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Coupled map lattices as models of deterministic and stochastic differential delay equations

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We discuss the probabilistic properties of a class of differential delay equations (DDE's) by first reducing the equations to coupled map lattices, and then considering the spectral properties of the associated transfer operators. The analysis is carried out for the deterministic case and a stochastic case perturbed by additive or multiplicative white noise. This scheme provides an explicit description of the evolution of phase space densities in DDE's, and yields an evolution equation that approximates the analog for delay equations of the generalized Liouville and Fokker-Planck equations. It is shown that in many cases of interest, for both stochastic and deterministic delay equations, the phase space densities reach a limit cycle in the asymptotic regime. This statistical cycling is observed numerically in continuous time systems with delay and discussed in light of our analytical description of the transfer operators.

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I. INTRODUCTION

In this paper, we study the statistical properties of first order differential delay equations (DDE). These equations routinely appear as realistic models in mathematical biology [1–3], in nonlinear optics [4,5], and in the description of agricultural commodity markets [6,7] to mention a few applications. In the deterministic case, these models are of the form

$$\frac{dx(t)}{dt} = -\alpha x(t) + F(x(t-1)) ,$$

and if noise enters the problem, they can be written as

$$dx(t) = [-\alpha x(t) + F(x(t-1))]dt + G(x(t))\xi(t)dt ,$$

where $\alpha > 0$, $\xi(t)dt$ denotes a stochastic process whose characteristics will be discussed in Sec. II B below, and the initial condition for the system in both cases is a function φ defined on $[-1,0)$ (the delay is taken to be one without loss of generality).

The phase space of these systems is infinite dimensional. The ensemble density, which gives the probability of occupation of phase space is, therefore, a functional. The evolution equation for this functional, known as the Hopf equation [8–11], cannot be integrated due to the lack of a theory of integration with respect to arbitrary functional measures.

In this paper, we propose a reduction of the original DDE to a finite-dimensional system that is arbitrarily accurate. This approximation is framed in both the stochastic and the deterministic case as a coupled map lattice (CML). The work presented here strongly indicates that in many circumstances of interest (from a modeling perspective) the Hopf equation can be approximated by the Perron-Frobenius equation in $\mathbb{R}^N$ (or its stochastic equivalent). The resulting description of delayed dynamics is akin to the description of ordinary differential equations (ODE's) given by the generalized Liouville equation or of the Langevin equation by the Fokker-Planck equation. Once the reduction is completed, the analytical techniques available to describe the probabilistic properties of CML's can then be used to explain the presence of continuous-time statistical cycling numerically observed in the DDE's.

In Sec. II, the reduction of first order DDE's to CML's is described in both the presence and the absence of noise. Section III introduces basic concepts necessary to describe the evolution of ensemble densities in CML's: The Perron-Frobenius and transfer operators are defined. The link between various convergence properties of sequences of functions and the thermodynamic description of dynamical systems is described, and we briefly review the concept of variation in $\mathbb{R}^N$, which is central to our description of deterministic CML's. An important theorem due to Ionescu Tulcea and Marinescu is discussed. In Sec. IV B, the analysis of deterministic systems is presented. Numerical investigations of a particular model confirm analytical predictions. In Sec. V, we extend this presentation to stochastic models and explore the remarkable phenomenon of statistical cycling induced by noise.

II. FROM DDE'S TO CML'S

The link between hereditary dynamical systems (framed as functional or delay differential equations) and spatially extended models [hyperbolic partial differential equations (PDE's) to be precise] has been discussed extensively (cf. [12–15]). In a rather formal context, Fargue [16,17] argues that it is possible to interpret hereditary systems as being nonlocal or extended. This allows the introduction of a field that is intrinsic to the system, and

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the variable that satisfies the hereditary model is then a functional of this field. In other words, the memory in the system is interpreted as a nonlocality.

At a more applied level, Sharkovskiĭ, Maistrenko, and Ramanenko [18] have shown that systems of hyperbolic PDE's could, given appropriate boundary condition, be reduced *via* use of the method of characteristics to differential delay equations of the first order. Lukin and Shestopalov [19] have applied this reduction procedure to investigate the dynamics of electromagnetic fields confined to cavities possessing nonlinear reflection properties that are routinely used in the construction of radio-optical devices.

A. The deterministic case

The deterministic DDE's considered in this section are of the form

$$\frac{dx(t)}{dt}=-\alpha x(t)+F(x(t-1)) \tag{1}$$

with an initial function $\varphi(s)$ defined for $s\in[-1,0)$. There is a continuous-time semidynamical system associated with (1), given by

$$\frac{dx_\varphi(t)}{dt}=\begin{cases}\dfrac{d\varphi(t)}{dt} & \text{if } t\in[-1,0)\\ -\alpha x_\varphi(t)+F(x_\varphi(1-t)) & \text{if } t\geq 0\end{cases}$$

so that the DDE (1) defines a continuous time operator $\mathcal{S}_t$ acting on bounded functions defined everywhere on $[-1,0)$. For example, if φ denotes such an initial function,

$$\mathcal{S}_t\varphi=\{x_\varphi(s):\ s\in[t-1,t)\},\quad 0\leq t\leq 1 \tag{2}$$

(if $t>1$, the initial function is no longer φ).

The first step in the reduction of (1) to a coupled map lattice is to use the Euler approximation to dx/dt and write

$$\lim_{\Delta\to 0}\frac{x_\varphi(t)-x_\varphi(t-\Delta)}{\Delta}=-\alpha x_\varphi(t)+F(x_\varphi(t-1)),\quad \Delta>0. \tag{3}$$

Removing the limit, (3) can be approximated by

$$x_\varphi(t)=\frac{1}{(1+\alpha\Delta)}[x_\varphi(t-\Delta)+\Delta F(x_\varphi(t-1))], \tag{4}$$

where $0<\Delta\ll 1$.

Before describing the second step of the reduction, recall from (2) that Eq. (1) transforms an initial function φ defined on $[-1,0)$ into another function: the solution x_φ defined on $[-1+t,t)$, where $0<t\leq 1$ is continuous. Hence, if $t<1$, there is an overlap between φ and x_φ. It is possible to vary the extent of this overlap by restricting the values that can be assumed by the time t in the definition (2). For example, if $t=m\Delta$, with $0<\Delta\ll 1$ and $m=1,2,\ldots,$ the continuous-time definition (2) can be replaced by

$$\mathcal{S}_m\varphi=\{x_\varphi(s):\ s\in[m\Delta-1,m\Delta)\},\quad 0\leq m\Delta<1. \tag{5}$$

If $\Delta=1/N$, where $N\gg 1$, then $m=1,\ldots,N$.

The second step in the reduction consists of approximating the initial function φ by a set of N points (as illustrated in Fig. 1), and following the evolution of these points approximating the corresponding solution. Hence, if $m=1$ in (5), the initial function φ is replaced by a vector $\boldsymbol{\varphi}=(\varphi_1,\ldots,\varphi_N)$, and the solution $\{x_\varphi(s):\ s\in[\Delta-1,\Delta)\}$ by a vector $\mathbf{x}_1=(x_1^1,\ldots,x_1^N)$ (the subscript φ has been dropped to simplify the notation). Now define a discrete time transformation $\mathbf{\Phi}_1:\mathbb{R}^N\mapsto\mathbb{R}^N$ (the subscript indicates that $m=1$) such that

$$\overbrace{\mathbf{\Phi}_1\circ\cdots\circ\mathbf{\Phi}_1}^{n\text{ times}}\equiv\mathbf{\Phi}_1^n(\mathbf{x}_0)=\mathbf{x}_n,\quad n=1,2,\ldots,\quad \text{where } \mathbf{x}_0\equiv\boldsymbol{\varphi}.$$

To obtain an explicit expression for $\mathbf{\Phi}_1$, let $\Delta\equiv 1/N$, and suppose that $\varphi_j=\varphi(-1+j\Delta)$, so that in general, x_n^j approximates the value of solution $x(t)$ at time $t=-1+(n+j)\Delta$. Then, Eq. (4) can be approximated by an N-dimensional difference equation

$$\begin{aligned}x_1^1&=x_0^2,\\ \vdots&=\vdots,\\ x_1^{j-1}&=x_0^j,\\ \vdots&=\vdots,\\ x_1^{N-1}&=x_0^N,\\ x_1^N&=\frac{1}{(1+\alpha\Delta)}[x_0^N+\Delta F(x_0^1)].\end{aligned} \tag{6}$$

In vector notation, the system (6) can be written as

$$\mathbf{x}_{n+1}=\mathbf{A}_1\circ\mathbf{x}_n\quad \text{for } n=0,1,\ldots, \tag{7}$$

where the matrix $\mathbf{A}_1$ is given by

$$\mathbf{A}_1=\begin{bmatrix}0 & 1 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0\\ & & \ddots & & \\ \dfrac{\Delta F}{(1+\alpha\Delta)} & 0 & \cdots & 0 & \dfrac{1}{(1+\alpha\Delta)}\end{bmatrix}. \tag{8}$$

Equations (7) and (8) define a transformation $\mathbf{\Phi}_1$ which approximates the DDE (1). In the limit $N\to\infty$, the solution of the difference equation (7) converges to the solution of the DDE (1), because x is by definition always differentiable. $\mathbf{x}_n$ approximates the continuous time solution on the time interval $[n\Delta-1,n\Delta)$, and $\mathbf{x}_{n+1}$ approximates the solution on the time interval $[(n+1)\Delta-1,(n+1)\Delta)$. As illustrated schematically in Fig. 1, in general one can approximate the original DDE by a transformation $\mathbf{\Phi}_m$ such that $\mathbf{x}_{n+1}$ approximates the solution on $[(n+m)\Delta-1,(n+m)\Delta)$ [with m an integer such that $1\leq m\leq N$, as in (5)].

