ELSEVIER

Physica D 80 (1995) 1–17

PHYSICA D

Asymptotic stability of densities in coupled map lattices

Michael Mackey [a], John Milton [a,b,*]

[a] *Center for Nonlinear Dynamics and Department of Physics, McGill University, 3655 Drummond, Rm 1125, Montréal, Qué. H3G-1Y6 Canada*

[b] *Department of Neurology and Committee on Neurobiology, The University of Chicago Hospitals, MC2030, 5841 South Maryland, Chicago, IL 60637, USA*

Received 10 May 1993; revised 20 June 1994; accepted 22 June 1994

Communicated by K. Kaneko

Abstract

Sufficient conditions for the asymptotic stability (strong convergence) of density evolution in finite dimensional piecewise monotone map lattices with both constant and state dependent coupling are given. These conditions are quite useful in numerical work since they allow one to precisely define where one expects to see numerical signatures of asymptotic stability. Asymptotic stability is illustrated with several examples. It is also shown that in constantly coupled lattices the density formed by collapsing the higher dimensional density into one dimension can be approximated by the evolution of densities under the action of an appropriately perturbed one-dimensional map.

1. Introduction

Coupled map lattices have elicited substantial interest among applied scientists since they serve as paradigms for a variety of spatially extended dynamical systems such as interacting biological populations [27], neural networks [3,7], convection and fully developed turbulence in fluid flow [3,5,9,10] and pattern formation in chemical oscillators [9,14]. Typically, for one space dimension, diffusively coupled map lattices are written in the form

$$x_{t+1}^{j} = (1-\epsilon)F(x_t^j) + \frac{\epsilon}{2}\left[F(x_t^{j-1}) + F(x_t^{j+1})\right], \quad \epsilon \in (0,1] \quad j = 1,\cdots,L, \tag{1}$$

where the map $F : [0,1] \to [0,1]$ determines the local dynamics and ϵ is the coupling coefficient. The index $j = 1,\cdots,L$ specifies the lattice position. If the lattice is finite, it is often the case that $j = L+1$ is identified with $j = 1$, so the boundary conditions are periodic. These lattice equations can be derived from certain partial differential equations by discretizing physical space and time [23].

* To whom correspondence should be addressed.

SSDI 0167-2789(94)00159-N

Numerical studies have documented that coupled map lattices described by (1) can produce interesting and complex dynamics including spatio-temporal intermittency [3,5,6,10]. As the dynamics of the lattice are extended beyond the simplest bifurcations of F, it becomes more natural to use statistical approaches to extract physically meaningful information about the lattice dynamics. Bunimovich and Sinai [2], and subsequently Volevich [28], applied a variety of techniques from statistical mechanics to study the dynamics of a coupled map lattice with state dependent coupling

$$\begin{aligned} x_{t+1}^j &= \left[1 - \epsilon(S(x_t^j))\right] S(x_t^j) + \frac{\epsilon(S(x_t^j))}{2} \left[S(x_t^{j-1}) + S(x_t^{j+1})\right], \\ &\equiv \mathcal{W}(x_t^{j-1}, x_t^j, x_t^{j+1}) \qquad j = 1, \cdots, L \end{aligned} \tag{2}$$

on an *infinite dimensional lattice* under the following assumptions:

(i) S is piecewise monotone;

(ii) Each branch of S maps $[0,1]$ onto itself;

(iii) $\inf_x |S'(x)| \geq \alpha > 1$;

(iv) $\epsilon \in C^2([0,1])$, $0 \leq \epsilon \leq 1$ for $x \in [0,1]$, and $\epsilon(0) = \epsilon(1) = 0$.

For this special case they were able to prove that $\mathcal{W}$ is both ergodic and mixing for sufficiently small $\max_y \epsilon(y) > 0$ and $\max_y |\epsilon'(y)| > 0$. Recently, Keller and Künzle [11] have extended the work of Bunimovich and Sinai [2] by:

(i) eliminating assumption (2);

(ii) replacing (3) by the assumption that each branch of S is C^2 and $\inf_x |S'(x)| > 2$;

(iii) replacing (4) by $\epsilon \in C^2([0,1])$.

They were able to prove the ergodicity and mixing of $\mathcal{W}$ on *finite dimensional lattices* for sufficiently small $\max_y \epsilon(y) > 0$.

Numerical studies, on finite lattices by necessity, suggest that the properties of ergodicity and mixing hold for moderately large ranges of the map and coupling parameters [5, 21]. Despite the power of the results of Bunimovich and Sinai [2] and Keller and Künzle [11], a number of practical issues presently limit their usefulness for the interpretation of these numerical studies. Typically, the investigator must decide for a given choice of parameters whether his numerical studies reflect ergodicity-mixing, long transients, or possibly numerical artifacts. The fact that ergodicity is guaranteed provided certain parameters are sufficiently small provides little reassurance since in practice one does not know what "sufficiently small" means.

A second issue concerns the nature of the invariant measure of (2) whose existence is ensured by the property of ergodicity as well as that of mixing. The estimation of the density of this invariant measure is of paramount importance to experimentalists since it can be used to calculate measurable quantities using ensemble averages. How is this density to be calculated ?

A final issue concerns how measurable quantities determined from the density of the invariant measure evolve with time. Consider, for example, the Boltzmann-Gibbs entropy, $\mathcal{H}$. It has been shown that mixing is not sufficient to ensure that $\mathcal{H}$ converges to a maximum [19]. The convergence of $\mathcal{H}$ to a maximum requires demonstration of a stronger type of convergence than that guaranteed by the above theorems [2,11], namely asymptotic stability. Asymptotic stability is equivalent to the strong convergence of densities to a unique density. Although asymptotic stability implies ergodicity and mixing, the converse is not true.

