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Long memory and scaling for multiplicative stochastic processes with application to the study of population oscillations

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Abstract

An analytically tractable multiplicative random process is introduced based on an analogy between the random phase modulation, wave propagation in a random medium and the population growth in a fluctuating environment. It is assumed that the process depends on a multiplicative random parameter which can be eliminated by introducing an intrinsic time scale; the relation between the intrinsic and the physical (watch) time scales is determined by the stochastic properties of the random parameter. The multitime joint probability densities of the state variables expressed in terms of the physical time can be computed in a closed form in terms of the corresponding joint probability densities expressed in the intrinsic time scale. The theory is applied to the study of age-dependent population oscillations in a random environment. In this case the intrinsic time scale is a biological time which is the same for any physical realization of the random environment. The random fluctuations of the environment lead to a decrease of the intrinsic rate of population growth and generate a temporal analogue of Anderson localization: due to fluctuations the population oscillations are damped. The asymptotic behavior of the process depends on the range of the memory effects of the environmental fluctuations: for short memory the qualitative asymptotic behavior of the population size for large time is the same for a fluctuating as well as for a constant environment; for slowly decaying correlations, however, the exponential increase of the population is outweighed by a compressed exponential decay due to environmental fluctuations and the population eventually becomes extinct.

The study of multiplicative noise is a problem of topical interest, merely in connection with the investigation of fractal random processes [1–4]. Most systems with multiplicative noise cannot be solved exactly and thus the literature focuses mainly on the elaboration of approximate analytical and numerical techniques. Nevertheless the study of a physically significant class of multiplicative random processes amenable to analytical treatment would be of interest.

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In this Letter we investigate a class of physically and biologically interesting multiplicative random processes for which a fairly detailed analytical description is possible. We illustrate our approach by three different examples. The first example is the Anderson–Kubo oscillator (AK) [5],

$$\dot{X} = i\omega(t)X, \quad (1)$$

where the pulsation $\omega(t)$ is a stationary random function with known stochastic properties and the state variable X is a complex random function of time whose stochastic properties should be evaluated. The Anderson–Kubo model can be solved exactly. One possible way of solving the problem would be to introduce an intrinsic time scale $\theta(t)$ as the solution of the differential equation

$$d\theta(t)/dt = \omega(t)/\langle\omega\rangle, \quad \theta(0) = 0, \quad (2)$$

where, due to the stationarity of the random pulsation $\omega(t)$, the average value $\langle\omega\rangle$ is time-independent. As $\omega(t)$ is random the intrinsic time $\theta(t)$ is also a random function of the physical time t . However, in terms of θ , the state variable $X = X(\theta)$ becomes a deterministic function obeying the differential equation

$$dX(\theta)/d\theta = i\langle\omega\rangle X(\theta). \quad (3)$$

The moments of the state variable X as a function of the physical time t can be evaluated by solving the deterministic equation (3) and by taking an average over all random trajectories $\theta = \theta(t)$ compatible with the random differential equation (2). A similar problem arises in the study of wave propagation in random media [6]. In the eikonal approximation the properties of a wave can be expressed in terms of the optical path length

$$l = \int n(s) ds, \quad (4)$$

where s is the arc length along the wave and n is the refraction index of the medium. Due to the density fluctuations the refraction index n is a random function of the current value of the arc length s . The analogy between the Anderson–Kubo model and the random wave propagation is clear: by writing Eq. (4) in the form

$$dl(s)/ds = n(s), \quad (5)$$

and comparing Eqs. (2) and (5) we note that the physical and the intrinsic times, t and $\theta(t)$, are the analogs of the physical and the optical path lengths s and $l(s)$, respectively, and that the fluctuating refraction index $n(s)$ is the analogue of the ratio $\omega(t)/\langle\omega\rangle$ between the fluctuating pulsation $\omega(t)$ and its average value $\langle\omega\rangle$.