If $m>1$ in (5), the set of difference equations (6) becomes

$$\begin{aligned}
x_1^1 &= x_0^{1+m}\,, \\
\vdots &= \vdots\,, \\
x_1^j &= x_0^{j+m}\,, \\
\vdots &= \vdots\,, \\
x_1^{N-m+1} &= \frac{1}{(1+\alpha N)}[x_0^N+\Delta F(x_0^1)]\,, \qquad (9)\\
\vdots &= \vdots\,, \\
x_1^i &= \frac{1}{(1+\alpha\Delta)}[x_1^{i-1}+\Delta F(x_0^{m+1+(N-i)})]\,, \\
\vdots &= \vdots\,, \\
x_1^N &= \frac{1}{(1+\alpha\Delta)}[x_1^{N-1}+\Delta F(x_0^{m+1})]\,.
\end{aligned}$$

Therefore, in vector notation, the equation that generalizes (7) is

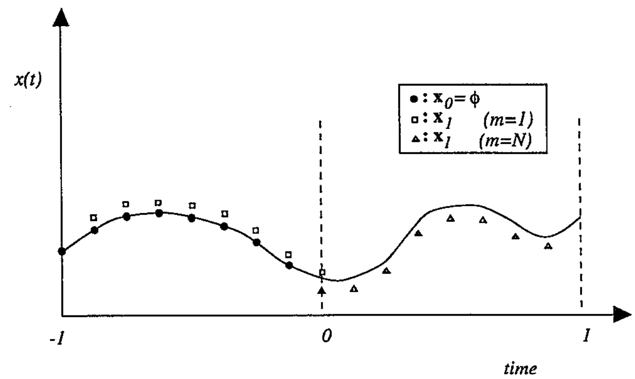

FIG. 1. Schematic illustration of the approximation of the differential delay equation (1) by a coupled map lattice: The initial function is replaced by a set of N points, and these N points form a vector that evolves in time under the action of an N-dimensional discrete time transformation (the coupled map lattice). The parameter $1\le m\le N$ denotes the number of elements of $\mathbf{x}_n$ which are not elements of $\mathbf{x}_{n+1}$. See text for details.

$$\mathbf{x}_{n+1}=\mathbf{B}_m\mathbf{x}_{n+1}+\mathbf{A}_m\circ\mathbf{x}_n\,, \qquad (10)$$

where the $N\times N$ matrices $\mathbf{A}_m$ and $\mathbf{B}_m$ are given by

$$\mathbf{B}_m=\begin{bmatrix}
0 & \cdots & & & & \cdots & 0\\
\vdots & & & & & \vdots & \\
0 & \cdots & & & & \cdots & 0\\
0 & \cdots & 0 & \frac{1}{(1+\alpha\Delta)} & & \cdots & 0\\
\ddots & & & & & \ddots & \\
0 & \cdots & & 0 & \frac{1}{(1+\alpha\Delta)} & 0 & 0\\
0 & \cdots & & & 0 & \frac{1}{(1+\alpha\Delta)} & 0
\end{bmatrix} \qquad (11)$$

where there are $N-(m-1)$ empty rows and $N-m$ zeros in the center row before $1/(1-\alpha\Delta)$ and

$$\mathbf{A}_m=\begin{bmatrix}
0 & \cdots & & 0 & 1 & 0 & \cdots & \\
0 & \cdots & & & 0 & 1 & 0 & \cdots\\
\ddots & & & \ddots & & & & \\
0 & \cdots & & & & & 0 & 1\\
\frac{\Delta F}{(1+\alpha\Delta)} & 0 & \cdots & & & & 0 & \frac{1}{(1+\alpha\Delta)}\\
0 & \frac{\Delta F}{(1+\alpha\Delta)} & 0 & \cdots & & & & 0\\
\ddots & & & \ddots & & & & \\
0 & \cdots & 0 & \frac{\Delta F}{(1+\alpha\Delta)} & 0 & \cdots & & 0
\end{bmatrix}, \qquad (12)$$

where there are m zeros in the first row before 1. In the case where $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ have no overlap, which corresponds to choosing $m=N$, these matrices become:

$$\mathbf{B}_N=\begin{bmatrix} 0 & \cdots & & 0 \\ \frac{1}{(1+\alpha\Delta)} & 0 \cdots & 0 & \\ \ddots & & & \ddots \\ 0 & \cdots & \frac{1}{(1+\alpha\Delta)} & 0 \end{bmatrix},$$

$$\mathbf{A}_N=\begin{bmatrix} \frac{\Delta F}{(1+\alpha\Delta)} & 0 & \cdots & 0 & \frac{1}{(1+\alpha\Delta)} \\ 0 & \frac{\Delta F}{(1+\alpha\Delta)} & 0 & \cdots & 0 \\ \ddots & & & \ddots & \\ 0 & \cdots & & 0 & \frac{\Delta F}{(1+\alpha\Delta)} \end{bmatrix}. \tag{13}$$

given the form of the matrix $\mathbf{B}_m$, it is possible to write

$$\mathbf{x}_{n+1}=(\mathbf{I}-\mathbf{B}_m)^{-1}\mathbf{A}_m\circ\mathbf{x}_n\equiv\mathbf{\Phi}_m(\mathbf{x}_n). \tag{14}$$

We will assume from now on that F is piecewise linear, because when this is the case, (10) can be simplified by replacing the composition in the right hand side by a simple multiplication,

$$\mathbf{x}_{n+1}=(\mathbf{I}-\mathbf{B}_m)^{-1}\mathbf{A}_m\mathbf{x}_n\equiv\mathbf{\Phi}_m(\mathbf{x}_n). \tag{15}$$

We have, therefore, reduced the differential delay equation (1) with a piecewise linear F to a piecewise linear CML $\mathbf{\Phi}_m$ which can be analyzed from a probabilistic point of view.

This probabilistic analysis is done *via* an investigation of the spectral properties of the Perron Frobenius operator associated with $\mathbf{\Phi}_m$. Before considering the Perron-Frobenius operator, we extend the reduction of deterministic DDE's to CML's to the case where the DDE's are subjected to stochastic perturbations.

B. The stochastic case

In this section, we explore the approximation of various stochastic DDE's by stochastic CML's. The stochastic DDE's we are concerned with are of the form

$$dx(t)=[-\alpha x(t)+F(x(t-1))]dt + G(x(t))\xi(t)dt, \tag{16}$$

where the stochastic process $\xi(t)dt$ will be either a δ-correlated stationary white noise process, or an Ornstein-Uhlenbeck process [20]. For both types of noise, the stochastic process $x(t)$ is called a solution of the differential equation (16) when it satisfies, with probability 1, the integral equation

$$x(t)=x(t_*)+\int_{t_*}^{t}[-\alpha x(s)+F(x(s-1))]ds + \int_{t_*}^{t}G(x(s))d\xi(s), \tag{17}$$

where $0<t_*<t$ and the second integral is a stochastic integral interpreted in either the Itô or the Stratonovich sense [20].

Define a partition of (t_*,t) by $t_*=s_0<s_1<\cdots<s_i<\cdots<s_k=t$. In the Stratonovich calculus, the stochastic integral in (17) is defined as the limit

$$\int_{t_*}^{t}G(x(s))d\xi(s)=\lim_{k\to\infty}\sum_{i=1}^{k}G\left[\frac{x(s_{i-1})-x(s_i)}{2}\right]\times[\xi(s_i)-\xi(s_{i-1})]. \tag{18}$$

Similarly, the Itô stochastic integral is defined to be

$$\int_{t_*}^{t}G(x(s))d\xi(s)=\lim_{k\to\infty}\sum_{n=1}^{k}G(x(s_{i-1}))\times[\xi(s_i)-\xi(s_{i-1})]. \tag{19}$$

Both definitions are clearly not equivalent. Unlike the usual Riemann or Lebesgue integrals which yield the same results when the integrand is such that both are defined, the Itô and Stratonovich integrals of the same function can differ. The choice of either definition must be motivated by careful analysis of the physical situation under consideration [21–23]. The exact formulation of a CML that results from a discrete time approximation depends on whether the stochastic integral in (17) is interpreted in the Itô or Stratonovich sense. However, as the reduction schemes for both cases are similar, we will illustrate it with the Itô interpretation of (17), which yields a more concise expression for the resulting CML.

