbf{x})q(\mathbf{x})d\mathbf{x} \quad \text{for all } q\in L^{\infty}(\mathbf{X})$$

and all initial probability densities $f_0(\mathbf{x})$. An even stronger type of chaotic behavior, known as *exactness* (or asymptotic stability) is reflected by the *strong convergence* of the sequence $\{f_t\}$ to the invariant density f_*:

$$\lim_{t\to\infty}\|\mathcal{P}_{\mathcal{T}} f_t - f_*\|_{L^1}$$

for all initial probability densities $f_0(\mathbf{x})$. Exactness implies mixing and is interesting from a physical point of view because it is the only one of the properties discussed so far that guarantees the evolution of the thermodynamic entropy of $\mathcal{T}$ to a global maximum, irrespective of the initial condition f_0 [34].

The hierarchy of chaotic behaviors

$$\text{exactness} \Longrightarrow \text{mixing} \Longrightarrow \text{ergodicity}$$

is discussed here because it is shown in Sec. V that a large class of stochastic CML's is either exact or possesses another dynamical property, known as *asymptotic periodicity,* of which exactness is a special case.

Asymptotic periodicity is a property of certain Markov operators that ensures that the density sequence $\{f_t\}$ converges strongly to a periodic cycle [recall that $\mathcal{P}$ is a Markov operator if it is linear and if for all probability densities f it satisfies (i) $\mathcal{P}f \geq 0$ for $f \geq 0$ and (ii) $\|\mathcal{P}f\|_{L^1} = \|f\|_{L^1}$].

Definition 1: Asymptotic periodicity. A Markov operator $\mathcal{P}$ is asymptotically periodic if there exist finitely many distinct probability density functions $v_1, \ldots, v_r$ with disjoint supports, a unique permutation γ of the set $\{1, \ldots, \}$, and positive linear continuous functionals $\Gamma_1, \ldots, \Gamma_r$, on $L_1(\mathbf{X})$ such that, for almost all initial densities f_0,

$$\lim_{t\to\infty} \left\| \mathcal{P}^t \left[f_0 - \sum_{i=1}^{r} \Gamma_i[f_0] v_i \right] \right\|_{L^1} = 0 \tag{15}$$

and

$$\mathcal{P}v_i = v_{\gamma(i)}, \quad i = 1, \ldots, r .$$

Clearly, if $\mathcal{P}$ satisfies these conditions with $r=1$, it is exact (or *asymptotically* stable). If $r>1$ and the permutation γ is cyclical, asymptotic periodicity also implies ergodicity [34]. A rigorous discussion of asymptotically periodic Markov operators is given in [35]. A more intuitive presentation is given in [32]. ■

The dependence of the functionals Γ_i on the initial density f_0 implies a dependence on the initial conditions much stronger than that usually discussed in relation to chaotic dynamical systems. Here the ensemble properties depend on the initial ensemble. Though there appears to be no rigorous characterization of this sensitivity, extensive numerical experiments suggest that apparently "similar" initial densities can give rise to very different cycles [the difference being always confined to the coefficients Γ_i weighing the various v_i's in the linear combination (15)].

V. ANALYTIC RESULTS

Section III numerically illustrated the influence of noise on the evolution of several CML's. The numerics demonstrated the ubiquity of statistical cycling and that of the statistical hysteresis displayed in Fig. 2. In this section, these results are examined in light of a theoretical framework based on the properties of the transfer operator (denote by $\mathcal{P}_{\mathcal{T}}$ above). Using basic results from the theory of functions of bounded variation, the Perron-Frobenius operator associated with certain deterministic chaotic CML's was shown in [12] to be asymptotically periodic. The approach presented here to determine the spectral characteristics of the Perron-Frobenius operator associated with stochastic CML's is somewhat more "indirect" since it relies on proving a property, known as *constrictiveness* (defined in the Appendix A) which, by a theorem due to Komorník [35], implies asymptotic periodicity and the decomposition (15).

First, we derive the evolution equation for the density f_t introduced in Sec. IV, for CML's perturbed by various types of noise. This evolution equation implicitly defines the transfer operator. Second, it is shown that $\mathcal{P}_{\mathcal{T}}$ is a Markov operator defined by a stochastic kernel and we use this fact to investigate the convergence properties of the sequence $\{f_t\}$. The relation between these properties and the numerical results of Sec. III is then discussed.

Before proceeding, we remind the reader that a function $K: \mathbf{X}\times\mathbf{X} \mapsto \mathbb{R}$ is a stochastic kernel if it satisfies

$$K(\mathbf{x},\mathbf{y}) \geq 0 , \quad \int_{\mathbf{X}} K(\mathbf{x},\mathbf{y})d\mathbf{x} = 1 .$$

(The integral is understood as a Lebesgue integral with respect to the Lebesgue measure, but in general the measure of integration can be different. Consult [32] for details.) To simplify the algebra the model (2) is replaced by a one-dimensional version so that the double superscripts of (2) and (4) are replaced by a single space index denoted i,

$$\begin{aligned} x_{t+1}^{(i)} &= \Phi^{(i)}(\mathbf{x}_t) + \xi_t^{(i)} \\ &= (1-\varepsilon)S(x_t^{(i)}) + \frac{\varepsilon}{p} \sum_{\substack{p \text{ nearest} \\ \text{neighbors}}} S(x_t^{(j)}) + \xi_t^{(i)} , \\ & \qquad \varepsilon \in (0,1) . \end{aligned}$$

The boundary conditions need not be specified explicitly. The only requirement they must meet is that the evolution of the elements on the boundaries be well defined (the frequently encountered periodic and no-flux boundary conditions obviously satisfy this requirement.)