A third example comes from biology: it is the growth of an organism, of an individual or of a population in a random environment [7,8]. In this context a common assumption is that there is an intrinsic biological time scale for which the growth process can be represented by the same function for any kind of environment and the random behavior of the environment determines the stochastic connection between the physical and the biological time scales. There is observational evidence supporting this assumption [9].

For all three examples presented above there is an intrinsic time or space scale for which the dynamics of the process is the same for any fluctuation of the environment and the fluctuations of the environment are completely described by the stochastic properties of the intrinsic time or space scale, which are random functions of the corresponding physical scales. In some cases the intrinsic scale has a direct physical or biological significance (as in the case of the optical path length or of the biological time). However this is not necessary; in some cases the intrinsic scale can be formally introduced by eliminating a random parameter from the evolution equations (similar to the elimination of the random pulsation $\omega(t)$ from the evolution equation (1) for the Anderson–Kubo oscillator).

In general we consider a set of random variables $\mathbf{X} = (X_1, X_2, \dots)$ and assume that there are two sources of stochasticity in the system. A first source is due to the process itself (internal noise) and can be described in terms of an intrinsic time scale θ by means of an infinite chain of probability densities

$$P_m^*(X_1, \theta_1; \dots; X_m, \theta_m) dX_1 \dots dX_m, \quad \text{with } \int \dots \int P_m^* dX_1 \dots dX_m = 1. \quad (6)$$

The second source of stochasticity is due to the environment (external noise) and can be described in terms of the stochastic properties of the intrinsic time θ as a random function of the physical time $\theta = \theta(t)$. $\theta(t)$ is a random function of the real time t obeying a stochastic differential equation similar to Eqs. (2) or (5),

$$d\theta(t)/dt = \nu(t), \quad \theta(0) = 0, \quad (7)$$

where the differential rate $\nu(t)$ is a stationary random function of time. Without loss of generality we can always choose the intrinsic time scale so that if the differential rate $\nu(t)$ is non-random (constant), then the intrinsic and the physical time scales are the same: $\theta = t$. This is done by choosing

$$\langle \nu(t) \rangle = 1. \quad (8)$$

A supplementary advantage of convention (8) is that if it is fulfilled then both the intrinsic and the physical times are measured in the same units. We assume that the stochastic properties of the differential rate $\nu(t)$ are known; they can be characterized by the characteristic functional

$$g[\varepsilon(t')] = \left\langle \exp\left(i \int \varepsilon(t') \nu(t') dt'\right) \right\rangle, \quad (9)$$

where $\varepsilon(t')$ is a suitable test function conjugate to the differential rate $\nu(t')$. We are interested in solving the following problem. If the intrinsic joint probability densities P_m^* , $m = 1, 2, \dots$ and the characteristic functional $g[\varepsilon(t')]$ are known, which are the stochastic properties of the random vector $X = X(t)$ in real time, that is, which are the joint probability densities

$$P_m(X_1, t_1; \dots; X_m, t_m) dX_1 \dots dX_m, \quad \text{with } \int \dots \int P_m dX_1 \dots dX_m = 1, \quad (10)$$

defined in terms of the physical time t ?

To answer this question we express the intrinsic joint probability densities P_m^* as multiple inverse Fourier transforms

$$P_m^*(X_1, \theta_1; \dots; X_m, \theta_m) = (2\pi)^{-m} \int \dots \int \exp\left(-i \sum_{\rho} \omega_{\rho} \theta_{\rho}\right) \bar{P}_m^*(X_1, \omega_1; \dots; X_m, \omega_m) d\omega_1 \dots d\omega_m, \quad (11)$$

where

$$\bar{P}_m^*(X_1, \omega_1; \dots; X_m, \omega_m) = \int \dots \int \exp\left(i \sum_{\rho} \omega_{\rho} \theta_{\rho}\right) P_m^*(X_1, \theta_1; \dots; X_m, \theta_m) d\theta_1 \dots d\theta_m \quad (12)$$