Here we give an alternate approach to the determination of the statistical properties of (1) with both variable (i.e. (2)) and constant coupling. This approach is based on older results of Krzyzewski and Szlenk [13] and Krzyzewski [11] which are summarized in [15]. The main advantages of this approach are that estimates of the parameter ranges for which invariant measures exist can be obtained, as well as conditions which ensure asymptotic stability. Moreover an estimate of the density of the invariant measure can be obtained for coupled map lattices with constant coupling.

The concepts of ergodicity, mixing, and asymptotic stability are reviewed in Section 2 from the point of view of the evolution of densities under the action of a discrete-time dynamical system. In Section 3 we derive sufficient conditions for the asymptotic stability of lattices with both state dependent and constant coupling. Our results are illustrated by coupled map lattices composed of the Rényi map [24,25], of a map introduced by Manneville [20] for the study of intermittency, the tent map, and the Mori [22] map in Section 4. A consideration of other coupling schemes, including mean field coupling, is given briefly in Section 5. Finally, in Section 6 we develop an approach for analytically calculating approximations to the one-dimensional density formed by collapsing the L-dimensional density in lattices with constant coupling.

2. Mathematical background

We briefly review the evolution of densities [15] under the action of a map $\mathcal{W} : X \to X$. By a density we mean a positive normalized L^1 function $f : X \to R$, *i.e.* f is a density if $f \geq 0$ and $\int_X f\,dx = 1$. Given a density f, then the corresponding measure $\mu_f(A)$ of a set $A \subset X$ is defined by $\mu_f(A) = \int_A f(x)\,dx$, and f is called the density of the measure μ_f. Having a density f, the associated measure μ_f, and a non-singular map $\mathcal{W} : X \to X$ then $\mathcal{W}$ is said to be measure preserving with respect to μ_f if $\mu_f(\mathcal{W}^{-1}(A)) = \mu_f(A)$, where $\mathcal{W}^{-1}(A)$ is the counterimage of the set A. Alternately, this is expressed by saying that μ_f is an invariant measure with respect to $\mathcal{W}$. Any set $A \subset X$ such that $\mathcal{W}^{-1}(A) = A$ is called an invariant set. Given a density f, any invariant set A such that $\mu_f(A) = 0$ or $\mu_f(X \setminus A) = 0$ is called trivial.

The dynamics $\mathcal{W}$ are said to be ergodic if every invariant subset A is trivial. If $\mathcal{W}$ is measure preserving with respect to μ_{f_*}, then $\mathcal{W}$ is mixing if $\lim_{t\to\infty} \mu_{f_*}(A \cap \mathcal{W}_t^{-1}(B)) = \mu_{f_*}(A)\mu_{f_*}(B)$ for all $A, B \subset X$. If $\mathcal{W}$ is measure preserving with respect to μ_{f_*} then we say it is asymptotically stable if $\lim_{t\to\infty} \mu_{f_*}(\mathcal{W}_t(A)) = 1$ for all $A \subset X$ such that $\mu_{f_*}(A) > 0$. Asymptotic stability implies mixing which implies ergodicity, but not *vice versa*.

Since $x_{t+1} = \mathcal{W}(x_t)$, the evolution of a density f under the action of $\mathcal{W}$ is formally given by

$$\int_A P_{\mathcal{W}} f(u)\,du = \int_{\mathcal{W}^{-1}(A)} f(u)\,du \qquad A \subset X,$$

where the operator $P_{\mathcal{W}}$ is known as the Frobenius-Perron operator corresponding to $\mathcal{W}$. If there is a density f_* such that $P_{\mathcal{W}} f_* = f_*$, then we call f_* a stationary density of $P_{\mathcal{W}}$. If f_* exists, then it can be shown that this is equivalent to the invariance of the measure μ_{f_*} with respect to the dynamics $\mathcal{W}$. If $\mathcal{W}$ is ergodic, mixing, or asymptotically stable, then we say that the corresponding Frobenius Perron operator $P_{\mathcal{W}}$ has the same property.

The dynamics $\mathcal{W}$ generate a sequence of densities $\{P_{\mathcal{W}}^t f\}_{t=0}^{\infty}$. From a technical point of view three types of convergence of this sequence of densities can be distinguished. First, if $\mathcal{W}$ is measure preserving with respect to μ_{f_*}, then there may be weak Cesàro convergence of $P_{\mathcal{W}}^t f$ to f_*, i.e.

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \langle P_{\mathcal{W}}^k f, g\rangle = \langle f_*, g\rangle \qquad \text{for all initial densities } f \text{ and } g \in L^{\infty}.$$

Weak Cesàro convergence of $P_{\mathcal{W}}^t f$ to f_* is equivalent to the ergodicity of $\mathcal{W}$.

The second type of convergence is weak convergence, i.e.

$$\lim_{t\to\infty} \langle P^t_{\mathcal{W}} f, g\rangle = \langle f_*, g\rangle \qquad \text{for all initial densities } f \text{ and } g \in L^\infty,$$

and is equivalent to mixing. The work of Bunimovich and Sinai [2] and Keller and Künzle [11] demonstrate that this type of convergence can occur in coupled map lattices described by (2).

Finally, there can be strong convergence of $P^t_{\mathcal{W}} f$ to f_*, i.e.

$$\lim_{t\to\infty} ||P^t_{\mathcal{W}} f - f_*||_{L^1} = 0 \qquad \text{for all initial densities } f.$$

Strong convergence is equivalent to asymptotic stability.

Though the concepts of ergodicity and mixing are relatively familiar, a few examples may be helpful to recall these for the reader as well as illustrate the less well known asymptotic stability [15].