A. Additive noise

In this case, the lattice transformation is (4). Consider an initial density f_0: $\mathbf{X} \mapsto \mathbb{R}$ that describes an ensemble of initial lattices. If the noise perturbation is distributed according to density g [cf. (3)] the evolution equation for this density is [30]

$$f_{t+1}(\mathbf{x}) = \int_{\mathbf{X}} f_t(\mathbf{y}) g(\mathbf{x} - \Phi(\mathbf{y})) d\mathbf{y} , \quad n = 0, 1, \ldots , \tag{16}$$

which also defines the transfer operator $\mathcal{P}_{\Phi_{\text{add}}}$ for CML's perturbed as in (4) since $\mathcal{P}_{\phi_{\text{add}}} f_t(\mathbf{x}) = f_{t+1}(\mathbf{x})$. Without loss of generality, in the remainder of this section we will assume that the density associated with the stochastic perturbation is piecewise constant and given by

$$g(\xi) = \prod_{i=1}^{N} \chi_{[b,c]}(\xi^{(i)}) , \quad 0 \leq b < c \leq 1 , \tag{17}$$

where the indicator function χ is defined by $\chi_{[b,c]}(x) = (c-b)^{-1}$ if $x \in [b,c]$ and $\chi_{[b,c]}(x) = 0$ otherwise.

1. Statement of the result

If the CML Φ_{add} is written in the form (4), where the density of the perturbation ξ is given by (17), and the local map S of (2) is bounded and nonsingular, then $\mathcal{P}_{\Phi_{\text{add}}}$ defined by (16) is symptotically periodic. This is a consequence of the following theorem.

Theorem 1. (Lasota and Mackey [32]). Let $K: \mathbb{X}\times\mathbb{X}\mapsto\mathbb{R}$ be a stochastic kernel and $\mathcal{P}$ a Markov operator defined by

$$\mathcal{P}f(\mathbf{x})=\int_{\mathbb{X}}K(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y}\ . \tag{18}$$

Assume that there is a non-negative $\lambda<1$ such that for every bounded $\mathbb{B}\subset\mathbb{X}$ there is a $\delta=\delta(\mathbb{B})>0$ for which

$$\int_{\mathbb{A}}K(\mathbf{x},\mathbf{y})d\mathbf{x}\leq\lambda\quad\text{for } \mu(\mathbb{A})<\delta\ ,\ \mathbf{y}\in\mathbb{B}\ ,\ \mathbb{A}\subset\mathbb{B}\ . \tag{19}$$

Assume further that there exists a Lyapunov function $V: \mathbb{X}\mapsto\mathbb{R}$ such that

$$\int_{\mathbb{X}}V(\mathbf{x})\mathcal{P}f(\mathbf{x})d\mathbf{x}\leq\alpha\int_{\mathbb{X}}V(\mathbf{x})f(\mathbf{x})d\mathbf{x}+\beta\ ,\qquad \alpha\in[0,1)\ ,\ \beta>0 \tag{20}$$

for every density f. Then $\mathcal{P}$ is asymptotically periodic and therefore admits the representation (15). [Recall that a non-negative function $V: \mathbb{X}\mapsto\mathbb{R}$ is known as a *Lyapunov function* if it satisfies $\lim_{|\mathbf{x}|\to\infty}V(\mathbf{x})=\infty$.]

To show that the operator $\mathcal{P}_{\Phi_{\text{add}}}$ defined in (16) satisfies the conditions of Theorem 1, note that (16) can be written in the form (18) with

$$K(\mathbf{x},\mathbf{y})=g(\mathbf{x}-\Phi(\mathbf{y}))\ . \tag{21}$$

Clearly $g>0$ and since it is a normalized probablity density, $\int_{\mathbb{X}}(K(\mathbf{x},\mathbf{y})d\mathbf{x}=1$, so (21) defines a stochastic kernel. In addition, it is straightforward to show that $\mathcal{P}_{\Phi_{\text{add}}}$ is a Markov operator. To verify that (19) holds when K is given by (21), remember that since g is integrable, for every $\lambda>0$, there is a $\delta>0$ such that

$$\int_{\mathbb{A}}g(\mathbf{x})d\mathbf{x}<\lambda\quad\text{for } \mu(\mathbb{A})<\delta\ .$$

Hence

$$\int_{\mathbb{A}}K(\mathbf{x},\mathbf{y})d\mathbf{x}=\int_{\mathbb{A}}g(\mathbf{x}-\Phi(\mathbf{y}))d\mathbf{x}=\int_{\mathbb{A}-\Phi(\mathbf{y})}g(\mathbf{x})d\mathbf{x}<\lambda$$

for $\mu(\mathbb{A}(\mathbf{y}))=\mu(\mathbb{A})<\delta$. Thus (19) holds for all bounded sets $\mathbb{B}$.

We now check that when S is bounded, (20) holds. To do this we pick

$$V(\mathbf{x})=\prod_{i=1}^{N}\exp|x^{(i)}|\ . \tag{22}$$

Using (17) and (22), one obtains

$$\begin{aligned}\int_{\mathbb{X}}g(\mathbf{x}-\Phi(\mathbf{y}))V(\mathbf{x})d\mathbf{x}&=\int_{\mathbb{X}}\prod_{i=1}^{N}\chi_{b,c}(x^{(i)}=\Phi^{(i)}(\mathbf{y}))V(\mathbf{x})d\mathbf{x}\\&=(c-b)^{-N}\prod_{ki=1}^{N}\int_{b+\Phi^{(i)}(\mathbf{y})}^{c+\Phi^{(i)}(\mathbf{y})}V(x^{(i)})dx^{(i)}\\&=\left[\frac{e^{c}-e^{b}}{c-b}\right]^{N}\prod_{i=1}^{N}[\exp|\Phi^{(i)}(\mathbf{y})|]\\&=\left[\frac{e^{c}-e^{b}}{c-b}\right]^{N}\prod_{i=1}^{N}[\exp|S(y^{(i)})|]\ .\end{aligned} \tag{23}$$

By definition, S is bounded, so $\prod_{i=1}^{N}[\exp|S(y^{(i)})|]$ is finite. As a result, it is always possible to choose $\alpha\in(0,1)$ and a finite but arbitrarily large β such that

$$\left[\frac{e^{c}-e^{b}}{c-b}\right]^{N}\prod_{i=1}^{N}[\exp|S(y^{(i)})|]\leq\alpha\prod_{i=1}^{N}[\exp|y^{(i)}|]+\beta\ .$$

Thus we have

$$\int_{\mathbb{X}}g(x-\Phi(y))V(\mathbf{x})d\mathbf{x}\leq\alpha V(\mathbf{y})+\beta\ ,\qquad \alpha\in(0,1)\ ,\ \beta>0\ , \tag{24}$$

which implies (20). Hence all conditions of Theorem 1 are met and $\mathcal{P}_{\Phi_{\text{add}}}$ is asymptotically periodic.