are the Fourier transforms of P_m^* and ω_{ρ} , $\rho = 1, \dots, m$ are the Fourier variables conjugate to the intrinsic times θ_{ρ} , $\rho = 1, \dots, m$. By integrating Eq. (7) for θ_{ρ} , $\rho = 1, \dots, m$ we have

$$\theta_{\rho}(t_{\rho}) = \int_0^{t_{\rho}} \nu(t') dt'. \quad (13)$$

The real time joint probability densities P_m can be expressed as

$$P_m(X_1, t_1; \dots; X_m, t_m) = \langle P_m^*(X_1, \theta_1(t_1); \dots; X_m, \theta_m(t_m)) \rangle, \quad (14)$$

where the brackets mean a multiple average over all possible random paths $\theta_{\rho}(t'_{\rho})$, $0 \leq t'_{\rho} \leq t_{\rho}$, $\rho = 1, \dots, m$. By using Eqs. (11)–(14) and expressing the path averages in terms of the characteristic functional $g[\varepsilon(t')]$ we get the following expressions for the real time joint probability densities,

$$P_m(X_1, t_1; \dots; X_m, t_m) = (2\pi)^{-m} \int \dots \int g\left(\varepsilon(t') = - \sum_{\rho} \omega_{\rho} h(t' - t_{\rho})\right) d\omega_1 \dots d\omega_m \\ \times \int \dots \int \exp\left(i \sum_{\rho} \omega_{\rho} \theta_{\rho}\right) P_m^*(X_1, \theta_1; \dots; X_m, \theta_m) d\theta_1 \dots d\theta_m, \quad (15)$$

where $h(x)$ is the usual Heaviside step function.

Eq. (15) is the main result of this Letter; it gives an integrated account of the interaction between the internal fluctuations expressed by the intrinsic joint probability densities P_m^* and the external fluctuations expressed by the characteristic functional $g[\varepsilon(t')]$. Eq. (15) allows us to express all real time moments of the random variables $X = (X_1, X_2, \dots)$

$$\langle (X_{q_1}(t_1))^{M_1} \dots (X_{q_m}(t_m))^{M_m} \rangle = \int \dots \int (X_{q_1})^{M_1} \dots (X_{q_m})^{M_m} P_m \, dX_1 \dots dX_m, \quad (16)$$

in terms of the corresponding intrinsic time moments

$$\langle (X_{q_1}(\theta_1))^{M_1} \dots (X_{q_m}(\theta_m))^{M_m} \rangle^* = \int \dots \int (X_{q_1})^{M_1} \dots (X_{q_m})^{M_m} P_m^* \, dX_1 \dots dX_m. \quad (17)$$

By combining Eqs. (15)–(17) we obtain

$$\begin{aligned} & \langle (X_{q_1}(t_1))^{M_1} \dots (X_{q_m}(t_m))^{M_m} \rangle \\ &= (2\pi)^{-m} \int \dots \int g\left(\varepsilon(t') = -\sum_{\rho} \omega_{\rho} h(t' - t_{\rho})\right) d\omega_1 \dots d\omega_m \\ & \times \int \dots \int \exp\left(i \sum_{\rho} \omega_{\rho} \theta_{\rho}\right) \langle (X_{q_1}(\theta_1))^{M_1} \dots (X_{q_m}(\theta_m))^{M_m} \rangle^* d\theta_1 \dots d\theta_m. \end{aligned} \quad (18)$$

In order to clarify the significance of Eqs. (15) and (17) we introduce the joint probability densities of the values $\theta_1, \dots, \theta_m$ of the intrinsic time θ at different real times t_1, \dots, t_m .

$$\phi_m(\theta_1, t_1; \dots; \theta_m, t_m) \, d\theta_1 \dots d\theta_m, \quad \text{with} \quad \int \dots \int \phi_m(\theta_1, t_1; \dots; \theta_m, t_m) \, d\theta_1 \dots d\theta_m = 1. \quad (19)$$