The map corresponding to rotation on the circle, $\mathcal{W} : [0, 2\pi) \to [0, 2\pi)$, defined by

$$\mathcal{W}(x) = x + \phi \qquad \text{mod } 2\pi,$$

is ergodic (but neither mixing nor asymptotically stable) when $\phi/2\pi$ is irrational, and is not ergodic in all other cases. The same is true of the transformation

$$\mathcal{W}(x, y) = \left(\sqrt{2} + x, \sqrt{3} + y\right) \qquad \text{mod } 1,$$

where $\mathcal{W} : X \to X$, where $X = [0, 1] \times [0, 1]$. For both of these ergodic transformations, the unique stationary density is $f_* = 1_X$ where

$$1_X = \begin{cases} 1 & x \in X \\ 0 & \text{otherwise.} \end{cases}$$

The baker transformation $\mathcal{W} : X \to X$, where $X = [0, 1] \times [0, 1]$ and

$$\mathcal{W} = \begin{cases} (2x, \frac{1}{2}y) & 1 \le x < \frac{1}{2}, \quad 0 \le y \le 1 \\ (2x - 1, \frac{1}{2}y + \frac{1}{2}) & \frac{1}{2} \le x \le 1, \quad 0 \le y \le 1 \end{cases}$$

is mixing (and hence ergodic) but not asymptotically stable. The same is true of

$$\mathcal{W} = (x + y, x + 2y) \qquad \text{mod } 1,$$

with X defined as for the baker transformation. (This latter map has gained some considerable fame as the 'Arnold cat map' [1].) Both of these have the uniform density 1_X as a unique stationary density.

Finally, the map $\mathcal{W} : X \to X$ given by

$$\mathcal{W} = (3x + y, x + 3y) \qquad \text{mod } 1,$$

with X again the unit square, is asymptotically stable with the uniform density as the stationary density. Both the hat map (when $a = 2$) and quadratic map (with $r = 4$), considered in the following section, are also asymptotically stable. The hat map has a unique uniform stationary density on the unit interval ($a = 2$), while the unique stationary density of the quadratic map with $r = 4$ is given by

$$f_*(x) = \frac{1}{\pi\sqrt{x(1-x)}}.$$

As we will demonstrate, the following theorem concerning asymptotic stability in discrete-time dynamical systems is quite useful for studying the statistical properties of coupled map lattices.

Let $\mathcal{M}$ be a finite dimensional smooth connected compact C^∞ manifold with a Riemannian metric and tangent space $\Gamma_{\mathcal{M}}$. A mapping $\mathcal{W}:\mathcal{M}\to\mathcal{M}$ is said to be **expanding** if there exists a constant $\lambda > 1$ such that the differential $d\mathcal{W}(m)$ satisfies $||d\mathcal{W}(m)\xi|| \geq \lambda||\xi||$ at each $m \in \mathcal{M}$ for every tangent vector $\xi \in \Gamma_{\mathcal{M}}$, where $||\xi||$ denotes the norm of the vector ξ so $||\xi|| \equiv \langle\cdot,\cdot\rangle^{\frac{1}{2}}$ and $\langle\cdot,\cdot\rangle$ denotes the scalar product. Then we have

Theorem 1. *([15], Theorem 6.8.1). Let $\mathcal{W}:\mathcal{M}\to\mathcal{M}$ be a C^2 mapping and $P_{\mathcal{W}}$ the corresponding Frobenius Perron operator. If $\mathcal{W}$ is expanding then $P_{\mathcal{W}}$ is asymptotically stable.*

3. Asymptotic stability in coupled map lattices

3.1. Variable coupling

Here we first use Theorem 1 to show that a coupled map lattice with variable coupling, i.e. (2), can exhibit asymptotic stability. This result extends the findings of Bunimovich and Sinai [2] to finite lattices with variable coupling.

To apply Theorem 1 to (finite) L-dimensional lattices like (2), we assume periodic boundary conditions and identify $\mathcal{M}$ as the L-dimensional torus formed by taking the Cartesian product of L circles of unit circumference:

$$\mathcal{M} = \{(m^1,\dots,m^L) : m^1 = \mathrm{e}^{2\pi i x^1},\cdots, m^L = \mathrm{e}^{2\pi i x^L}, x^1,\dots,x^L \in R\}.$$

Thus in the local coordinates x^j the map $\mathcal{W}$ is given by

$$\mathcal{W}(\cdots,x_t^j,\cdots) = \left(\cdots,\frac{\epsilon(S(x_t^j))}{2}S(x_t^{j-1}) + \left[1-\epsilon(S(x_t^j))\right]S(x_t^j) + \frac{\epsilon(S(x_t^j))}{2}S(x_t^{j+1}),\cdots\right),$$

and $\mathcal{W}$ maps each point $(m^1,\dots,m^L)$ into the point $(\tilde{m}^1,\dots,\tilde{m}^L)$ where

$$\tilde{m}^j = \exp\left\{2\pi i\left[\frac{\epsilon(S(x_t^j))}{2}S(x_t^{j-1}) + \left[1-\epsilon(S(x_t^j))\right]S(x_t^j) + \frac{\epsilon(S(x_t^j))}{2}S(x_t^{j+1})\right]\right\}.$$

Using these conventions, it is a straightforward consequence (see Appendix A) of Theorem 1 that for the coupled map lattice (2), when the map $S:[0,1)\to[0,1)$ and coupling ϵ satisfy:

(i) There is a finite partition of $[0,1)$, denoted by $0 = a_0 < a_1 < \cdots < a_r = 1$, such that for each integer $i = 1,\dots,r$ the restriction of S to the interval $[a_{i-1},a_i)$ is a C^2 function;

(ii) For every i, $S([a_{i-1},a_i)) = [0,1)$ so S is onto;