This is a general result. The two main assumptions that are necessary for its derivation are that S be nonsingular and bounded. In light of this result, the numerics of Sec. III on the dynamics of the tent, quadratic, and Keener map lattices are seen to reflect the cyclical spectral decomposition (15) of the transfer operator associated with these CML's. In particular, the statistical hysteresis of Fig. 2 is a consequence of the dependence in (15) of the functionals Γ_i on the initial density f_0.

The presence of asymptotic periodicity in systems of the form (4) also has important consequences for the thermodynamics of CML's and for the proper interpretation of the behavior of their statistical quantifiers. Before discussing these consequences in detail, we consider the behavior of CML's perturbed by multiplicative noise.

B. Multiplicative noise

Here the transformation Φ_{mul} is given by (5). In this section it is proved that the effects of multiplicative noise on CML dynamics are usually similar to the effects of additive noise discussed above. However, the formalism describing these two situations is different since the mechanisms by which additive and multiplicative perturbations operate are themselves different. Our discussion is inspired by the treatment of the effects of parametric noise on one dimensional maps given by Horbacz [36].

To derive the expression for the operator that governs the evolution of ensemble densities, we introduce an arbitrary bounded measurable function $h: \mathbb{X}\mapsto\mathbb{R}$, which can be written

$$h(\mathbf{x})=\prod_{i=1}^{N} h^{(i)}(x^{(i)}) .$$

The expectation value of $H(\mathbf{x}_{t+1})$ is given by

$$E(h(\mathbf{x}_{t+1}))=\int_{\mathbf{X}} h(\mathbf{x})f_{t+1}(\mathbf{x})d\mathbf{x} . \tag{25}$$

However, from (5), we have

$$E(h(\mathbf{x}_{t+1}))=E(h(\boldsymbol{\Phi}_{\mathrm{mul}}(\mathbf{x}_t)))$$
$$=\int_{\mathbf{X}}\int_{\mathbf{X}} f_t(\mathbf{y})\prod_{i=1}^{N} h^{(i)}(z^{(i)}\Phi^{(i)}(\mathbf{y}))\times g(z^{(i)})d\mathbf{z}\,d\mathbf{y} . \tag{26}$$

Equation (26) can be simplified by recalling the following identity. Let Φ: $\mathbf{X}\mapsto\mathbf{X}$ be a nonsingular dynamical system that induces the Perron-Frobenius operator $\mathcal{P}_\Phi$. Then for all L^1 functions $u:\mathbf{X}\mapsto\mathbb{R}$ and an L^∞ function $v:\mathbf{X}\mapsto\mathbb{R}$, we have

$$\int_{\mathbf{X}}\mathcal{P}_\Phi u(\mathbf{x})v(\mathbf{x})d\mathbf{x}=\int_{\mathbf{X}} u(\mathbf{x})v(\Phi(\mathbf{x}))d\mathbf{x} . \tag{27}$$

The operator $\mathcal{K}_\Phi$ such that

$$\mathcal{K}_\Phi v(\mathbf{x})=v(\Phi(\mathbf{x})) , \quad \text{for all } \mathbf{x}\in\mathbf{X} \text{ and all } v\in L^\infty$$

is called the *Koopman* operator induced by Φ. Equation (27) expresses the fact that $\mathcal{P}_\Phi$ and $\mathcal{K}_\Phi$ are adjoint and is written

$$\langle \mathcal{P}_\Phi u,v\rangle=\langle u,\mathcal{K}_\Phi v\rangle ,$$

where $\langle\ ,\ \rangle$ denotes the scalar product between two functions.

Applying this identity to Eq. (26) yields

$$E(h(\mathbf{x}_{t+1}))=\int_{\mathbf{X}}\int_{\mathbf{X}}\mathcal{P}_\Phi f_t(\mathbf{y})\prod_{i=1}^{N} h^{(i)}(z^{(i)}y^{(i)})\times g(z^{(i)})d\mathbf{z}\,d\mathbf{y} , \tag{28}$$

where $\mathcal{P}_\Phi$ denotes the Perron-Frobenius operator induced by the deterministic CML Φ. Now let

$$z^{(i)}y^{(i)}=x^{(i)}\Longrightarrow dz^{(i)}=\frac{ds^{(i)}}{y^{(i)}} .$$

Using this change of variables, (28) becomes

$$E(h(\mathbf{x}_{t+1}))=\int_{\mathbf{X}}\int_{\mathbf{X}'}\mathcal{P}_\Phi f_t(\mathbf{y})\prod_{i=1}^{N}\left[h^{(i)}(x^{(i)})g\left[\frac{x^{(i)}}{y^{(i)}}\right]\frac{1}{y^{(i)}}\right]d\mathbf{x}\,d\mathbf{y} \tag{29}$$

where $\mathbf{X}'=X_1'\times\cdots\times X_N'$ is a product space with each interval in the product defined to be $I_i\equiv[0,y^{(i)}]$. Changing the order of integration in (29) yields

$$E(h(\mathbf{x}_{t+1}))=\int_{\mathbf{X}}\int_{\mathbf{Y}'}\mathcal{P}_\Phi f_t(\mathbf{y})\prod_{i=1}^{N}\left[h^{(i)}(x^{()})g\left[\frac{x^{(i)}}{y^{(i)}}\right]\frac{1}{y^{(i)}}\right]d\mathbf{y}\,d\mathbf{x} , \tag{30}$$

where $\mathbf{Y}'$ is a product space of intervals $Y_i'\equiv[x^{(i)},1]$. By assumption h is arbitrary. Therefore, comparing (30) with (25) and (28), we obtain