The first step in the reduction procedure involves replacing the integrals in (17) by the appropriate sums,

$$x(t)=x(t_*)+\lim_{k\to\infty}\sum_{i=1}^{k}\Delta[-\alpha x(s_i)+F(x(s_i-1))] + \lim_{k\to\infty}\sum_{i=1}^{k}Q(x(s_{i-1}))[\xi(s_i)-\xi(s_{i-1})], \tag{20}$$

where by definition of the Riemann integral $\Delta\equiv(s_i-s_{i-1})>0$. The precise value of Δ depends on the difference $t-t_*$ and will be given below.

The sums in (20) are over a partition of the interval (t_*,t). Hence by choosing $k=1$, $t_*=s_0\simeq t-\Delta$, and $s_1\simeq t$, we obtain

$$\sum_{i=1}^{k}\Delta[-\alpha x(s_i)+F(x(s_i-1))] \to\Delta[-\alpha x(t)+F(x(t-1))],$$

$$\sum_{i=1}^{k}Q(x(s_{i-1}))[\xi(s_i)-\xi(s_{i-1})] \to Q(x(t_*))[\xi(t)-\xi(t_*)].$$

Therefore, (20) becomes

$$x(t)=x(t_*)+\Delta[-\alpha x(t)+F(x(t-1))] + Q(x(t_*))[\xi(t)-\xi(t_*)]. \tag{21}$$

As in the deterministic case, the second step of the reduction consists in approximating the function $\{x(s):$

$s\in[n\Delta-1,n\Delta)\}$ by an N-dimensional vector $\mathbf{x}_n$ as illustrated in Fig. 1. If the time t is discretized as in Sec. II A, the approximating solution (21) becomes the N-dimensional difference equation

$$\mathbf{x}_{n+1}=\mathbf{B}_m\mathbf{x}_{n+1}+\mathbf{A}_m\mathbf{x}_n+[\mathbf{Q}_m^{(1)}\circ\mathbf{x}_m+\mathbf{Q}_m^{(2)}\circ\mathbf{x}_{n+1}]\cdot[\xi_{n+1}-\xi_n]\ , \qquad (22)$$

where the matrices $\mathbf{A}_m$ and $\mathbf{B}_m$ are given in (12) and (11), respectively, and the matrices $\mathbf{Q}_m^{1,2}$ are given by

$$\mathbf{Q}_m^{(1)}=\begin{bmatrix} 0 & \cdots & & & & \cdots & & 0 \\ \vdots & & & & & \vdots & & \\ 0 & \cdots & & & & \cdots & & 0 \\ 0 & \cdots & 0 & Q & 0 & \cdots & & 0 \\ \ddots & & & & & \ddots & & \\ 0 & \cdots & & & & Q & 0 & 0 \\ 0 & \cdots & & & & 0 & Q & 0 \end{bmatrix}, \qquad (23)$$

where there are $N-(m-1)$ empty rows and $N-m$ zeros in the center row before Q and

$$\mathbf{Q}_m^{(2)}=\begin{bmatrix} 0 & \cdots & & & 0 & 1 & 0 & \cdots & \\ 0 & \cdots & & & & 0 & 1 & 0 & \cdots \\ \ddots & & & & \ddots & & & & \\ 0 & \cdots & & & & & & 0 & 1 \\ Q & 0 & \cdots & & & & & & 0 \\ 0 & Q & 0 & \cdots & & & & & 0 \\ \ddots & & & & \ddots & & & & \\ 0 & \cdots & 0 & Q & 0 & \cdots & & & 0 \end{bmatrix}. \qquad (24)$$

where there are m zeros in the first row before 1. The entries of the N-dimensional vector ξ_n are random variables that are independent of one another. Hence, we define the density $g(\xi_n)$ to be

$$g(\xi_n)\equiv\prod_{i=1}^{N}g(\xi_n)\ ,\quad n=0,1,2,\ldots\ . \qquad (25)$$

It is possible to study the probabilistic properties of the CML's defined in (15) and (22) by investigating the spectral characteristics of the operator that governs the evolution of probability densities in these systems. When the CML is deterministic as in (15), this operator is known as the Perron Frobenius operator. When the CML is stochastic, it will be referred to as the transfer operator.

III. THE EVOLUTION OF PROBABILITY DENSITIES IN CML'S

In this section, we recall some basic definitions associated with the evolution of probability densities under the action of discrete-time transformations in $\mathbb{R}^N$.

A. The deterministic case

A discrete-time nonsingular transformation Φ: $\mathbf{X}\mapsto\mathbf{X}(\mathbf{X}\subset\mathbb{R}^N)$ induces an operator denoted $\mathcal{P}_\Phi$ which acts on probability densities, and which is defined implicitly by the relation

$$\int_{\mathbf{A}}\mathcal{P}_\Phi f(\mathbf{x})d\mathbf{x}=\int_{\Phi^{-1}(\mathbf{A})}f(\mathbf{x})d\mathbf{x}\quad\text{for all } \mathbf{A}\subset\mathbf{X}\ , \qquad (26)$$

and all probability densities f. $\mathcal{P}_\Phi$ is called the Perron Frobenius operator induced by Φ, and a study of its properties will be the cornerstone of our probabilistic description of deterministic CML's. If the transformation Φ is piecewise monotone, it is possible to give a more explicit definition of $\mathcal{P}_\Phi$.

Let $\Phi_{|i}$ be the monotone restriction of Φ to the set $\pi_i\subset\mathbf{X}$, $i=1,\ldots,M$ (with of course $\cup_{i=1}^{M}\pi_i=\mathbf{X}$). Let $\tilde{\pi}_i$ denote the image of the set π_i: $\tilde{\pi}_i\equiv\Phi_{|i}(\pi_i)$. The Perron Frobenius operator induced by Φ can be written

$$f_{n+1}(\mathbf{x})\equiv\mathcal{P}_\Phi f_n(\mathbf{x})=\sum_{i=1}^{M}\frac{f_n(\Phi_{|i}^{-1}(\mathbf{x}))}{\mathcal{J}_\Phi(\Phi_{|i}^{-1}(\mathbf{x}))}\chi_{\tilde{\pi}_i}(\mathbf{x})\ , \qquad (27)$$

where $\chi_{\tilde{\pi}_i}(\mathbf{x})\equiv1$ if $\mathbf{x}\in\tilde{\pi}_i$, and 0 otherwise, and $\mathcal{J}_\tau(\mathbf{y})$ is the absolute value of the Jacobian of the transformation τ evaluated at $\mathbf{y}$. It is well understood [24–26] that the asymptotic properties of the sequence $\{f_n\}$ of the iterates of an initial density f_0 under the action of $\mathcal{P}_\Phi$ determine the thermodynamic behavior of the dynamical system Φ.

These asymptotic properties are themselves dependent on the spectral characteristics of the operator $\mathcal{P}_\Phi$. Before we describe these characteristics in more detail, we discuss the properties of the operator analogous to the Perron-Frobenius operator when the CML under consideration is stochastic rather than deterministic.

B. The stochastic case

In this case, the evolution of probability densities depends both on the deterministic part of the transformation, and on the type of noise present in the system. We distinguish two types of noise, which model different perturbation mechanisms: additive noise and multiplicative (or parametric) noise. The expressions for the transfer operators induced by nonsingular CML's perturbed by these types of noise are derived in [27].

1. Additive noise

In this case, the evolution of an element of the lattice transformation is given by a relation of the form

$$x_{n+1}^{(i)}=\Phi^{(i)}(\mathbf{x}_n)+\xi_n^{(i)}\equiv\Phi_{\text{add}}^{(i)}(\mathbf{x}_n)\ , \qquad (28)$$

where the density g of the vector random variable ξ is the product (25). The evolution equation for phase space probability densities in this case is written [28]

$$f_{n+1}(\mathbf{x})=\int_{\mathbf{X}}f_n(\mathbf{y})g(\mathbf{x}-\Phi(\mathbf{y}))d\mathbf{y},\quad n=0,1,\ldots\ . \qquad (29)$$

Equation (29) implicitly defines the transfer operator $\mathcal{P}_{\Phi_{\text{add}}}$ for CML's perturbed as in (28) since $\mathcal{P}_{\Phi_{\text{add}}}f_n(\mathbf{x})$

$=f_{n+1}(\mathbf{x})$.

We will return to a discussion of the convergence properties of the sequence $\{f_n\}$ in Sec. III C, and discuss the connection between these properties and the nonequilibrium thermodynamics of the associated CML's. We now give the analog of $\mathcal{P}_{\Phi_{\rm add}}$ when the noise is multiplicative.

2. *Multiplicative noise*

In this case, the evolution of a lattice site is given by

$$x_{n+1}^{(i)}=\Phi^{(i)}(\mathbf{x}_n)\times\xi_n^{(i)}\equiv\Phi_{\rm mul}^{(i)}(\mathbf{x}_n)\ , \tag{30}$$

and the transfer operator $\mathcal{P}_{\Phi_{\rm mul}}$ is given by [27]

$$f_{n+1}=\int_{x^{(N)}}^{1}\cdots\int_{x^{(1)}}^{1}f_n(\mathbf{y})\prod_{i=1}^{N}\left[g\left[\frac{x^{(i)}}{\Phi^{(i)}(\mathbf{y})}\right]\frac{1}{\Phi^{(i)}(\mathbf{y})}\right]d\mathbf{y},\quad n=0,1,\ldots\ . \tag{31}$$

It was shown in [27] that both $\mathcal{P}_{\Phi_{\rm add}}$ and $\mathcal{P}_{\Phi_{\rm mul}}$ are Markov operators defined by stochastic kernels. This property was then used to describe the asymptotic behavior of the sequence of densities $\{f_n\}$ (i.e., the convergence to a fixed point or to a limit cycle). But the evolution of $\{f_n\}$ reflects the spectral characteristics of the transfer operators defined in (29) and (31). It is, therefore, useful to briefly review these characteristics, and to discuss their connection with the nonequilibrium thermodynamics of CML's.