(iii) There are constants α and M such that $1 < \alpha \leq |S'(x)| \leq M < \infty$ for $0 \leq x < 1$; and

(iv) The coupling satisfies $\epsilon(0) = \epsilon(1) = 0$, $0 \leq \epsilon(y) \leq \epsilon_{\max} \leq 1$ and $|\epsilon'(y)| \leq \epsilon'_{\max}$ for $y \in (0,1)$,

then $P_{\mathcal{W}}$ is asymptotically stable for

$$(1-\epsilon_{\max})^2\alpha^2 - 2\epsilon'_{\max}(1+\epsilon_{\max})M^2 - 2\gamma M^2\Delta > 1, \tag{3}$$

where $\gamma = 1 - \min_{x\in[0,1)} 1_{R^+}(S'(x))$ and

$$\Delta = \begin{cases} \epsilon_{\max}(1-\epsilon_{\max}) & 0 < \epsilon_{\max} < \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \leq \epsilon_{\max} < 1. \end{cases} \tag{4}$$

3.2. Constant coupling

Many investigators have performed numerical studies on coupled map lattices with constant coupling (see, for example, [3,5-10,21,28]). Consider lattices

$$x_{t+1}^j = (1-\epsilon)S(x_t^j) + \frac{\epsilon}{2}[S(x_t^{j-1}) + S(x_t^{j+1})],$$
$$\equiv \mathcal{W}(x_t^{j-1}, x_t^j, x_t^{j+1}) \qquad j = 1, \cdots, L \tag{5}$$

of nonsingular maps $S : [0,1] \to [0,1]$ with constant coupling $\epsilon \in (0,1)$. We assume that the full lattice dynamics $\mathcal{W}$ defined by (5) operate in a phase space X_L consisting of the L-dimensional unit cube $X_L = [0,1] \times \cdots \times [0,1]$ and are nonsingular, $\mathcal{W} : X_L \to X_L$ is onto, and we specifically do not associate the point 1 with 0. $\mathcal{W}$ maps the triple $(x_t^{j-1}, x_t^j, x_t^{j+1})$ to a point x_{t+1}^j so as before $x_{t+1}^j = \mathcal{W}(x_t^{j-1}, x_t^j, x_t^{j+1})$, or $x_{t+1} = \mathcal{W}(x_t)$ where $x_t = (x_t^1, \cdots, x_t^L)$.

It is straightforward to show that if the lattice dynamics are given by (5) and the dynamics of the map $S : [0,1) \to [0,1)$ at each lattice site have the properties:

(i) There is a finite partition of $[0,1]$, denoted by $0 = a_0 < a_1 < \cdots < a_r = 1$, such that for each integer $i = 1, \ldots, r$ the restriction of S to the interval $[a_{i-1}, a_i)$ is a C^2 function;

(ii) For at least one i, $S([a_{i-1}, a_i)) = [0,1)$; and

(iii) There are constants α and M such that $1 < \alpha \leq |S'(x)| \leq M < \infty$ for $0 \leq x < 1$,

then $P_{\mathcal{W}}$ is asymptotically stable whenever

$$\alpha^L \prod_{j=0}^{L-1} \left| 1 - 2\epsilon \sin^2\left(\frac{\pi j}{L}\right) \right| > 1. \tag{6}$$

The demonstration follows directly by calculating the Jacobian of the transformation $\mathcal{W}$, and using the property of circulant determinants to show that $|\mathcal{W}'| \geq \lambda > 1$ when (6) is satisfied, and then applying an easily proved extension of Theorem 6.8.1 of Ref. [15].

The above results for constant and variable coupling do not apply to lattices composed of maps such as the quadratic map

$$S(x) = rx(1-x)$$

for $0 < r \leq 4$ since the slope does not have absolute value greater than one. Moreover, these results do not apply to maps which are not onto, e.g. the quadratic map with $r < 4$.

4. Specific illustrations of the results

In this section we illustrate the content of the results of the previous section with four examples. The first two are for situations in which the map has strictly positive slope.

Example 1. Consider the Rényi map [24]

$$S(x) = \alpha x \qquad \text{mod } 1, \tag{7}$$

which is known to be asymptotically stable for $\alpha > 1$ ([15], Theorem 6.2.1). Let $\alpha \geq 2$ be an integer. From (3) and (4) we know that when Rényi maps are in a lattice with variable coupling of the type (2), the map $\mathcal{W}$ is expanding and $P_{\mathcal{W}}$ is also asymptotically stable whenever

$$0 < \epsilon'_{\max} < \frac{(1-\epsilon_{\max})^2\alpha^2 - 1}{2(1+\epsilon_{\max})\alpha^2}$$

is satisfied. However, when the Rényi map is in a constantly coupled lattice of the type (5), then $P_{\mathcal{W}}$ is asymptotically stable whenever (6) holds (note that α is not restricted to integer values).•

Example 2. Intermittency is often observed in models of fluid flow and Manneville [20] has argued that continuous time models for turbulence have a Poincaré section approximated by

$$S(x) = (1+\delta)x + (1-\delta)x^2 \qquad \text{mod } 1, \tag{8}$$

where $\delta > 0$ is proportional to the degree of turbulence and x is a normalized fluid velocity. If we consider a coupled lattice of Manneville maps, then the entire lattice is a spatially discrete approximation to a continuous system in which the velocity within fluid macrocells interacts with nearest neighbors. Since $S'(x) = (1+\delta) + 2(1-\delta)x$, it is clear that $\alpha = \min(1+\delta, 3-\delta)$ and $M = \max(1+\delta, 3-\delta)$. If $\alpha = 1+\delta$ and $M = 3-\delta$, then (3) implies $P_{\mathcal{W}}$ for the lattice will be asymptotically stable whenever

$$0 < \epsilon'_{\max} < \frac{(1+\delta)^2(1-\epsilon_{\max})^2 - 1}{2(3-\delta)^2(1+\epsilon_{\max})} \qquad 0 < \delta \le 1.$$