$$f_{t+1}(\mathbf{x})\equiv\mathcal{P}_{\Phi_{\mathrm{mul}}}f_t(\mathbf{x})=\int_{x^{(N)}}^{1}\cdots\int_{x^{(1)}}^{1}\mathcal{P}_\Phi f_t(\mathbf{y})\prod_{i=1}^{N}\left[g\left[\frac{x^{(i)}}{y^{(i)}}\right]\frac{1}{y^{(i)}}\right]d\mathbf{y} \tag{31}$$

$$=\int_{x^{(N)}}^{1}\cdots\int_{x^{(1)}}^{1}f_t(\mathbf{y})\prod_{i=1}^{N}\left[g\left[\frac{x^{(i)}}{\Phi^{(i)}(\mathbf{y})}\right]\frac{1}{\Phi^{(i)}(\mathbf{y})}\right]d\mathbf{y} , \tag{32}$$

which is the expression for the Perron-Frobenius operator induced by the stochastic CML (5). We are now in a position to discuss the asymptotic properties of the iterates of $\mathcal{P}_{\Phi_{\mathrm{mul}}}$.

1. Statement of result

A CML of the form (5), perturbed by the noise term ξ_t distributed with density (3) will induce a transfer operator $\mathcal{P}_{\Phi_{\mathrm{mul}}}$ defined by (32). If the deterministic part of the transformation (denoted Φ) is bounded and nonsingular, then $\mathcal{P}_{\Phi_{\mathrm{mul}}}$ is asymptotically periodic. This result follows from the application of the following theorem, proved in the Appendix.

Theorem 2. Let $K:\mathbf{X}\times\mathbf{X}\mapsto\mathbb{R}$ be a stochastic kernel and $\mathcal{P}$ the Markov operator defined by

$$\mathcal{P}_f(\mathbf{x})=\int_{x^{(N)}}^{1}\cdots\int_{x^{(1)}}^{1}K(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y} , \tag{33}$$

and assume that inequalities (19) and (20) are satisfied. Then $\mathcal{P}$ is asymptotically periodic.

In the remainder of this section, it is shown that $\mathcal{P}_{\Phi_{\mathrm{mul}}}$, defined in (32), is of the form (33), that the corresponding kernel satisfies (19), and that inequality (20) holds if Φ is bounded.

To see that $\mathcal{P}_{\Phi_{\mathrm{mul}}}$ is a Markov operator, note first that from (32)

$$
\begin{aligned}
\int_{\mathbf{X}} \mathcal{P}_{\Phi_{\text{mul}}} f(\mathbf{x}) d\mathbf{x} &= \int_{\mathbf{X}} \int_{x^{(N)}}^{1} \cdots \int_{x^{(1)}}^{1} \mathcal{P}_{\Phi} f(\mathbf{y}) \prod_{i=1}^{N} \left[g\left[\frac{x^{(i)}}{y^{(i)}}\right] \frac{1}{y^{(i)}} \right] d\mathbf{y}\, d\mathbf{x} \\
&= \int_{\mathbf{X}} \int_{0}^{y^{(N)}} \cdots \int_{0}^{y^{(1)}} \mathcal{P}_{\Phi} f(\mathbf{y}) \prod_{i=1}^{N} \left[g\left[\frac{x^{(i)}}{y^{(i)}}\right] \frac{1}{y^{(i)}} \right] d\mathbf{x}\, d\mathbf{y} \\
&= \int_{\mathbf{X}} \mathcal{P}_{\Phi} f(\mathbf{y}) \left\{ \int_{0}^{y^{(N)}} \cdots \int_{0}^{y^{(1)}} \prod_{i=1}^{N} \left[g\left[\frac{x^{(i)}}{y^{(i)}}\right] \frac{1}{y^{(i)}} \right] d\mathbf{x} \right\} d\mathbf{y} \\
&= \int_{\mathbf{X}} \mathcal{P}_{\Phi} f(\mathbf{y}) \left\{ \int_{0}^{1} \cdots \int_{0}^{1} \prod_{i=1}^{N} [g(z^{(i)})] d\mathbf{z} \right\} d\mathbf{y} \\
&= \int_{\mathbf{X}} \mathcal{P}_{\Phi} f(\mathbf{y}) d\mathbf{y} \\
&= \int_{\mathbf{X}} f(\mathbf{y}) d\mathbf{y} = 1
\end{aligned} \tag{34}
$$

since $\mathcal{P}_{\Phi}$ is itself a Markov operator. Clearly $\mathcal{P}_{\Phi_{\text{mul}}}$ is linear and this completes the demonstration that it is Markovian. Furthermore, from (32), $\mathcal{P}_{\Phi_{\text{mul}}}$ can be written as in (33) with the kernel given by

$$
K(\mathbf{x},\mathbf{y}) = \prod_{i=1}^{N} \left[g\left[\frac{x^{(i)}}{\Phi^{(i)}(\mathbf{y})} \right] \frac{1}{\Phi^{(i)}(\mathbf{y})} \right] . \tag{35}
$$

Since g is a normalized probability density on $\mathbf{X}$, $K(\mathbf{x},\mathbf{y}) \geq 0$ and $\int_{\mathbf{X}} K(\mathbf{x},\mathbf{y}) d\mathbf{x} = 1$ and therefore K is a stochastic kernel.