We have

$$\phi_m(\theta_1, t_1; \dots; \theta_m, t_m) = \left\langle \prod_{\rho=1}^m \delta\left(\theta_{\rho} - \int_0^{t_{\rho}} \nu(t') \, dt'\right) \right\rangle, \quad (20)$$

where the average of the product of delta functions is computed by taking into account all possible paths $\theta(t'_{\rho})$, $0 \leq t'_{\rho} \leq t_{\rho}$, $\rho = 1, \dots, m$. By expressing this average in terms of the characteristic functional $g[\varepsilon(t')]$ we obtain

$$\phi_m(\theta_1, t_1; \dots; \theta_m, t_m) = (2\pi)^{-m} \int \dots \int \exp\left(i \sum_{\rho} \omega_{\rho} \theta_{\rho}\right) g\left(\varepsilon(t') = -\sum_{\rho} \omega_{\rho} h(t' - t_{\rho})\right) d\omega_1 \dots d\omega_m, \quad (21)$$

and therefore Eqs. (15) and (17) can be rewritten as

$$P_m(X_1, t_1; \dots; X_m, t_m) = \int \dots \int d\theta_1 \dots d\theta_m \phi_m(\theta_1, t_1; \dots; \theta_m, t_m) P_m^*(X_1, \theta_1; \dots; X_m, \theta_m), \quad (22)$$

$$\begin{aligned} & \langle (X_{q_1}(t_1))^{M_1} \dots (X_{q_m}(t_m))^{M_m} \rangle \\ &= \int \dots \int d\theta_1 \dots d\theta_m \phi_m(\theta_1, t_1; \dots; \theta_m, t_m) (X_{q_1}(\theta_1))^{M_1} \dots (X_{q_m}(\theta_m))^{M_m}. \end{aligned} \quad (23)$$

Now the physical significance of Eqs. (15) and (17) is clear: they are finite-dimensional integral representations of the more complicated path integrals expressing the averages over $\theta(t'_{\rho})$, $\rho = 1, \dots, m$.

As a simple illustration of our approach we consider the age-dependent population growth [10] in a random environment [7,8]. We assume that the intrinsic population dynamics is deterministic and described by the Lotka–von Foerster equations

$$(\partial_\theta + \partial_a)n^*(a, \theta) = -\mu(a)n^*(a, \theta), \tag{24}$$

$$B^*(\theta) = \int \lambda(a)n^*(a, \theta) da, \tag{25}$$

where $n^*(a, \theta)$ is the density of individuals with age between a and $a + da$ at the intrinsic (biological) time θ , $\lambda(a)$, $\mu(a)$ are age-dependent birth and death rates and

$$B^*(\theta) = n^*(a=0, \theta), \tag{26}$$

is the flux of births at the intrinsic time θ . By integrating Eq. (24) along the characteristics and inserting the result of integration into Eq. (25) we get an integral equation for the flux of births $B^*(\theta)$,

$$B^*(\theta) = \int_0^\theta \lambda(a)l(a)B^*(\theta - a) da + A(\theta), \tag{27}$$

where

$$l(a) = \exp\left(-\int_0^a \mu(a) da\right) \tag{28}$$

is the survival function of an individual at age a and the function $A(\theta)$ depends on the initial age density $n^*(a, \theta=0)$. The solution of Eq. (27) has the form

$$B^*(\theta) = b_0 \exp(r_0 \theta) + \sum_{\rho=1}^{\infty} [b_\rho^+ \cos(v_\rho \theta) + b_\rho^- \sin(v_\rho \theta)] \exp(u_\rho \theta), \tag{29}$$