C. The convergence of the sequence $\{f_n\}$

At time n, the thermodynamic state of a nonsingular dynamical system Φ: $\mathbf{X}\mapsto\mathbf{X}$ (whether it be deterministic or stochastic) is completely described by the probability density f_n [24,25,26,29]. To see this, note that the probability $p(\mathbf{x}_n')$ of finding $\mathbf{x}_n\in\mathbf{X}$ between $\mathbf{x}_n'$ and $\mathbf{x}_n'+\delta\mathbf{x}_n'$ is

$$p(\mathbf{x}_n')=\int_{\mathbf{x}_n'}^{\mathbf{x}_n'+\delta\mathbf{x}_n'}f_n(\mathbf{y})d\mathbf{y}\ .$$

Hence, all the statistical quantifiers of the dynamics of Φ are computed with respect to f_n. If the sequence $\{f_n\}$ converges to a probability density f_* in the limit $n\to\infty$, the statistical quantifiers will be computed, for asymptotically large times, with respect to this equilibrium density f_*. On the other hand, if $\{f_n\}$ converges to a limit cycle, the quantifiers will remain time dependent for asymptotically large times, and the corresponding dynamical system will possess a thermodynamic equilibrium unlike those usually described in statistical mechanics. In this case, the notion of equilibrium must be extended to include sets of states visited sequentially in time. It is possible to formalize this heuristic discussion by considering specific convergence properties of $\{f_n\}$ and their link to the thermodynamics of Φ.

We mention four cases [30]. Φ is ergodic if and only if the sequence is *weak Cesàro convergent* to the invariant density f_*,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int_{\mathbf{X}}f_k(\mathbf{x})q(\mathbf{x})d\mathbf{x}=\int_{\mathbf{X}}f_*(\mathbf{x})q(\mathbf{x})d\mathbf{x}\qquad\text{for all } q\in L^\infty(\mathbf{X})\ ,$$

and all initial probability densities f_0. A stronger (but familiar) property, mixing, is equivalent to the *weak convergence* of the sequence to f_*,

$$\lim_{n\to\infty}\int_{\mathbf{X}}f_n(\mathbf{x})q(\mathbf{x})d\mathbf{x}=\int_{\mathbf{X}}q(\mathbf{x})f_*(\mathbf{x})d\mathbf{x}\qquad\text{for all } q\in L^\infty(\mathbf{X})$$

and all initial probability densities f_0. An even stronger type of chaotic behavior, known as *exactness* (or asymptotic stability) is reflected by the *strong convergence* of the sequence $\{f_t\}$ to the invariant density f_*,

$$\lim_{n\to\infty}\|\mathcal{P}_\Phi f_n-f_*\|_{L^1}=0$$

for all initial densities f_0 ($\|\cdot\|_{L^1}$ denotes the usual L^1 norm). We note without proof that exactness is interesting from a physical point of view because it is the only one of the properties discussed above that is a necessary and sufficient condition for the evolution of the thermodynamic entropy of the system Φ to a global maximum irrespective of the initial condition f_0. For details consult [26]. Exactness is also a special case of a more general convergence property of the iterates of f_0 which is known as asymptotic periodicity, and which has been discussed in detail in [31,32].

Definition 1: Asymptotic periodicity. A Markov operator $\mathcal{P}$ is asymptotically periodic if there exist finitely many distinct probability density functions $v_1,\ldots,v_r$ with disjoint supports, a unique permutation γ of the set $\{1,\ldots,r\}$ and positive linear continuous functionals $\Gamma_1,\ldots,\Gamma_r$, on $L^1(\mathbf{X})$ such that, for almost all initial densities f_0,

$$\lim_{n\to\infty}\left\|\mathcal{P}^n\left[f_0-\sum_{i=1}^{r}\Gamma_i[f_0]v_i\right]\right\|_{L^1}=0 \tag{32}$$

and

$$\mathcal{P}v_i=v_{\gamma(i)},\quad i=1,\ldots,r\ .$$

Clearly, if $\mathcal{P}$ satisfies these conditions with $r=1$, it is exact (or *asymptotically stable*). If $r>1$ and the permutation γ is cyclical, asymptotic periodicity implies ergodicity [26]. A rigorous discussion of asymptotically periodic Markov operators is given in [31]. A more intuitive presentation is given in [30]. □

The representation (32) implies that the ensemble densities asymptotically reach a stable limit cycle, and, therefore, that the equilibrium thermodynamic properties describing an ensemble of CML's will cycle periodically in time [29]. In order to give conditions on the parameters of (10) that guarantee the cyclical spectral decomposition (32), it is useful to recall some basic definitions concerning the notion of variation in $\mathbb{R}^N$ and some properties of linear operators.

D. Functions of bounded variation

The transfer operators discussed in this paper act on functions that are elements of normed linear spaces. The metric properties of these spaces depend on the choice of the norm. For reasons that will become clear in the next section, two natural norms arise in the descriptions of CML's: The familiar L^1 norm, and the so-called *bounded variation* norm. To introduce the latter, it is necessary to recall the definition of the variation of a high-dimensional function. The short discussion given here is based on the presentations of [33–35].

First, we define the gradient in the distributional sense ∇_d. Let f be a real-valued function defined on an open set $\mathbf{X}\subset\mathbb{R}^N$, and $\mathcal{C}^1(\mathbf{X})$ denote the space of differentiable functions from $\mathbf{X}$ to $\mathbf{X}$ having compact support. Then the operator ∇_d is the vector-valued measure defined by

$$\nabla_d f \equiv \left[\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_N}\right] .$$

With this definition, it is possible to define the *variation* of f,

$$\mathcal{V}(f)\equiv\|\nabla_d f\|_1$$

$$\equiv\sup\left\{\int_{\mathbf{X}} f\,\mathrm{div}h\,d\mu_L^{\mathbf{X}}:\ h\in\mathcal{C}^1(\mathbf{X}),|h|\le 1\right\},$$

$\mathbf{X}$. If f is $\mathcal{C}^1(\mathbf{X})$, it is straightforward to show [35] that this definition reduces to

$$\mathcal{V}(f)=\int_{\mathbf{X}}|\nabla_d f|_1 d\mu_L^{\mathbf{X}} , \tag{33}$$

where

$$|\nabla_d f|_1\equiv\sum_{i=1}^{N}\left|\frac{\partial f}{\partial x_i}\right| .$$

A more detailed presentation is given in Chap. 5 of [35], and in [34]. Giusti introduces functions of bounded variation in Chap. 1 of [33] in a somewhat more intuitive manner.

With the definition (33), it is possible to introduce the bounded variation norm:

$$\|\cdot\|_{BV}\equiv\mathcal{V}(\cdot)+\|\cdot\|_{L^1} . \tag{34}$$

The space of functions of bounded variation defined on $\mathbf{X}$ is a Banach space (cf. [33]) denoted $BV(\mathbf{X})$. In addition, the definition of variation given here implies that the space $BV(\mathbf{X})$ is relatively compact in L^1. Therefore, the probability densities describing the ensemble properties of our CML's are elements of a compact function space, and since such spaces are finite, the densities can be represented in terms of "basis states." To formalize this discussion, we recall the following result from the theory of linear operators due to Ionescu Tulcea and Marinescu.

E. The result of Ionescu Tulcea and Marinescu

This result was originally published in [36], and is of fundamental importance for our analytic description of the probabilistic properties of deterministic CML's.

Theorem 1: (adapted from Ionescu Tulcea and Marinescu [36]). Consider two Banach spaces

$$(\mathbb{A},\|\cdot\|_{\mathbb{A}})\subset(\mathbb{Y},\|\cdot\|_{\mathbb{Y}})$$

with the properties:

(a) If $\{f_n\}$ is a bounded sequence of elements of $\mathbb{A}$ such that

$$\lim_{n\to\infty}\|f_n-f\|_{\mathbb{Y}}=0 ,$$

where $f\in\mathbb{Y}$, then f is also an element of $\mathbb{A}$, and $\|f\|_{\mathbb{A}}\le\sup_n\|f_n\|_{\mathbb{A}}$.

(b) Let $\mathcal{P}$: $(\mathbb{A},\|\cdot\|_{\mathbb{A}})\mapsto(\mathbb{A},\|\cdot\|_{\mathbb{A}})$, be a bounded operator that can be extended to a bounded operator in $(\mathbb{Y},\|\cdot\|_{\mathbb{Y}})$.

(c) Suppose that there is an integer n such that

(1) If X is a $\|\cdot\|_{\mathbb{A}}$-bounded subset, then $\mathcal{P}^n X$ is compact in $\mathbb{Y}$.