To verify that inequality (19) holds when $K(\mathbf{x},\mathbf{y})$ is given by (35), fix an arbitrary $\lambda < 1$. Choose $\mathbb{B} \subset \mathbf{X}$, bounded, and $\mathbb{A} \subset \mathbb{B}$. The function g is integrable and so there must be $\delta_1 > 0$ such that

$$
\int_{\mathbb{A}} g(\mathbf{x}) d\mathbf{x} \leq \lambda \quad \text{for } \mu(\mathbb{A}) < \delta_1 \text{ , } \mathbb{A} \subset \mathbb{B} \text{ .} \tag{36}
$$

Define

$$
\delta = \delta_1 \inf_{\substack{\mathbf{y} \in \mathbb{B} \\ i=1,\ldots,N}} \Phi^{(i)}(\mathbf{y}) \text{ .}
$$

Since $\mathbf{X} \subset \mathbb{R}^N$ is finite, the set $\mathbb{A}$ is a direct product of the form $\mathbb{A} = \prod_{i=1}^{N} A_i$, where each $A_i \subset \mathbb{R}$ is also finite. Denote by $\mathbb{A} - \Phi(\mathbf{y})$ the direct product $\prod_{i=1}^{N} A_i / \Phi^{(i)}(\mathbf{y})$. If $\mu(\mathbb{A}) < \delta$, $\mu(\mathbb{A}/\Phi(\mathbf{y})) < \delta_1$ and therefore

$$
\begin{aligned}
\int_{\mathbb{A}} K(\mathbf{x},\mathbf{y}) d\mathbf{x} &= \int_{\mathbb{A}} \prod_{i=1}^{N} \left[g^{(i)} \left[\frac{x^{(i)}}{\Phi^{(i)}(\mathbf{y})} \right] \frac{1}{\Phi^{(i)}(\mathbf{y})} \right] d\mathbf{x} \\
&= \int_{\mathbb{A} - \Phi(\mathbf{y})} g(\mathbf{x}) d\mathbf{x} \\
&\leq \lambda
\end{aligned}
$$

by (36). This verifies inequality (19).

If Φ is bounded, it is possible to choose $\alpha_1 \in (0,1)$ and$\beta_1 > 0$ such that

$$
0 < \prod_{i=1}^{N} \Phi^{(i)}(\mathbf{x}) \leq \alpha_1 \prod_{i=1}^{N} x^{(i)} + \beta_1 \quad \text{for } x^{(i)} \in [0,1] \text{ and all } i \text{ .} \tag{37}
$$

We now show that when Φ satisfies (37) the inequality (20) is valid. Choose a Lyapunov function $V(\mathbf{xx}) = \prod_{i=1}^{N} x^{(i)}$. From (32),

$$
\begin{aligned}
\int_{\mathbf{X}} V(\mathbf{x}) \mathcal{P}_{\Phi_{\text{mul}}} f(\mathbf{x}) d\mathbf{x} &= \int_{\mathbf{X}} \prod_{i=1}^{N} x^{(i)} \int_{x^{(N)}}^{1} \cdots \int_{x^{(1)}}^{1} f(\mathbf{y}) \prod_{i=1}^{N} \left[g\left[\frac{x^{(i)}}{\Phi^{(i)}(\mathbf{y})} \right] \frac{1}{\Phi^{(i)}(\mathbf{y})} \right] d\mathbf{y}\, d\mathbf{x} \\
&= \int_{\mathbf{X}} \int_{x^{(N)}}^{1} \cdots \int_{x^{(1)}}^{1} \mathcal{P}_{\Phi} f(\mathbf{y}) \prod_{i=1}^{N} \left[g\left[\frac{x^{(i)}}{y^{(i)}} \right] \frac{x^{(i)}}{y^{(i)}} \right] d\mathbf{y}\, d\mathbf{x} \\
&= \int_{\mathbf{X}} \int_{0}^{y^{(N)}} \cdots \int_{0}^{y^{(1)}} \mathcal{P}_{\Phi} f(\mathbf{y}) \prod_{i=1}^{N} \left[g\left[\frac{x^{(i)}}{y^{(i)}} \right] \frac{x^{(i)}}{y^{9i)}} \right] d\mathbf{x}\, d\mathbf{y} \\
&= \int_{\mathbf{X}} \mathcal{P}_{\Phi} f(\mathbf{y}) \prod_{i=1}^{N} y^{(i)} \int_{0}^{1} \cdots \int_{0}^{1} \prod_{i-1}^{N} [g(z^{(i)}) z^{(i)}] d\mathbf{z}\, d\mathbf{y} \\
&= \int_{\mathbf{X}} f(\mathbf{y}) \prod_{i=1}^{N} \Phi^{(i)}(y) \int_{\mathbf{X}} \prod_{i=1}^{N} [g(z^{(i)}) z^{(i)}] d\mathbf{z}\, d\mathbf{y} \text{ ,}
\end{aligned} \tag{38}
$$

where the change of variables $z^{(i)}=x^{(i)}/y^{(i)}$ was used between the third and fourth equalities. Since $\mathbf{X}$ is the N-dimensional unit hypercube,

$$\langle \mathbf{z}\rangle=\int_{\mathbf{X}}\prod_{i=1}^{n}[g(z^{(i)})z^{(i)}]d\mathbf{z}<1 .$$

Therefore, from (37)

$$\langle \mathbf{z}\rangle\int_{\mathbf{X}}f(\mathbf{y})\prod_{i=1}^{N}\Phi^{(i)}(\mathbf{y})\le\langle \mathbf{z}\rangle\alpha_1\int_{\mathbf{X}}f(\mathbf{y})V(\mathbf{y})d\mathbf{y}+\langle \mathbf{z}\rangle\beta_1$$

and, as a consequence, there is a $\alpha=\alpha_1\langle \mathbf{z}\rangle\in[0,1)$ and a $\beta=\beta_1\langle \mathbf{z}\rangle>0$ such that

$$\int_{\mathbf{X}}V(\mathbf{x})\mathcal{P}_{\Phi_{\rm mul}}f(\mathbf{x})d\mathbf{x}\le\alpha\int_{\mathbf{X}}V(\mathbf{x})f(\mathbf{x})d\mathbf{x}+\beta ,$$

thus proving that (20) is satisfied. To summarize, all the conditions of Theorem 2 are met by the transfer operator associated with the stochastic coupled map lattice (5) and it is therefore asymptotically periodic when the deterministic part of this CML satisfies condition (37).

So far, we have considered the statistical behavior of CML's perturbed by noise at each and every time step. As mentioned in Sec. II A, these perturbations are known as "constantly applied perturbations." There is another class of perturbations, known as "randomly applied," which were considered in [32]. These results are briefly reviewed in the next subsection.