where

$$r_{\pm \rho} = u_\rho \pm i v_\rho, \quad \rho = 0, 1, 2, \dots, \tag{30}$$

are the solutions of Lotka's transcendental equation

$$\int_0^\infty \exp(-ra) \lambda(a)l(a) da = 1, \tag{31}$$

and b_ρ^\pm , $\rho = 1, 2, \dots$ and b_0 are constants depending on the initial age density $n^*(a, \theta=0)$. It has been shown in the literature of population dynamics [10] that Eq. (31) has a unique real root r_0 , which is simple and which is positive or negative depending on the value of the integral $\int_0^\infty \lambda(a)l(a) da$,

$$r_0 \gtrless 0 \quad \text{for} \quad \int_0^\infty \lambda(a)l(a) da \gtrless 1. \tag{32}$$

The real parts u_ρ of the complex roots $r_{\pm \rho} = u_\rho \pm i v_\rho$ are at most equal to the real root r_0 ,

$$r_0 \geq u_\rho, \quad \rho = 1, 2, \dots \tag{33}$$

From the above-mentioned properties of the solutions of Lotka's equation (31) it follows that the real root r_0 determines the large time behavior of the population; depending on whether the real root r_0 is positive or negative the population either increases exponentially to infinity or decreases exponentially to zero. In this Letter we limit ourselves to an exponentially increasing population in the intrinsic time scale for which $r_0 > 0$, that is $\int_0^\infty \lambda(a)l(a) da > 1$, i.e., during its life each individual has at least only one offspring.

Concerning the environmental fluctuations we assume that the differential rate $\nu(t)$ is a stationary Gaussian random function with the average value equal to unity (Eq. (8)) and with the second cumulant (correlation function) given by

$$\langle\langle \nu(t)\nu(t') \rangle\rangle = C(|t-t'|), \tag{34}$$

where $C(|t-t'|)$ is a non-negative function depending only on the absolute value of the time difference $t-t'$ and which fulfills the initial and boundary conditions

$$C(0) = 1, \quad C(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty, \tag{35}$$

It follows that the characteristic functional $g[\varepsilon(t')]$ is given by

$$g[\varepsilon(t')] = \exp\left(i \int \varepsilon(t') dt' - \frac{1}{2} \iint \varepsilon(t'_1)\varepsilon(t'_2)C(|t'_1-t'_2|) dt'_1 dt'_2\right). \tag{36}$$

The real time dependence of the moments $\langle B(t_1)\dots B(t_m) \rangle = \langle B(\theta_1(t_1))\dots B(\theta_m(t_m)) \rangle$ can be evaluated from Eqs. (18), (29) and (36). After lengthy but standard manipulations we come to

$$\begin{aligned} \langle B(t_1)\dots B(t_m) \rangle = & \sum_{q_1=-\infty}^{\infty} \dots \sum_{q_m=-\infty}^{\infty} 2^{-m} \prod_{\rho=1}^m [b_{q_\rho}^+ - i b_{q_\rho}^- \operatorname{sgn} q_\rho] \\ & \times \exp\left(\sum_{\rho=1}^m r_{q_\rho} t_\rho - \frac{1}{2} \sum_{\rho_1=1}^m \sum_{\rho_2=1}^m r_{q_{\rho_1}} r_{q_{\rho_2}} [f(t_{\rho_1}) + f(t_{\rho_2}) - f(|t_{\rho_1}-t_{\rho_2}|)]\right), \end{aligned} \tag{37}$$

where

$$b_0^+ = b_0, \quad b_0^- = 0, \tag{38}$$

$\operatorname{sgn} q_\rho$ denotes the sign of q_ρ and the function $f(t)$ is given by

$$f(t) = \int_0^t (t-x)C(x) dx. \tag{39}$$

In particular for $m = 1$ the average flux of births in real time is given by

$$\begin{aligned} \langle B(t) \rangle = & b_0 \exp[r_0 t - (r_0)^2 f(t)] + \sum_{\rho=1}^{\infty} \exp\left\{u_\rho t + f(t) [(v_\rho)^2 - (u_\rho)^2]\right\} \\ & \times \left\{b_\rho^+ \cos[v_\rho t - 2f(t)u_\rho v_\rho] + b_\rho^- \sin[v_\rho t - 2f(t)u_\rho v_\rho]\right\}. \end{aligned} \tag{40}$$