Alternately, if $\alpha = 3-\delta$ and $M = 1+\delta$, then (3) implies that $P_{\mathcal{W}}$ for the lattice will be asymptotically stable whenever

$$0 < \epsilon'_{\max} < \frac{(3-\delta)^2(1-\epsilon_{\max})^2 - 1}{2(1+\delta)^2(1+\epsilon_{\max})} \qquad 1 < \delta < 2.$$

(Note that in both of these inequalities, it is necessary that $0 < \epsilon_{\max} < \frac{1}{2}$.) If the Manneville map is in the constantly coupled lattice (5), then (6) gives the condition for $P_{\mathcal{W}}$ to be asymptotically stable. Asymptotic stability in the dynamics may correspond to what is usually called fully developed turbulence [3,5,20].•

To illustrate the results for situations in which the map has both positive and negative slopes, we have the next example.

Example 3. The tent map

$$S(x) = \begin{cases} ax, & \text{for } 0 \le x \le \frac{1}{2} \\ a(1-x), & \text{for } \frac{1}{2} < x \le 1 \end{cases} \tag{9}$$

when $a = 2$ and $\alpha = M = 2$ satisfies all of the conditions for variable coupling lattices for

$$0 < \epsilon'_{\max} < \frac{4(1-\epsilon_{\max})^2 - 8\epsilon_{\max}(1-\epsilon_{\max}) - 1}{8(1+\epsilon_{\max})}.$$

Thus we know that a variable coupling lattice of tent maps will display asymptotic stability for this range of coupling slopes and

$$0 < \epsilon_{\max} < \frac{4-\sqrt{7}}{6} \simeq 0.226.$$

For the constant coupling case, the sufficient condition (6) for asymptotic stability is consistent with the numerical results of [8].•

Finally, we give an example for which asymptotic stability can be demonstrated for constant, but not variable, coupling.

Example 4. The Mori [22] map

$$S_c(x) = \begin{cases} \dfrac{1-c}{c}x + c & 0 \le x \le c \\ \dfrac{1}{1-c} - \dfrac{1}{1-c}x & c \le x \le 1, \end{cases} \tag{10}$$

where $c \in (0,1)$, has a corresponding Frobenius Perron operator given by

$$P_{S_c}f(x) = \frac{c}{1-c} f\left(\frac{c}{1-c}(x-c)\right) 1_{[c,1]}(x) + (1-c)\, f(1-(1-c)x),$$

with a unique parametrized stationary density

$$f_*(x;c) = \frac{1}{1+c}1_{[0,c]}(x) + \frac{1}{1-c^2}1_{(c,1]}(x), \qquad c \in (0,1).$$

Ito et al. [4] have shown that the Mori map is asymptotically stable. If we identify

$$\alpha = \frac{1}{1-c} \quad \text{and} \quad M = \frac{1-c}{c}, \qquad c \in (0, c_*)$$

or

$$\alpha = \frac{1-c}{c} \quad \text{and} \quad M = \frac{1}{1-c}, \qquad c \in (c_*, \tfrac{1}{2}),$$

where

$$c_* = \frac{3-\sqrt{5}}{2} \simeq 0.382$$

is the value of c at which $\alpha = M$, then it is straightforward to show that the Mori map satisfies all of the conditions on S for constantly coupled lattices when $c \in (0, \frac{1}{2})$. Hence, for a lattice of constantly coupled Mori maps we know that $P_{\mathcal{W}}$ will be asymptotically stable whenever inequality (6) is satisfied with α and M appropriately identified depending on the value of c as given above. Note in particular that the results for variable coupling lattices cannot be applied to the Mori map since S_c is not onto for $x \in [0, c]$. •

5. Other coupling schemes

We next briefly comment on situations in which the coupling extends beyond nearest neighbors [7,8,11]. A variety of schemes may be considered, for example a variable coupling one in which each lattice site is coupled to an even number $\mathcal{L} \le L - 1$ nearest neighbors. In this case, the lattice equations (2) are replaced by

$$\begin{aligned} x_{t+1}^j &= \left[1 - \epsilon(S(x_t^j))\right] S(x_t^j) + \frac{\epsilon(S(x_t^j)}{\mathcal{L}} \sum_{k=j-\mathcal{L}/2, k\neq j}^{j+\mathcal{L}/2} S(x_t^k) \\ &\equiv \mathcal{W}(x_t^1, \ldots, x_t^L) \qquad j = 1, \cdots, L. \end{aligned}$$

For this "boxcar" coupling, $d\mathcal{W}(m)$ maps the jth component of the vector $\xi = (\xi^1, \ldots, \xi^L)$ into

$$\Gamma_j = (A_j + B_j)\xi^j + \frac{\epsilon_j}{\mathcal{L}} \sum_{k=j-\mathcal{L}/2, k\neq j}^{j+\mathcal{L}/2} S_k' \xi^k,$$

where $A_j = (1-\epsilon_j)S'_j$ as before and now

$$B_j = S'_j \frac{\epsilon'_j}{\mathcal{L}} \left(\sum_{k=j-\mathcal{L}/2, k\neq j}^{j+\mathcal{L}/2} S_k - \mathcal{L} S_j \right).$$

Identical results, like (3) and (4), for variable coupling lattices follow much as before when one notes that

$$\sum_{j=1}^{L} \xi^j \left(\sum_{k=j-\mathcal{L}/2, k\neq j}^{j+\mathcal{L}/2} \xi^k \right) \leq \mathcal{L} ||\xi||^2.$$

Similar procedures for the constantly coupled lattice (5) yields a generalization of (6). Mean field coupling occurs when each lattice site is coupled to all other elements so $\mathcal{L} = L - 1$.