C. Randomly applied perturbations

These perturbations are "strong" in the sense that when they are applied at a time t_*, the value $x_{t_*+1}^{(kl)}$ becomes independent of its preimages. Mathematically, a CML Φ with randomly applied perturbations is written

$$x_{t+1}^{kl}=\Phi_{\rm ran}^{(kl)}(\mathbf{x}_t)=v_t^{(kl)}\Phi(\mathbf{x}_t)+(1-v_t^{(kl)})\xi_t^{(kl)} , \quad (39)$$

where $v_t^{(kl)}$ is a random variable that takes the two values 0 or 1 with the probabilities

$$\text{prob}(v_t^{(kl)}=1)=(1-q) , \quad \text{prob}(v_t^{(kl)}=0)=q ,$$

where $q\in(0,1]$ is a control parameter, which is itself distributed with density $\mathbf{g}$. Randomly applied perturbations are in a category that is apart from the (more familiar) constant perturbations of Sec. II A. However, their influence on CML dynamics can be investigated using analytical tools related to those presented in the previous sections. In fact, there are strong results concerning the behavior of the iterates of the transfer operator for these systems[32].

Systems of the form (2) with randomly applied perturbations are *always* asymptotically stable. To see this, note that the transfer operator for (39) is [32]

$$\mathcal{P}_{\Phi_{\rm ran}}f=(1-q)\mathcal{P}_\Phi f+qg . \quad (40)$$

Clearly, $\mathcal{P}_{\Phi_{\rm ran}}f>qg$, which implies that qg is a nontrivial lower bound function for $\mathcal{P}_{\Phi_{\rm ran}}f$ (since $q>0$) and this in turn implies [32] that there exists a unique density f_* such that $\mathcal{P}_{\Phi_{\rm ran}}f_*=f_*$, and

$$\lim_{n\to\infty}\|\mathcal{P}_{\Phi_{\rm ran}}^n f-f_*\|_{L^1}=0 \quad \text{for every density } f . \quad (41)$$

This property (exactness, sometimes referred to as asymptotic stability) implies mixing (which in turn implies ergodicity) of the CML. Hence we have the rather general result that the slightest perturbation of any nonsingular CML by a stochastic term as in (39) yields a system that is always exact, irrespective of the statistical properties of the original lattice transformation.

It is possible to go one step further in our discussion of randomly perturbed CML's and give an exact expression for the invariant density f_* of the operator $\mathcal{P}_{\Phi_{\rm ran}}$. Recall that if $\mathcal{P}$ is a Markov operator, it satisfies

$$\|\mathcal{P}f\|_{L^1}\le\|f\|_{L^1} \quad \text{for all } f\in L^1 \quad (42)$$

(this property is known as the *contractiveness* of Markov operators, not to be confused with *constrictiveness* defined in the Appendix). Applying (2) to the Perron-Frobenius operator $\mathcal{P}_\Phi$ associated with the deterministic transformation gives $\|(1-q)^k\mathcal{P}_\Phi^k g\|_{L^1}\le(1-q)^k\|g\|_{L^1}$. Hence the series $\sum_{k=0}^{\infty}(1-q)^k\mathcal{P}_\Phi^k g$ is absolutely convergent. Substituting this series into the expression (40) yields

$$\begin{aligned}\mathcal{P}_{\Phi_{\rm ran}}q\sum_{k=0}^{\infty}(1-q)^k\mathcal{P}_\Phi^k g&=(1-q)\mathcal{P}_\Phi\left[q\sum_{k=0}^{\infty}(1-q)^k\mathcal{P}_\Phi^k g\right]+qg\\&=q\left[\sum_{k=0}^{\infty}(1-q)^k\mathcal{P}_\Phi^k g-g\right]+qg\\&=q\sum_{k=0}^{\infty}(1-q)^k\mathcal{P}_\Phi^k g . \quad (43)\end{aligned}$$

In other words, if $f_*=\sum_{k=0}^{\infty}(1-q)^k\mathcal{P}_\Phi^k g$, we have $\mathcal{P}_{\Phi_{\rm ran}}f_*=f_*$, and f_* describes the state of thermodynamic equilibrium of the CML (39).

VI. DISCUSSION

The presence of asymptotic periodicity in CML's has important consequences for our understanding of their thermodynamic behavior. It is well known [37] that if a transformation $\mathcal{T}$ is ergodic, the average of an observable $\mathcal{O}$ along a trajectory $\{\mathbf{x}_t\}_0^\infty$ of $\mathcal{T}$ is equal to its average with respect to the probability density of occupation of the phase space (the so-called ensemble density). Mathematically, if f_* denotes the invariant ensemble density, we have

$$\lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{t-1}\mathcal{O}(\mathbf{x}_t)=\int_{\mathbf{X}}\mathcal{O}(\mathbf{x})f_*(\mathbf{x})d\mathbf{x} . \quad (44)$$

If, as in all the numerical examples of Sec. III, $\mathcal{T}$ is asymptotically periodic with γ of (15), a cyclical permutation of the set $\{1,\dots,r\}$, then $\mathcal{T}$ is also ergodic and (44) holds. However, almost all initial densities converge in a strong sense (cf. Definition 1) to a cycle in density space. Hence, from relation (15), if a sequence of average $\{E(\mathcal{O}_t)\}$ is computed with respect to a sequence of density iterates $\{f_t\}$, where $f_t=\mathcal{P}_\Phi^t(f_0)$, then $\{E(\mathcal{O}_t)\}$ will, for almost all f_0, be time periodic (for t large enough). This oscillatory behavior is the origin of the statistical cycling described numerically in Sec. III. On the other hand, if statistics are computed with respect to a single trajectory the time dependence is lost because in this case the statistical quantifiers are time averaged. Thus, for practical applications, when considering ergodic systems that are also asymptotically periodic (which, by the results of Sec. III, are not uncommon), computing phase space averages will not, in general, yield the same answer as computing time averages. There is no inconsistency here since ergodic systems are not guaranteed to converge (in a strong sense; cf. Sec. IV) to their invariant measure when they are prepared out of equilibrium. The presence of asymptotic periodicity renders this apparently technical observation relevant because systems possessing this property are likely *not* to converge to their invariant measure.