(2) $\sup_n\|\mathcal{P}^n\|_{\mathbb{Y}}<\infty$.

(3) There exists $\omega\in(0,1)$ and $\Omega\ge 0$ such that

$$\|\mathcal{P}^n f\|_{\mathbb{A}}\le\omega\|f\|_{\mathbb{A}}+\Omega\|f\|_{\mathbb{Y}},\quad\text{for all } f\in\mathbb{A} . \tag{35}$$

If conditions (a)–(c) are satisfied, then the operator $\mathcal{P}$ is asymptotically periodic, and admits the spectral decomposition (32).

In order to apply this theorem to the study of CML's, we follow Gorá and Boyarski [37] in choosing $(\mathcal{A},\|\cdot\|_{\mathbb{A}})=(BV(\mathbf{X}),\|\cdot\|_{BV})$ included in $(\mathbb{Y},\|\cdot\|_{\mathbb{Y}})=(L^1(\mathbf{X}),\|\cdot\|_{L^1})$.

Verifying (a). By Theorem 1.9 of [33], if $\{f_n\}\in BV(\mathbf{X}),\|f_n\|_{BV}\le K$ for $n=1,2\dots$ and $f_n\to f$ in L^1 then $f\in BV(\mathbf{X})$ and $\|f_n\|_{BV}\le K$.

Verifying (b) and (c)(2). The operators under consideration here are Markov [30,27], and their operator norm is 1, hence (b) and (c)(2) are both verified.

Verifying (c)(1). This property follows from the compactness Theorem 1.19 of [33].

Hence, the theorem of Ionescu Tulcea and Marinescu guarantees that the transfer operators associated with CML's admits the spectral decomposition (32) if the condition (c3) is satisfied. By definition, the elements of $(BV(\mathbf{X}),\|\cdot\|_{BV})$ are of finite L^1 norm, so from (34), the inequality in (c)(3) becomes

$$\mathcal{V}(\mathcal{P}f)\le\omega\mathcal{V}(f)+\tilde{\Omega},\quad\tilde{\Omega}>0 , \tag{36}$$

where $\tilde{\Omega}=\Omega+\omega$. Inequality (34) guarantees that for all $f\in BV(\mathbf{X})$, the iterates $\mathcal{P}f$ will always remain in $BV(\mathbf{X})$: In some sense, the operator has a "smoothing" effect on the densities. In concrete examples, conditions on the parameters of the CML's under consideration will be obtained such that (36) holds, and, therefore, such that the corresponding transfer operator is asymptotically periodic.

IV. APPLICATIONS TO DETERMINISTIC DDE'S

In this section, we derive conditions on the control parameters of deterministic CML's which guarantee that the associated Perron Frobenius operator is asymptotically periodic [i.e., satisfies (32)].

A. Oscillatory solutions and expansion requirements

We say that a DDE possesses nontrivial statistical behavior when its solution is oscillatory and bounded (whether they are periodic, quasiperiodic or chaotic). Hence, for a given equation, we restrict our attention to the regions of parameter space in which the trajectories are oscillatory. To illustrate this point, we use a model with a piecewise linear transformation F similar to a DDE previously considered by Ershov [38]:

$$F(x)=\begin{cases} ax & \text{if } x<1/2 \\ a(1-x) & \text{if } x\geq 1/2 \end{cases} \quad a\in(1,2] . \tag{37}$$

The rationale for choosing this nonlinearity is that the resulting DDE displays a wide array of behaviors which is generic in more general (smooth) systems, while remaining amenable to analytic investigations. In addition, since F maps [0:1] into itself, we know (cf. Sec. 2.1 of [38]) that the solutions of the DDE will be bounded if the initial function takes value in [0,1] and if $a/\alpha\leq 2$.

The first fixed point of Eq. (1) with (37) is $x_*^{(1)}=0$. It is locally stable when $a<\alpha$, and unstable when $a>\alpha$. When $a>\alpha$, the equation possesses another fixed point

$$x_*^{(2)}=\frac{a}{a+\alpha} ,$$

which is linearly stable when

$$2\alpha\geq a>\alpha \quad\text{and}\quad \sqrt{a^2-\alpha^2}<\cos^{-1}\left[\frac{\alpha}{a}\right] .$$

When $\sqrt{a^2-\alpha^2}=\cos^{-1}(\alpha/a)$, the fixed point becomes unstable *via* a Hopf bifurcation, and the solutions of the DDE no longer converge to $x_*^{(2)}$. As mentioned above, the solutions must remain bounded when the initial function belongs to the interval [0,1], and since they do not converge to the fixed point, they must oscillate. We restrict our discussion of the dynamics of (1) to regions of parameter space in which the solutions are oscillatory, because stationary solutions are trivial from a statistical perspective. Hence our description of the probabilistic properties of (1) with (37) will focus on a region of parameter space in which

$$2\alpha\geq a>\alpha \quad\text{and}\quad \sqrt{a^2-\alpha^2}>\cos^{-1}\left[\frac{\alpha}{a}\right] . \tag{38}$$

Examples of oscillatory solutions of Eq. (1) with F given by (37) are shown in Fig. 2. The parameters used to produce that figure are the same as the ones used to produce the "ensemble density" results presented in Fig. 4 below. As expected, the remarkable agreement between the solutions obtained by both methods breaks down when N becomes too small (i.e., of order 10^2), and for large times when the solution is chaotic. Having derived

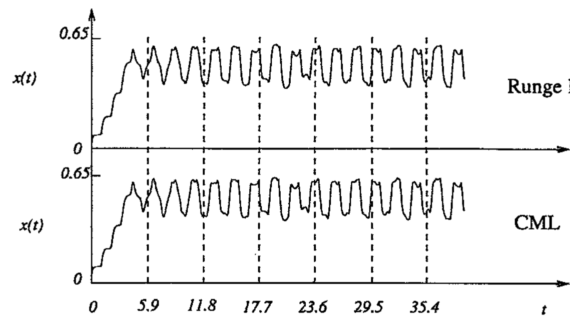

FIG. 2. Two numerical solutions of the DDE (1) with the nonlinearity given by (38), when $a=13$, $\alpha=10$, and a constant initial function $\varphi(s)=0.2$ for $s\in[-1,0)$. Top: The solution was produced by a standard adaptation of the fourth order Runge-Kutta method, with 40 points per delay. Bottom: The solution was produced by the CML approximation (16), with $m=N=1000$. As expected, although both solutions are in excellent agreement with one another, the Runge-Kutta method is numerically more efficient than the Euler approximation which underlies our derivation of the CML. The motivation for the CML approach is that it yields a system that is amenable to analytic investigations (cf. Sec. IV B).

the CML approximation to the DDE, we now use this expression to rigorously discuss the thermodynamic (or probabilistic) behavior of the equation.

B. The result

Suppose the parameters α and a of the DDE (1), with nonlinearity F given by (37), satisfy (38). Suppose further that

$$\lim\frac{N}{\alpha}=0 ,$$

where N denotes the number of elements in the approximating CML [i.e., $m=N$ in (15)]. Then if the initial function φ for the equation belongs to the interval [0,1], the corresponding CML (15) induces a Perron Frobenius operator that is asymptotically periodic and, therefore, admits the spectral decomposition (32). We now prove this statement.

Using the definition (27) and basic properties of the variation

$$\begin{aligned}\mathcal{V}(\mathcal{P}_\Phi f)&=\mathcal{V}\left[\sum_{i=1}^{M}\frac{f(\Phi_{|i}^{-1}(\mathbf{x}))}{\mathcal{J}_\Phi(\Phi_{|i}^{-1}(\mathbf{x}))}\chi_{\pi_i}(\mathbf{x})\right]\\ &\leq\sum_{i=1}^{M}\mathcal{V}\left[\frac{f(\Phi_{|i}^{-1}(\mathbf{x}))}{\mathcal{J}_\Phi(\Phi_{|i}^{-1}(\mathbf{x}))}\chi_{\pi_i}(\mathbf{x})\right]\\ &=\mathcal{J}_{\Phi^{-1}}\sum_{i=1}^{M}\mathcal{V}(f(\Phi_{|i}^{-1}(\mathbf{x}))\chi_{\pi_i}(\mathbf{x}))\end{aligned} \tag{39}$$

since $\mathcal{J}_\Phi(\Phi_i^{-1}(\mathbf{x}))$ is independent of both $\mathbf{x}$ and i when $\mathbf{\Phi}$ describes a lattice of coupled tent maps [39]. Each term in the sum on the right hand side of (39) can now be evaluated explicitly. From the definition (33),

$$\begin{aligned}\mathcal{V}(f(\Phi_{|i}^{-1}(\mathbf{x}))\chi_{\pi_i}(\mathbf{x}))&=\int_{\mathbf{X}}|\nabla_d[f(\Phi_{|i}^{-1}(\mathbf{x}))\chi_{\pi_i}(\mathbf{x})]|_1\\&\le\int_{\pi_i}|\nabla_d f(\Phi_{|i}^{-1}(\mathbf{x}))|_1 d\mu_L^{\mathbf{X}}+\int_{\mathbf{X}}|f(\Phi_{|i}^{-1}(\mathbf{x}))\nabla_d[\chi_{\pi_i}(\mathbf{x})]|_1 d\mu_L^{\mathbf{X}}\\&=\mathcal{V}(f(\Phi_{|i}^{-1}(\mathbf{x})))|_{\mathbf{x}\in\pi_i}+\int_{\mathbf{X}}|f(\Phi_{|i}^{-1}(\mathbf{x}))\nabla_d[\chi_{\pi_i}(\mathbf{x})]|_1 d\mu_L^{\mathbf{X}}\ .\end{aligned}\tag{40}$$