If one utilizes a coupled map lattice as a paradigm for the behavior of a collection of interconnected neurons, then based on anatomical evidence it would be more sensible to examine a symmetric coupling that decayed exponentially [26] as one moved away from a given site. This could be captured by

$$x^j_{t+1} = \left[1 - \epsilon(S(x^j_t))\right] S(x^j_t) + \frac{\epsilon(S(x^j_t)}{\mathcal{L}} \sum_{k=j-\mathcal{L}/2, k\neq j}^{j+\mathcal{L}/2} S(x^k_t)\, e^{-\kappa|k-j|}$$
$$\equiv \mathcal{W}(x^1_t, \cdots, x^L_t) \qquad j = 1, \cdots, L.$$

Inequality (3), with (4), is immediately applicable to this exponentially weighted coupling, and results analogous to (6) may also be derived in a straightforward manner.

6. Numerical determination of densities from coupled map lattices

6.1. Collapsed densities

The asymptotic stability guaranteed by the results of the previous sections means that the sequence of densities $\{f_t(x^1, \cdots, x^L)\}_{t=1}^{\infty}$ will converge strongly to a unique limiting density $f_*(x^1, \cdots, x^L)$ that is a stationary density of $P_{\mathcal{W}}$ or $\mathcal{P}_\epsilon$. However, it is inherently difficult to deal with this L-dimensional density in numerical studies. Here we present an approximation technique to determine the evolution of a collapsed one-dimensional density in lattices (5) with constant coupling.

Following the suggestion of Kaneko [6], at any given time we consider the collapsed density

$$f_t(x) = \int_0^1 \overset{L}{\cdots} \int_0^1 f_t(x^1, \cdots, x^L) \prod_{j=1}^{L} \delta(x^j - x)\, dx^j, \tag{11}$$

and then form the trajectory average

$$\langle f_*(x) \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} f_t(x) \tag{12}$$

from the sequence of collapsed densities $\{f_t\}_{t=1}^{\infty}$.

In studying (5) numerically, we approximate $\langle f_*(x)\rangle$ in the following way. First, we partition the interval $[0,1]$ into $n >> 1$ subintervals

$$\left[\frac{k-1}{n}, \frac{k}{n}\right) \qquad k = 1, \cdots, n.$$

Then from (5) and an initial vector $x_0 = (x_0^1, \cdots, x_0^L)$ we calculate a sequence of vectors $\{x_t\}_{t=1}^{N+\mathcal{T}}$, where $x_t = (x_t^1, \cdots, x_t^L)$ and $N >> n$. We discard a transient of length $\mathcal{T}$ and from the remaining N system states calculate the collapsed fraction of these states in the kth interval of the partition from

$$f^k = \frac{n}{N} \sum_{t=\mathcal{T}}^{N+\mathcal{T}} \left(\frac{1}{L} \sum_{j=1}^{L} 1_{\left[\frac{k-1}{n}, \frac{k}{n}\right)}(x_t^j) \right) \qquad k = 1, \cdots, n. \tag{13}$$

The vector $(f^1, \cdots, f^L)$ is a good approximation to $\langle f_*(x)\rangle$ when $N >> n$. When (5) is ergodic, the value of $\langle f_*(x)\rangle$ defined in (12) will coincide with

$$\tilde{f}(x) = \int_0^1 \overset{L}{\cdots} \int_0^1 f_*(x^1, \cdots, x^L) \prod_{j=1}^{L} \delta(x^j - x)\, dx^j. \tag{14}$$

Examples of the collapsed density obtained from numerical studies when S is given by the tent map (9) with $a = 2$ for constant and variable coupling are given in Fig. 1. For the case of variable coupling we took $\epsilon(S(x_t)) = rS(x_t)(1 - S(x_t))$. In these numerical experiments, initial densities were taken to be uniform on $[0,1]$ or on a subset of $[0,1]$. In all cases shown the collapsed density was independent of the choice of initial density. For $\epsilon < 0.226$ (or $r < 0.904$), a unique collapsed density exists which does not depend on L (compare Fig. 1a to 1b and Fig. 1e to 1f), but the shape of the collapsed density is clearly a function of ϵ. Examples of unique collapsed densities which were independent of the choice of initial density also occurred for ϵ larger than those calculated in Example 3 (see Figs. 1d and 1h).

6.2. Analytic approximation of collapsed densities

Kaneko [5] conjectured that under certain conditions it might be possible to approximate the coupling term of (5) by a random variable. In developing this conjecture to provide an analytic approximation to the numerically computed collapsed densities of coupled map lattices, we first consider the system

$$x_{t+1}^i = T(x_t^i) + \eta_t^i, \tag{15}$$

where $T : R \to R$ is a measurable transformation, and the η_t are random variables distributed with density g independent of time.

To derive an evolution operator for densities under the action of (15), let $h : R \to R$ be a bounded measurable arbitrary function. When T and η are independent of each other, then the expected value of $h(x_{t+1})$ is given by [15,16]

$$\begin{aligned} E(h(x_{t+1})) &= \int_R h(x_{t+1}) f_{t+1}(x)\, dx = E(h(T(x_t) + \eta_t)) \\ &= \int_R\int_R h(T(y) + z) f_t(y) g(z)\, dz\, dy = \int_R\int_R h(x) f_t(y) g(x - T(y))\, dx\, dy. \end{aligned} \tag{16}$$