As discussed in Sec. IV, the notion of equilibrium is usually associated in statistical mechanics with that of an invariant measure f_*, which describes the ensemble properties of a given physical system. The results presented here indicate that this paradigm can sometimes be too restrictive and needs to be extended for the proper description of the thermodynamics of nonlinear stochastic spatially extended systems modeled by coupled map lattices. More precisely, the results of Sec. V A 1 and V B 1 suggest that the notion of a "thermodynamic equilibrium state" for stochastic CML's should be extended to finite sets of states that are visited sequentially in time. The considerations of Sec. IV show that this statistical cycling at equilibrium reflects the underlying asymptotic periodicity of the transfer operator and provide an explanation for the presence of phase transitions in stochastic lattices. The period r of the statistical cycle (15) depends on the parameters of the transformation and when this period changes, the lattice undergoes a phase transition.

Another interesting consequence of asymptotic periodicity comes from the dependence of the Γ_i's of Definition 1 on the initial density f_0 since it implies a dependence on the initial conditions that is stronger than that usually discussed in reference to chaotic dynamics. Thus the ensemble properties of an asymptotically periodic CML depend on the initial ensemble. This explains the "statistical hysteresis" demonstrated in the bifurcation diagrams of Fig. 2. Similar properties of the transfer operator probably underlie the similar behavior reported earlier in deterministic lattices of logistic maps (cf. Sec. 3.2.2 of [18]).

The numerical investigations of Sec. III indicate that the presence of asymptotic periodicity also facilitates the formation of large-scale transient patterns in certain regions of parameter space. To understand the connection between asymptotic periodicity and pattern formation, note that asymptotically periodic systems occupy only small regions of their phase spaces determined by the support of the various v_i's of Eq. (15). The support of each v_i is a product of N intervals: $v_i = M_1 \times \cdots \times M_N$, where there are r distinct $M_k \subset [0,1]$ if i also runs from 1 to r (think of the situation for a lattice of tent maps when $r=2$ and each site belongs to one of two subintervals of [0,1]). Denote the r distinct attractive subsets of [0,1] by the symbols $\mathbb{I}_1, \ldots, \mathbb{I}_r$. Hence

$$v_i = \prod_{k=1}^{N} M_k = \prod_{j=1}^{r} \left[\prod_{k=1}^{N_j} \mathbb{I}_j \right], \quad \sum_{j=1}^{r} N_j = N .$$

The exact form of the $\mathbb{I}_j$'s depends on S and the coupling architecture. If the interelement coupling in the lattice tends to correlate the activity of neighboring sites (as is the case for diffusivelike couplings), the evolution of the lattice will be accompanied by the formation of clusters of sites whose values will tend to belong to the same $\mathbb{I}_j$. If the diffusive effects are too strong, all sites will eventually belong to the same $\mathbb{I}_j$ at the same time and there will be no pattern formation. On the other hand, if the diffusive effects are too weak, neighboring sites will not tend to belong to the same $\mathbb{I}_j$ and the lack of correlation between nearby elements will be reflected in the lack of large scale patterns. Thus the appearance of large-scale patterns would appear to be the result of a balance between: (i) a coupling induced tendency to synchronize neighboring sites and (ii) a tendency (due to the asymptotic periodicity) of all site activities to belong to small disjoint subsets of the phase space. If there is only one attractive $\mathbb{I}_j$ (i.e., $r=1$), so the system is statistically stable, patterns should not form spontaneously. This is numerically observed to be the case in all the systems we have investigated.

Finally, the presence of asymptotic periodicity in spatially extended dynamical systems points to the possibility of this behavior in related continuous-time models. For example, there is a strong connection (explored in detail in [38]) between dynamics in certain differential delay equations and coupled map lattices. Our results on stochastic CML's are presently being extended to the study of these functional equations, which play a prominent role in various fields in investigations, ranging from nonlinear optics to theoretical population biology.

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APPENDIX

In this appendix the proof of Theorem 2 is given. it rests on a result, originally published by Komoroník, which states that a Markov operator $\mathcal{P}$ is asymptotically periodic if and only if it is constrictive. We therefore show here that an operator $\mathcal{P}$ satisfying the conditions of Theorem 2 is constrictive and then invoke the Komorník result. The proof of the theorem is inspired by similar proofs given in Chap. 5 of [32]. Before proceeding, we introduce the notion of constrictiveness.

Definition 2: Constrictiveness. (Let $(\mathbb{X},\mathcal{B},\mu)$ be a finite measure space. A Markov operator $\mathcal{P}$ is said to be constrictive if there exists a $\delta > 0$) and $\kappa < 1$ such that for every density f, there is an integer $n_0(f)$ for which

$$\int_{\mathbb{E}} \mathcal{P}f(\mathbf{x})\mu(d\mathbf{x}) \le \kappa \quad \text{for } n \ge n_0(f), \quad \mu(\mathbb{E}) \le \delta . \tag{A1}$$

This property ensures that the iterates $\mathcal{P}^n f$ of any initial density f are not eventually concentrated on a set of small or zero measure. If the space $\mathbf{X}$ is not finite (as would be the case for CML's with a local transformation defined on $\mathbb{R}$ rather than [0,1]), a slight generalization of this definition is desired to rule out the possibility that $\mathcal{P}^n f$ be dispersed throughout the entire space. In this case $\mathcal{P}$ is said to be constrictive if there is a measurable set $\mathcal{B}$ of finite measure such that for every density f there is an integer $n_0(f)$ for which

$$\int_{\mathbf{X}\ openB \cup \mathbb{E}} \mathcal{P}f(\mathbf{x})\mu(d\mathbf{x}) \le \kappa$$
$$\text{for } n \ge n_0(f) , \ \mu(\mathbb{E}) \le \delta . \tag{A2}$$

If $\mathbf{X}$ is finite and $\mathbb{B} = \mathbf{X}$, (A2) reduces to (A1). ■

Next, we recall the Chebyshev inequality. Let $(\mathbf{X},\mathcal{B},\mu)$ be a measure space, V: $\mathbf{X} \mapsto \mathbb{R}$ a non-negative measurable function, and for all densities f set

$$E(V|f) = \int_{\mathbf{X}} V(\mathbf{x})f(\mathbf{x})\mu(d\mathbf{x}) .$$

If $G_a = \{\mathbf{x}\colon V(x) < a\}$, then

$$\int_{G_a} f(\mathbf{x})\mu(d\mathbf{x}) \ge 1 - \frac{E(V|f)}{a}$$
$$\text{(the Chebyshev inequality)} .$$