Since $\mathcal{J}_\Phi$ is independent of $\mathbf{x}$, a simple change of variables yields

$$\mathcal{V}(f(\Phi_{|i}^{-1}(\mathbf{x})))|_{\mathbf{x}\in\pi_i}\le\mathcal{J}_\Phi\times|D\Phi^{-1}|_1\times\mathcal{V}(f(\mathbf{x}))|_{\mathbf{x}\in\pi_i}\ ,\tag{41}$$

where $|D\Phi^{-1}|_1$ is the norm of the derivative matrix of the transformation Φ^{-1} (i.e., the sum of the absolute value of its entries). The integral in the right-hand side of Eq. (40) can be simplified using example 1.4 of [33], which states that for any $\mathbf{u}\in BV(\mathbf{X})$, and $\mathbf{A}\subset\mathbf{X}$ with piecewise C^2 boundaries of finite $(N-1)$-dimensional measure,

$$\int_{\mathbf{X}}|u(\mathbf{x})\nabla_d[\chi_{\mathbf{A}}(\mathbf{x})]|_1 d\mu_L^{\mathbb{R}^N}=\int_{\partial_{\mathbf{A}}}|u(\mathbf{x})|_1 d\mu_L^{\mathbb{R}^{N-1}}\ .$$

Choosing $u(\mathbf{x})=f(\Phi_{|i}^{-1}(\mathbf{x}))$, and $\mathbf{A}=\pi_i$, one obtains

$$\int_{\mathbf{X}}|f(\Phi_{|i}^{-1}(\mathbf{x}))\nabla_d[\chi_{\pi_i}(\mathbf{x})]|_1 d\mu_L^{\mathbb{R}^N}=\int_{\partial\pi_i}|f(\Phi_{|i}^{-1}(\mathbf{x}))|_1 d\mu_L^{\mathbb{R}^{N-1}}\ .\tag{42}$$

Furthermore, for any $u\in BV(\mathbf{X})$, and any $\mathbf{A}$ as specified above, we have from Lemma 3 of [37]

$$\int_{\partial\mathbf{A}}|u(\mathbf{x})|_1 d\mu_L^{\mathbb{R}^{N-1}}\le\frac{1}{\sin\theta(\mathbf{A})}\mathcal{V}(u(\mathbf{x}))|_{x\in\mathbf{A}}+\mathcal{K}_{\mathbf{A}}\ ,\tag{43}$$

where $\mathcal{K}_{\mathbf{A}}>0$ is bounded and $\sin\theta(\mathbf{A})$ depends on the smallest angle subtended by intersecting edges of the set $\mathbf{A}$. Letting $u(\mathbf{x})=f(\Phi_{|i}^{-1}(\mathbf{x}))$ and $\mathbf{A}=\pi_i$ as before, and recalling identity (41), the integral in (40) satisfies

$$\begin{aligned}\int_{\mathbf{X}}|f(\Phi_{|i}^{-1}(\mathbf{x}))\nabla_d[\chi_{\pi_i}(\mathbf{x})]|_1 d\mu_L^{\mathbb{R}^N}&\le\frac{1}{\sin\theta(\pi_i)}\mathcal{V}(f(\Phi_{|i}^{-1}(\mathbf{x})))|_{\mathbf{x}\in\pi_i}+\mathcal{K}_i\\&\le\frac{\mathcal{J}_\Phi\times|D\Phi^{-1}|_1}{\sin\theta(\pi_i)}\mathcal{V}(f(\mathbf{x}))|_{\mathbf{x}\in\pi_i}+\mathcal{K}_i\ .\end{aligned}\tag{44}$$

Letting $\sin\theta(\pi)\equiv\min_i\sin\theta(\pi_i)$, and using (40), (41), and (44), (39) becomes

$$\begin{aligned}\mathcal{V}(\mathcal{P}_\Phi f)&\le\mathcal{J}_{\Phi^{-1}}\times|D\Phi^{-1}|_1\sum_{i=1}^{M}\left[\mathcal{J}_\Phi\mathcal{V}(f(\mathbf{x}))|_{\mathbf{x}\in\pi_i}+\frac{\mathcal{J}_\Phi}{\sin\theta(\pi_i)}\mathcal{V}(f(\mathbf{x}))|_{\mathbf{x}\in\pi_i}+\mathcal{K}_i\right]\\&\le|D\Phi^{-1}|_1\left[1+\frac{1}{\sin\theta(\pi)}\right]\mathcal{V}(f)+M\max_i\mathcal{K}_i\ .\end{aligned}\tag{45}$$

Therefore, comparing (45) with (36), the theorem of Ionescu Tulcea and Marinescu guarantees the asymptotic periodicity of the Perron-Frobenius operator when

$$|D\Phi^{-1}|_1\left[1+\frac{1}{\sin\theta(\pi)}\right]<1\ .$$

Using $m=N$ in Eq. (15) and the matrices in (13), it is straightforward to calculate the norm $|D\Phi^{-1}|_1$ when the parameters α and a satisfy (38), and when in addition, we take the limit,

$$a,\alpha\to\infty,\quad\frac{\alpha}{N}\to\infty\ .\tag{46}$$

In this case, the norm of $D\Phi^{-1}$ is easily shown to be α/Na, and so the CML associated with the DDE induces an asymptotically periodic Perron-Frobenius operator when

$$\frac{\alpha}{Na}\left[1+\frac{1}{\sin\theta(\pi)}\right]<1\ .\tag{47}$$

In order for this inequality to yield precise conditions on the control parameters of the CML, it is necessary to evaluate the quantity $\sin\theta(\pi)$ in terms of these parameters. This calculation is straightforward but lengthy, and has been published for general situations of which the present problem is a special case [40]. If the boundaries of the sets π_i intersect at an angle that is bounded below by $\rho>0$ (for all $i=1,\dots,M$), then when $\mathbf{X}$, and thus the π_i, are subsets of $\mathbb{R}^N$, we have [40]

$$\sin\theta(\pi)=\left[\frac{1-\cos\rho}{N[1+(N-2)\cos\rho]}\right]^{1/2}\ .\tag{48}$$

Note that if the boundaries of the image sets intersect at right angles so that $\rho=90°$, we have $\sin\theta(\pi)=1/\sqrt{N}$. In general, the image partition is not rectangular, and the angle ρ must be determined from the definition of the CML under consideration, but in the limit (46), the image is rectangular since the local maps are effectively decoupled. Hence, our criterion for asymptotic periodicity becomes

$$\frac{\alpha(1+\sqrt{N})}{Na}<1 . \tag{49}$$

The inequality is always satisfied for N large enough and $a>\alpha$. Theorem 1, therefore, implies that $\mathcal{P}_{\Phi_N}$ is always asymptotically periodic, when Φ_N approximates the DDE (1) with F given by (37) under the conditions (38) and (46) (though the period could be 1). This result is expected in the limit (46) because in that limit, the coupling between the elements of the CML effectively vanishes, and the resulting lattice transformation could have been obtained by a singular perturbation limit procedure (cf. [46]). In that case, the CML (15) behaves like a collection of *uncoupled* tent maps with slope $\in[1,2]$ (see the discussion in subsection V on noise-induced statistical cycling), and it is well known that such a system should be asymptotically periodic (although for $a>\sqrt{2}$ the period is 1).

Numerically, the presence of asymptotic periodicity of the Perron-Frobenius operator should be reflected by the temporally periodic behavior of various statistical descriptors of the motion. As an example, consider the "ensemble sample density" $p(x,t)$. This function is obtained by integrating (1) with a large number of different initial functions $\{\varphi_1,\ldots,\varphi_E\}$ (E large) and then, at time t, binning the set of points $\{x_{\varphi_i}(t)\}$, where $x_{\varphi_i}(t)$ denotes the solution of the DDE corresponding to the initial function φ_i. Schematically, Fig. 3 displays this construction. To establish a parallel with more frequently discussed models, if the equation satisfied by $x(t)$ was an ordinary differential equation (rather than a DDE), the evolution of $p(x,t)$ would be described by the Liouville equation.

The statistical cycling predicted by (49) can be observed numerically by following the function $p(x,t)$ for successive times. Figure 4 displays such a numerical simulation for the DDE (1) with F defined in (37). We have chosen to numerically illustrate the presence of statistical cycling in a region in which the stringent requirements (46) are *not* satisfied (in Fig. 4, note that $N \gg a$).