By comparing Eqs. (29) and (40) for the flux of births expressed in terms of the intrinsic (biological) and physical times θ and t , respectively, we note that for a constant environment ($C = 0, f = 0$) the two equations are identical and the biological and physical times are the same, $\theta = t$. Otherwise the behavior of the flux of births expressed in terms of the two time scales is generally different. The behavior of the population described in terms of the real time depends on the statistical properties of the environment described by the correlation function $C(t)$. We distinguish two extreme cases:

(1) Short memory environmental fluctuations for which the decay of the correlation function $C(t)$ is exponential,

$$C(t) \sim \exp(-\lambda t), \tag{41}$$

where λ is a characteristic frequency describing the decay of fluctuations. In this case, according to the Doob theorem, the Gaussian fluctuations of the differential rate $\nu(t)$ are also Markovian.

(2) Long memory environmental fluctuations for which the tail of the correlation function obeys an inverse power law,

$$C(t) \sim \text{const} \times t^{-H}, \quad t \gg 0, \tag{42}$$

where H is a fractal exponent between zero and unity.

In this Letter we use a Gaussian model for the fluctuations of the differential rate $\nu(t)$ which includes both the short and long memory as particular cases. We consider a renormalized fluctuation theory of a scalar variable derived from the Uhlenbeck–Ornstein process by applying a probabilistic version of the Shlesinger–Hughes renormalization [11]. This type of approach has been suggested by one of the authors in connection with the study of Anderson–Kubo oscillator [12]. The corresponding correlation function is [12]

$$C(t) = \alpha H \gamma(H, \lambda t) (\lambda t)^{-H} + (1 - \alpha) \exp(-\lambda t), \tag{43}$$

where H is a fractal exponent between zero and unity, λ is the characteristic frequency of the non-renormalized Uhlenbeck–Ornstein process, α is a clustering (renormalization) probability and

$$\gamma(H, x) = \int_0^x x^{H-1} \exp(-x) dx, \tag{44}$$

is the incomplete gamma function. For $\alpha = 0$ the renormalization does not exist and the correlation function (43) becomes an exponential of the type (41); otherwise for $\alpha \neq 0$ due to renormalization the correlation function has a long tail

$$C(t) \sim \Gamma(1 + H) \alpha (\lambda t)^{-H}, \quad t \gg 1/\lambda, \tag{45}$$

where $\Gamma(1 + H) = \gamma(1 + H, \infty)$ is the complete gamma function.

The function $f(t)$ can be easily evaluated from Eqs. (39) and (43). We have

$$f(t) = \lambda^{-2} \left\{ \exp(-\lambda t) + \lambda t - 1 + \frac{\alpha}{(1-H)(2-H)} \left\{ [(\lambda t)^{2-H} f(1+H, \lambda t) + 2 - 2H - \lambda t(2-H) + [(\lambda t)^2 + \lambda t H + H - 2] \exp(-\lambda t)] \right\} \right\}. \tag{46}$$

Now we can investigate the asymptotic behavior of the average flux of births for large time.