The proof is given in [32]. For future reference, we define

$$E_n(V|f) = \int_{\mathbf{X}} V(x)\mathcal{P}^n_{\Phi_{\mathrm{mul}}} f(\mathbf{x})d\mathbf{x}$$

Proof of Theorem 2. From (20),

$$E_n(V|f) \le \alpha E_{n-1}(V|f) + \beta .$$

By induction,

$$E_n(V|f) \le \beta \sum_{k=1}^{n} \alpha^{i-1} + \alpha^n E_0(V|f) < \frac{\beta}{1-\alpha} + \alpha^n E_0(V|f) .$$

Recall that $\alpha \in [0,1)$ and that $E_0(V|f)$ is finite for all f, so that there is a $n_0 = n_0(f)$ such that

$$E_n(V|f) \le \frac{\beta}{1-\alpha} + 1 \quad \text{for all } n \ge n_0(f) .$$

Hence, using G_a defined as in the Chebyshev inequality, one finds

$$\int_{\mathbf{X}\setminus G_a} \mathcal{P}^n_{\Phi_{\mathrm{mul}}} f(\mathbf{x})d\mathbf{x}$$
$$= 1 - \int_{G_a} \mathcal{P}^n_{\Phi_{\mathrm{mul}}} f(\mathbf{x})d\mathbf{x} \le \frac{E_n(V|f)}{a}$$
$$\le \frac{1}{a}\left[1 + \frac{\beta}{1-\alpha}\right] \quad \text{for all } n \ge n_0(f) . \tag{A3}$$

Let $\varepsilon = \frac{1}{3}(1-\lambda)$. If we choose a such that

$$a \ge \frac{1}{\varepsilon}\left[1 + \frac{\beta}{1-\alpha}\right] , \tag{A4}$$

we have

$$\int_{\mathbf{X}\setminus G_a} \mathcal{P}_{\Phi_{\mathrm{mul}}} f(\mathbf{x})d\mathbf{x} \le \varepsilon \quad \text{for } n \ge n_0(f) .$$

Hence, using (33)

$$\begin{aligned}
\int_{\mathbf{X}\setminus G_a \cup \mathbf{A}} \mathcal{P}_{\Phi_{\mathrm{mul}}} f(\mathbf{x})d\mathbf{x} &\le \int_{\mathbf{X}\setminus G_a} \mathcal{P}_{\Phi_{\mathrm{mul}}} f(\mathbf{x})d\mathbf{x} + \int_{\mathbf{A}} \mathcal{P}_{\Phi_{\mathrm{mul}}} f(\mathbf{x})d\mathbf{x} \\
&\le \varepsilon + \int_{\mathbf{A}} \mathcal{P}_{\Phi_{\mathrm{mul}}} f(\mathbf{x})d\mathbf{x} \\
&\le \varepsilon + \int_{\mathbf{A}} \int_{x^{(N)}}^{1} \cdots \int_{x^{(1)}}^{1} K(\mathbf{x},\mathbf{y})\mathcal{P}^{n-1}_{\Phi_{\mathrm{mul}}} f(\mathbf{y})d\mathbf{y}\,d\mathbf{x} \\
&\le \varepsilon + \int_{\mathbf{A}} \int_{0}^{1} \cdots \int_{0}^{1} K(\mathbf{x},\mathbf{y})\mathcal{P}^{n-1}_{\Phi_{\mathrm{mul}}} f(\mathbf{y})d\mathbf{y}\,d\mathbf{x} \\
&\le \varepsilon + \int_{\mathbf{X}} \mathcal{P}^{n-1}_{\Phi_{\mathrm{mul}}} f(\mathbf{y})d\mathbf{y} \int_{\mathbf{A}} K(\mathbf{x},\mathbf{y})d\mathbf{x} \\
&\le \varepsilon + \int_{\mathbf{X}\setminus G_a} \mathcal{P}^{n-1}_{\Phi_{\mathrm{mul}}} f(\mathbf{y})d\mathbf{y} \int_{\mathbf{A}} K(\mathbf{x},\mathbf{y})d\mathbf{x} + \int_{G_a} \mathcal{P}^{n-1}_{\Phi_{\mathrm{mul}}}(\mathbf{y})d\mathbf{y} \int_{\mathbf{A}} K(\mathbf{x},\mathbf{y})d\mathbf{x} .
\end{aligned} \tag{A5}$$

From (19), the integrals over $\mathbf{A}$ in (A5) are bounded above by λ, the integral over $\mathbf{X}\setminus G_a$ is bounded above by ε, and the integral over G_a is bounded above by 1, so we have

$$\int_{\mathbf{X}\setminus G_a \cup \mathbf{A}} \mathcal{P}_{\Phi_{\mathrm{mul}}} f(\mathbf{x})d\mathbf{x} \le 2\varepsilon + \lambda = 1 - \varepsilon \quad \text{for } n \ge n_0(f) + 1 ,$$

which is the definition of constrictiveness when $\mathbf{X}$ is infinite. However, in our case, $\mathbf{X}$ is the N-dimensional hypercube, so it is not infinite and the definition of constrictiveness is given by (A1). To obtain this inequality, note that a as defined in (A4) is arbitrary and we can choose it to be large enough so that $G_a = \mathbf{X}$. If we do so, the previous inequality becomes

$$\int_{\mathbf{A}} \mathcal{P}_{\Phi_{\mathrm{mul}}} f(\mathbf{x})d\mathbf{x} \le 2\varepsilon + \lambda = 1 - \varepsilon \quad \text{for } n \ge n_0(f) + 1 ,$$

which is equivalent to (A1). Hence $\mathcal{P}_{\Phi_{\mathrm{mul}}}$ is constrictive. Since it is a Markov operator, it is also asymptotically periodic [35]. ■

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