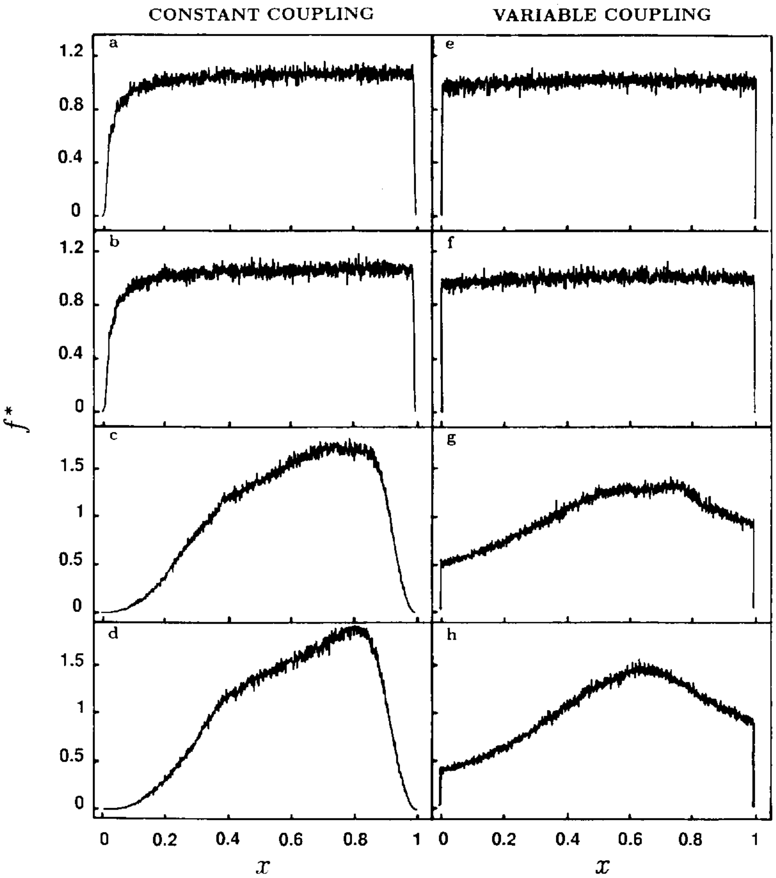

Fig. 1. The numerically computed collapsed density (13) when S is given by the tent map with constant coupling (a,b,c,d) and variable coupling (e,f,g,h). In both cases $a = 2$. For constant coupling the value of ϵ was (a) & (b) 0.01; (c) 0.20; (d) 0.30. For variable coupling, $\epsilon(S(x_t)) = rS(x_t)(1 - S(x_t))$ so $\epsilon_{\max} = r/4$, and r was equal to (a) & (b) 0.04; (c) 0.80; (d) 1.20. The lattice for a), e) contained 10 elements and in the remaining examples 100 elements. The collapsed density was constructed from a total of 10^6 iterations ($L \times$ number of iterations) after discarding a total of 10^5 iterations.

Since h was arbitrary, we obtain

$$f_{t+1}(x) = \int_R f_t(y) g(x - T(y))\, dy. \tag{17}$$

Defining a new density $Q_x(y) \equiv g(x-y)$ we have

$$g(x - T(y)) = Q_x(T(y)) = U_T Q_x(y), \tag{18}$$

so (17) is equivalent to (remember that the Koopman operator $U_{\mathcal{W}}$, defined by $U_{\mathcal{W}} f(x) = f(\mathcal{W}(x))$, is adjoint to the Frobenius Perron operator P since $\langle P_{\mathcal{W}} f, g\rangle = \langle f, U_{\mathcal{W}} g\rangle$)

$$f_{t+1}(x) = \langle f_t(y), U_T Q_x(y)\rangle = \langle P_T f_t(y), Q_x(y)\rangle = \int_R [P_T f_t(y)] g(x-y)\, dy. \tag{19}$$

Eq. (19) defines an operator $\mathcal{P}_\epsilon$,

$$\mathcal{P}_\epsilon f(x) = \int_R [P_T f(y)] g(x-y)\,dy, \qquad 0 \le \epsilon < 1. \tag{20}$$

It is easy to show that $\mathcal{P}_\epsilon$ is a Markov operator since, for every density f, we have $\mathcal{P}f \ge 0$ and $||\mathcal{P}f||_{L^1} = ||f||_{L^1} \equiv 1$, where $||\cdot||_{L^1}$ denotes the L^1 norm. Eq. (20) defines the evolution of densities of the iterates x determined by (15).

For the coupled map lattice (5), we set

$$T(x_t^j) = (1-\epsilon) S(x_t^j), \tag{21a}$$

$$\eta_t^j = \frac{\epsilon}{2}\left[S(x_t^{j-1}) + S(x_t^{j+1})\right], \tag{21b}$$

where we assume that $S : [0,1] \to [0,1]$ is an ergodic transformation with unique stationary density f_*. Clearly $T : [0,1] \to [0, 1-\epsilon] \subset [0,1]$ and $\eta_t \in [0,\epsilon]$. Since the domain and range of T are not R, the integration in Eq. (20) must be restricted [16] to the set $A(x) = \{y \in [0,1] : x > y\}$.

To use (20) to study the evolution of collapsed densities under the action of (5), we must deal with two issues. The first is that at any fixed time t both T and η, as defined in (21), are not independent even though the only quantity required to calculate $T(x_t^j)$ is x_t^j, and the quantities required to calculate η_t^j are x_t^{j-1} and x_t^{j+1}. However, we *assume* that for small ϵ the independence assumption is approximately true.

Secondly, we must know under what circumstances it is permissible to assume that the η_t are eventually distributed with density g independent of time, and what that density is. Since S is ergodic by assumption, the first point is immediate and we need only address the issue of the eventual density with which the η_t are distributed. Here we make our second assumption. Namely that for small ϵ we *assume* that the density g will be approximated by the convolution of the stationary density of each of the terms making up η:

$$g(z) \simeq \mathcal{G}(z) = \int_0^\epsilon f_*(z-y)\, f_*(y)\,dy.$$

If the support of the stationary density is $\operatorname{supp} f_* = [0,1]$ then $\operatorname{supp}\mathcal{G} = [0,\epsilon]$.