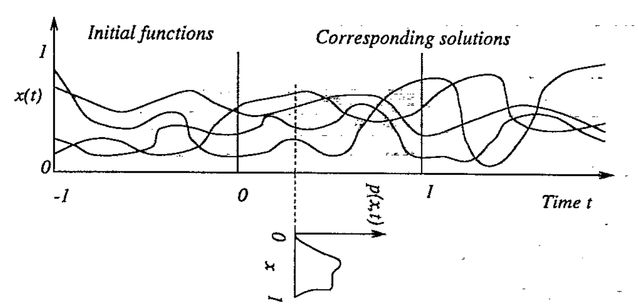

FIG. 3. Schematic illustration of the construction of the "sample ensemble density" $p(x,t)$. A set of initial conditions $\{\varphi_1,\ldots,\varphi_E\}$ generates a set of solutions $\{x_{\varphi_i}\}_{i=1}^{E}$, and at time t, the distribution of values $\{x_{\varphi_i}(t)\}_{i=1}^{E}$ is given by $p(x,t)$.

The reason for this choice of parameters is to illustrate that the cycling of densities is probably present in regions of parameter space larger than those in which both (38) and (46) are satisfied, although at present we are unable to prove this statement. The feature displayed in Fig. 4 is the dependence of the asymptotic density cycle on the initial density which describes the set of initial functions used to carry out a set of simulations. This property is not observed in continuous-time systems without delays, and it can be understood in light of the dependence of the functionals $\Gamma_{1,\ldots,r}$ of Eq. (32) on the initial density f_0. This dependence on initial conditions is in a sense much stronger than that usually discussed in relation to chaotic dynamical systems: Here the evolution of an ensemble of DDE's depends on the exact distribution of the initial ensemble.

Since the presence of stochastic fluctuations in experimental situations is ubiquitous, it is of interest to discuss the presence of statistical cycling and asymptotic periodicity in models that are stochastically perturbed.

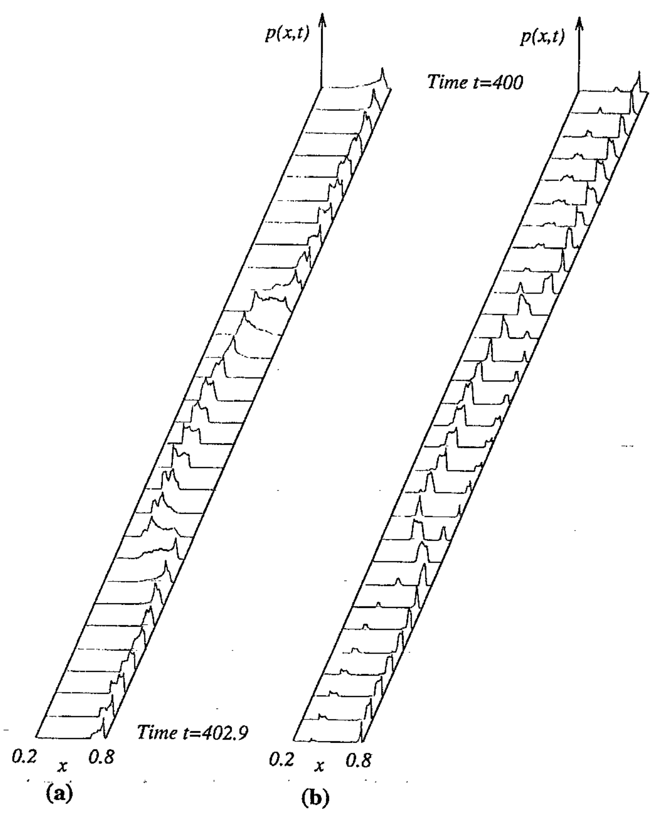

FIG. 4. Statistical cycling in an ensemble of DDE's of the form (1) with nonlinearity F given by (38). The parameters in the equation are $a=13$, $\alpha=10$, and the CML used for the solution contained $N=10^3$ sites. Both (a) and (b) were produced with 22 500 initial functions. (a) Each of the initial functions was a random process supported uniformly on [0.65,0.75]. (b) The initial functions were random processes supported either on [0.65,0.75] (for 17 000 cases) or on [0.35,0.45] (for the remaining 5500 initial functions). The cycling is not transient, and is observed for all times. The dependence of the density cycle on the initial density reflects the dependence of the Γ_i of (33) on f_0.

V. APPLICATION TO STOCHASTIC MODELS

In this section, we investigate the properties of the transfer operators (29) and (31) induced by the stochastic CML's (28) and (30). Our discussion is based on the results presented in [27]. In the deterministic case, deriving Eq. (49) required detailed knowledge of the function F. In the stochastic case, our results are, to a large extent, independent of the details of the model, and can be summarized as follows:

If the solution $x(t)$ of the stochastic DDE (16) belongs to [0,1] for all $t>0$, then the CML approximating (16) induces an asymptotically periodic transfer operator.

This statement stems from the application to Eq. (29) and (31) of results presented in [27]. We can state the following theorem, adapted from [27].

Theorem 2. Let $K:\mathbf{X}\times\mathbf{X}\mapsto\mathbb{R}$ be a stochastic kernel (recall $\mathbf{X}=[0,1]\times\cdots\times[0,1]$), $\mathcal{P}_1$ and $\mathcal{P}_2$ be two Markov operators defined by

$$\mathcal{P}_1 f(\mathbf{x})=\int_{\mathbf{X}} K(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y}\ ,$$
$$\mathcal{P}_2 f(\mathbf{x})=\int_{x^{(N)}}^{1}\cdots\int_{x^{(1)}}^{1} K(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y}\ .$$

Assume that there is a nonnegative $\lambda>1$ such that for every bounded $\mathbb{B}\subset\mathbf{X}$ there is a $\delta=\delta(\mathbb{B})>0$ for which

$$\int_{\mathbb{A}} K(\mathbf{x},\mathbf{y})d\mathbf{x}\le\lambda \quad \text{for } \mu(\mathbb{A})<\delta,\ \ \mathbf{y}\in\mathbb{B},\ \ \mathbb{A}\subset\mathbb{B}\ . \tag{50}$$

Assume further there exists a Lyapunov function V: $\mathbf{X}\mapsto\mathbb{R}$ such that

$$\int_{\mathbf{X}} V(\mathbf{x})\mathcal{P}_{1,2}f(\mathbf{x})d\mathbf{x}\le\alpha\int_{\mathbf{X}} V(\mathbf{x})f(\mathbf{x})d\mathbf{x}+\beta\ ,$$
$$\alpha\in[0,1),\ \ \beta>0 \tag{51}$$

for every density f. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are *asymptotically periodic,* and, therefore, admit the representation (32). [Recall that a nonnegative function V: $\mathbf{X}\mapsto\mathbb{R}$ is known as a *Lyapunov function* if it satisfies $\lim_{|\mathbf{x}|\to\infty}V(\mathbf{x})=\infty$.]

To apply this theorem to $\mathcal{P}_{\Phi_{\mathrm{add}}}$ and $\mathcal{P}_{\Phi_{\mathrm{mul}}}$, let

$$K_{\mathrm{add}}(\mathbf{x},\mathbf{y})=g(\mathbf{x}-\mathbf{\Phi})$$

for the additive noise case, and

$$K_{\mathrm{mul}}(\mathbf{x},\mathbf{y})=\prod_{i=1}^{N}\left[g\left[\frac{x^{(i)}}{\Phi^{(i)}(\mathbf{y})}\right]\frac{1}{\Phi^{(i)}(\mathbf{y})}\right]$$

for the multiplicative noise case. Hence, from (29) and (31)

$$\mathcal{P}_{\Phi_{\mathrm{add}}}f(\mathbf{x})=\int_{\mathbf{X}} K_{\mathrm{add}}(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y}$$
$$\mathcal{P}_{\Phi_{\mathrm{mul}}}f(\mathbf{x})=\int_{x^{(N)}}^{1}\cdots\int_{x^{(1)}}^{1} K_{\mathrm{mul}}(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y}\ .$$

Since g is a normalized probability density on $\mathbf{X}$, $K_{\mathrm{add,mul}}(\mathbf{x},\mathbf{y})>0$, and $\int_{\mathbf{X}} K_{\mathrm{add,mul}}(\mathbf{x},\mathbf{y})=1$, so that both $K_{\mathrm{add,mul}}(\mathbf{x},\mathbf{y})$ are stochastic kernels. In addition, it was shown in [27] that they both satisfy the inequalities (50) and (51). Hence, Theorem 2 [27] yields the desired results on the asymptotic periodicity of CML's approximating DDE's perturbed by additive and multiplicative noise.

This result depends on two assumptions: The noise density g must be normalized on the hypercube $\mathbf{X}$, and the solution $x(t)$ defined in (17) must remain in the interval [0,1] for all $t>0$. One of the intriguing consequences of the ubiquitous presence of asymptotic periodicity in stochastic CML's is explored in the next section.