(1) For short memory we have

$$f(t) \sim t/\lambda \quad \text{as } t \gg 1/\lambda, \tag{47}$$

and the average flux of births $\langle B(t) \rangle$ has a form similar to the intrinsic law (29) with the intrinsic time θ replaced by the real time t ,

$$\langle B(t) \rangle = b_0 \exp(\tilde{r}_0 t) + \sum_{\rho=1}^{\infty} [b_{\rho}^+ \cos(\tilde{v}_{\rho} t) + b_{\rho}^- \sin(\tilde{v}_{\rho} t)] \exp(\tilde{u}_{\rho} t), \tag{48}$$

where

$$\tilde{r}_0 = r_0 - (r_0)^2/\lambda < r_0, \tag{49}$$

$$\tilde{u}_{\rho} = u_{\rho} - [(u_{\rho})^2 - (v_{\rho})^2]/\lambda < u_{\rho}, \quad |u_{\rho}| > |v_{\rho}|, \tag{50}$$

$$\tilde{v}_{\rho} = v_{\rho}(1 - 2u_{\rho}/\lambda) < v_{\rho}, \quad 2u_{\rho} < \lambda, \quad u_{\rho} > 0. \tag{51}$$

We note the occurrence of a temporal analogue of the Anderson localization: due to the action of environmental fluctuations in the corresponding ensemble average for $\langle B(t) \rangle$ the superposition of the different oscillatory terms leads to damping resulting in the decrease of the rates of exponential variation ($\tilde{r}_0 < r_0$, $\tilde{u}_{\rho} < u_{\rho}$) and moreover the periods of oscillations increase and the oscillations become slower. Although the threshold condition for the passage from an exponential increase to an exponential decay is changed, otherwise the qualitative behavior of the dependence $\langle B \rangle = \langle B(t) \rangle$ remains the same as in the case of a constant environment.

(2) For long memory and large time $t \gg 1/\lambda$ the function $f(t)$ becomes

$$f(t) \sim \frac{\alpha \Gamma(1+H) \lambda^{-H}}{(1-H)(2-H)} t^{2-H}, \quad t \gg 1/\lambda. \quad (52)$$

In this case the large time behavior of the average flux of births $\langle B(t) \rangle$ is changed by the occurrence of environmental fluctuations. As $1 > H > 0$ the function $f(t)$ increases faster than linearly in time and for $t \gg 1/\lambda$ it outweighs the contribution of the linear functions in time and the population decreases to zero as a compressed exponential modulated by nonlinear oscillations,

$$\begin{aligned} \langle B(t) \rangle \simeq & b_0 \exp\left[-(r_0)^2 \beta t^{2-H}\right] + \sum_{\rho=1}^{\infty} \left(\left[b_{\rho}^{+} \cos(2u_{\rho} v_{\rho} \beta t^{2-H}) - b_{\rho}^{-} \sin(2u_{\rho} v_{\rho} \beta t^{2-H}) \right] \right. \\ & \left. \times \exp\left\{-\beta \left[(u_{\rho})^2 - (v_{\rho})^2 \right] t^{2-H}\right\} \right), \quad t \gg 1/\lambda, \end{aligned} \quad (53)$$

where

$$\beta = \alpha \frac{\Gamma(1+H) \lambda^{-H}}{(1-H)(2-H)}. \quad (54)$$

The general conclusion of the analysis of population oscillations in a random environment is that the external random fluctuations decrease the rate of population growth and slow down the population oscillations. There is an essential difference between the effects of short and long memory fluctuations. For Markovian fluctuations with short memory the qualitative behavior of the birth flux as a function of time is the same as in a constant environment, and only the threshold condition for the transition from exponential increase to exponential decay and the periods of oscillation are modified. For non-Markovian fluctuations with long memory, however, the large time behavior of the birth flux is different: it is given by a negative compressed exponential modulated by nonlinear oscillations in time and the population eventually becomes extinct for any value of the intrinsic rate of growth r_0 .

The main advantage of our approach for analysing the interference between the internal and external fluctuations is its simplicity. The disadvantage of being limited to systems for which a non-random intrinsic time scale can be found is compensated by the advantage of providing exact analytical solutions. The biological problem discussed before does not exhaust the possibilities of the application of the method. Further research should focus on the comparison with the more conventional approaches for the study of interference between internal and external noise [13,14] as well as on the analysis of other applications from physics, chemistry and biology.

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