Noise-induced statistical cycling

The effects of noise on dynamical systems have been the subject of intense investigations (an extensive overview of this literature is given in [41,42]). It is well understood that the presence of noise in nonlinear models can result in profound qualitative changes of the systems under study. The mechanisms that bring about these changes depend to a large extent on the specificities of the model under consideration. For example, Kapral and Celarier [43] have discussed the influence of additive noise on bistable systems, and showed in this case how the noise-induced transitions reflect a crossing of the basin boundaries. Here we describe a different class of noise-induced transitions in a DDE of the form (1) with a nonlinearity F given by

$$F(x)=(ax+b)\mathrm{mod}1,\ \ 0<a<1,\ \ 0<b<1\ . \tag{52}$$

As we will show, these transitions are best understood as resulting from a noise-induced bifurcation in the deterministic part of the model. The motivation for this choice of F is twofold. First, the behavior of the corresponding one-dimensional map $x_{n+1}=F(x_n)$ has been well documented [44]. Second, the presence of stochastic perturbations in Eq. (1) with F given by (52) can result in qualitative changes in the statistical behavior of the solutions [30,45]. Before describing the behavior of the DDE, it is helpful to recall several properties of the map $x_{n+1}=F(x_n)$, which are due to Keener [44].

As expected, the map $x_{n+1}=F(x_n)$ with F defined in (52) can possess stable limit cycle solutions, when $0<a<1$, $0<b<1$. Less expected is the presence in the same region of parameter space (i.e., when $0<a<1$, $0<b<1$), of aperiodic trajectories that are attracted to a Cantor set in [0,1]. In either case, the solutions cannot be described by probability densities. Since the trajectories visit only a Cantor set in the asymptotic regime, the probability of occupation of phase space is not a differentiable function. In fact, the probabilistic properties of the map (52) are described by the evolution of noncontinuous measures, rather than the more usual probability densities. For fixed $0<a<1$, a change in the parameter $0<b<1$ can, therefore, result in a bifurcation from a periodic solution to a chaotic one. A similar behavior is observed when the system is perturbed by noise, such that

$$F(x)=(ax+b+\xi)\mathrm{mod}1,\ \ 0<a<1,\ \ 0<b<1\ , \tag{53}$$

where ξ is a δ-correlated discrete time random process distributed uniformly on subintervals of [0,1]. If ξ has the "right amplitude," the map undergoes noise-induced bifurcations. These cannot be described deterministically however, since the map is then stochastic, but must be described in terms of the evolution of probability densities. Such noise-induced bifurcations have been discussed in the one-dimensional map [45], and in diffusively coupled lattices of the map (53) [27]. We now give numerical evidence that similar behavior is expected in the stochastic differential equation (1) with F given by (53).

Figure 5 displays the temporal evolution of the ensemble sample density $p(x,t)$ in the absence and the presence of noise, and clearly illustrates the presence of noise-induced statistical cycling in this equation. The behavior displayed in Fig. 5 can be understood by noting the similarities between one dimensional maps $x_{n+1}=F(x_n)$ and equations of the form (1). Ivanov and Sharkovskiĭ [46] have studied the dynamics of the DDE (1) by noting that in the limit $\alpha\to\infty$, and $F/\alpha\to\tilde{F}$ in (1), and with discrete time, one obtains the one-dimensional map $x_{n+1}=\tilde{F}(x_n)$. Although in general the bifurcation structure of the one-dimensional map need not survive this singular perturbation [47], Ivanov and Sharkovskiĭ showed that when the function $\tilde{F}$: $\mathbb{R}\mapsto\mathbb{R}$ leaves a subset of $\mathbb{R}$ invariant, (in our case, this subset is [0,1]), then the solutions of the DDE visit the locations on [0,1] which are visited by the iterates of the map (consult [46] for precise statements). On the basis of their analysis, one, therefore, expects that the bifurcation structure of the one-dimensional map will,

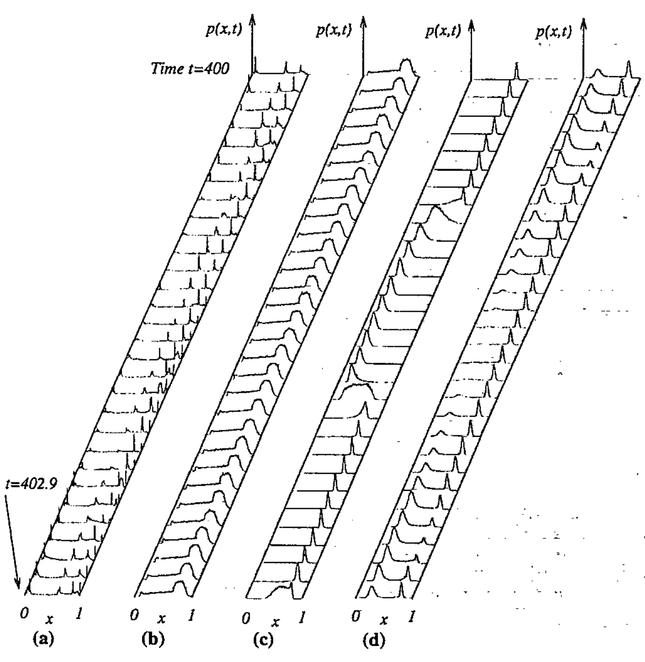

FIG. 5. Noise induced statistical cycling in (1) with F given by (57). As in Fig. 4, each simulation was performed with 22 500 random initial functions. In all four panels, the parameters of the equation were $a=0.5$, $b=0.567$, and $\alpha=10$. For panels (a)–(c) the initial density was as in Fig. 4(a). (a) No noise in the system: $p(x,t)$ is not a density, but a generalized function (see text for details). (b) Noise present as in (57), supported uniformly on [0,0.1]. The system is asymptotically stable, and $r=1$ in (33). (c) Noise uniformly supported on [0,0.2], and $r=2$ in (33). (d) Same noise as in (c), with an initial density as in Fig. 4(b). Here Γ_1 and Γ_2 of Eq. (33) are not the same as in panel (c) because the initial density $\mathbf{f}_0$ has changed.

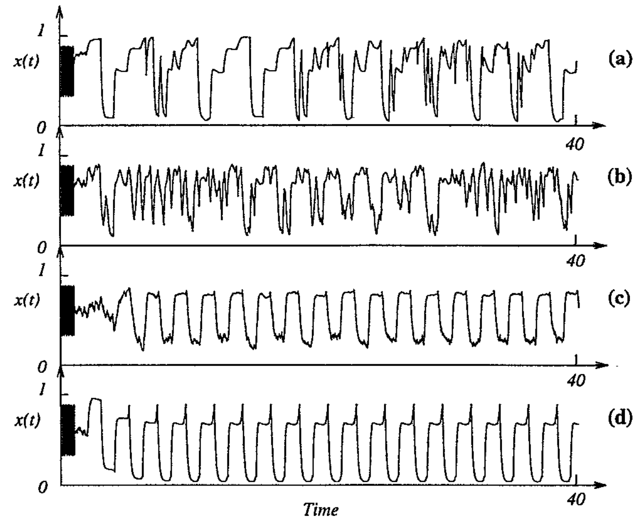

FIG. 6. Four solutions of the DDE (1) with the stochastic and deterministic forcing terms (57) and (56). Parameters for the top three panels are $a=0.5$, $b=0.567$, and $\alpha=10$. In all panels the initial function is a random process supported uniformly on [0.3,0.9]. (a) The function F is given by (56). (b) The function F is now given by (57), and the noise term ξ is supported uniformly on [0,0.1]. (c) Again, F is given by (57), but the noise term is supported uniformly on [0,0.2]. (d) Here F is given by (56), $\alpha=10$ as in panels (a)–(c), but $b=0.667$. This fourth solution shows by comparison to panel (c) that the parametric noise in (57) has an effect on the system which is akin to an increase of the parameter b from $b=0.567$ to $b=0.667$.

in some regions of parameter space, yield information on the bifurcation structure of the corresponding DDE.

Numerically, the bifurcations from chaotic to periodic attractors in the discrete map are also found in the deterministic DDE, and as illustrated in Fig. 6, they can be induced by the perturbation of the function F as in (53).

Hence, the noise-induced statistical cycling displayed in Fig. 5 probably reflects the presence in the DDE of a noise-induced bifurcation from a chaotic attractor to a periodic one. The presence of noise superimposed on the periodic solution could then explain the cyclical statistical behavior of Figs. 5(c) and 5(d). It is interesting to note that this phenomenon is consistent with the spectral decomposition (32) which was obtained using rather general considerations.

The behavior displayed in Fig. 5, though somewhat counterintuitive at first glance, is, therefore, not unexpected for systems possessing limit cycles and chaotic attractors that are close in some sense in the space of control parameters.

VI. DISCUSSION

The analytical results presented in this paper hold for CML's which approximate the dynamics of DDE's to an arbitrary degree of accuracy, but the techniques we used cannot obviously be extended to the infinite dimensional case (some extensions of the notion of variation in infinite dimensions have been considered [34], but the theory is not complete enough to allow the in-depth treatment possible with finite-dimensional maps). Hence, the present work opens the way for a more rigorous study of the present approximations, in the limit where the difference between the DDE and the CML vanishes. From a practical perspective however, the fact that there is no finite limit on the accuracy of our description of DDE's renders the presence of asymptotic periodicity in these systems inevitable. This is interesting because the spectral decomposition (32) has not been rigorously defined in continuous-time settings, and the present work clearly indicates that it should be possible to define it for certain semigroups of Markov operators which describe the evolution of ensembles of trajectories.

Finally, the strong connection demonstrated here between models framed as delay differential equations and models framed as coupled map lattices opens the way for the cross applications of techniques that have traditionally been used for the exclusive investigation of one or the other of these classes of models.

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