Example 5. We can test this approximation procedure for the collapsed density by returning to the tent map of Example 3. It is well known that the tent map (5) has a unique invariant density for all $1 < a \le 2$. Further, when $a = 2$, then $f_*(x) = 1_{[0,1]}(x)$ where $1_A(x)$ is the indicator function for the set A. If T and η were truly independent, this would imply that the covariance $\rho_{T\eta} = \langle T\eta\rangle - \langle T\rangle\langle\eta\rangle$ was identically zero. However, it is straightforward to calculate from Eqs. (21) that $\rho_{T\eta} \simeq \epsilon(1-\epsilon)[\langle x^2\rangle - \langle x\rangle^2]$. Since, for the tent map, $\langle x\rangle = \frac{1}{2}$ and $\langle x^2\rangle = \frac{1}{3}$ it is clear that the independence criterion is not precisely met except in the limit $\epsilon \to 0$ or $\epsilon \to 1$. For small ϵ and $a = 2$ the density g of the η for a constantly coupled tent map lattice (which we have shown is asymptotically stable when (6) holds) will be approximated by

$$g(z) \simeq \mathcal{G}(z) = \frac{4}{\epsilon^2}\int_0^\epsilon 1_{[0,\frac{\epsilon}{2}]}(y)\,1_{[0,\frac{\epsilon}{2}]}(z-y)\,dy \tag{22}$$

$$= \frac{4}{\epsilon^2}\begin{cases} z, & 0 \le z \le \frac{\epsilon}{2} \\ \epsilon - z, & \frac{\epsilon}{2} \le z \le \epsilon. \end{cases}$$

Fig. 2 shows that the actual density g of η is close to the approximation (22) for $\epsilon \le 10^{-2}$.

Using Eq. (20) in conjunction with (22) gives a method to calculate the approximation to the collapsed density for the tent map with $a = 2$ and small ϵ. In this case the Markov operator (20) is given explicitly by

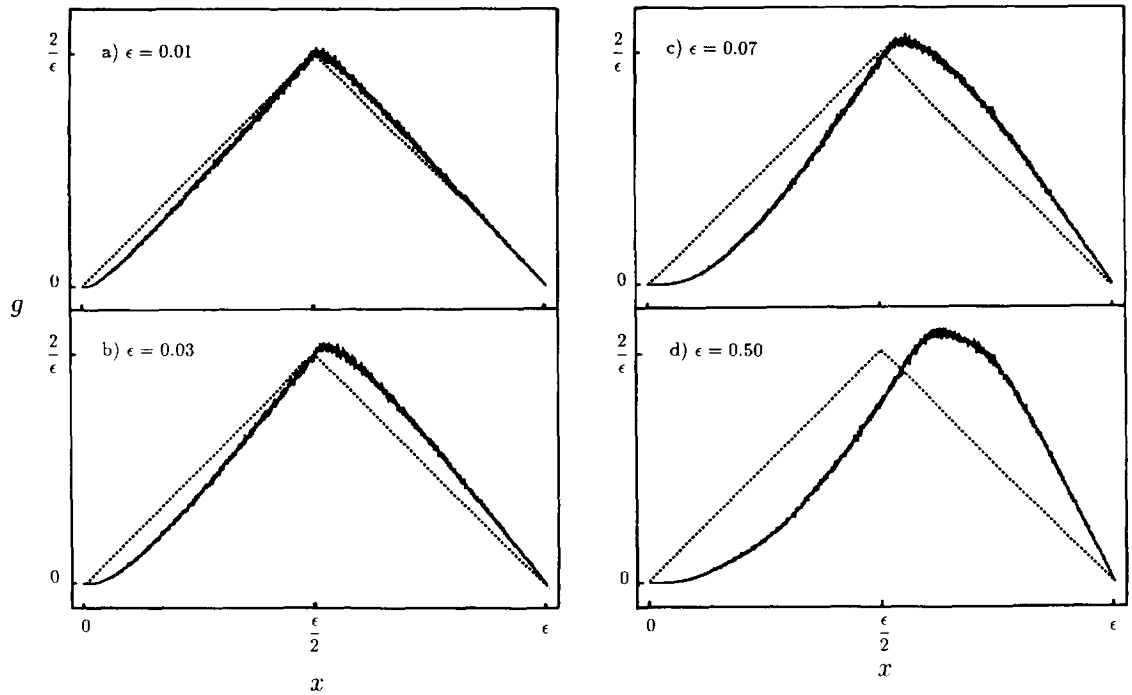

Fig. 2. The numerically computed density g of the perturbation term (21b) as a function of ϵ when S is given by the tent map with $a = 2$. In the four panels, ϵ has the values: (a) 0.01; (b) 0.03; (c) 0.07; and (d) 0.50. The lattice contained 100 elements, an initial 10^4 iterations were discarded and the next 10^5 iterations were used to compute the density. The dotted line is the approximating density $\mathcal{G}$ given by (22).

$$\mathcal{P}_\epsilon f(y) = \frac{1}{2(1-\epsilon)} \left[f\left(\frac{y}{2(1-\epsilon)}\right) + f\left(1 - \frac{y}{2(1-\epsilon)}\right)\right]. \tag{23}$$

Fig. 3 compares the analytic expressions for f_1 and f_2 (see Appendix B), calculated using (23), to the collapsed density determined numerically from (13) for $L = 2 \times 10^5$. As can be seen there is excellent agreement between the analytically calculated and numerically estimated results. The analytically calculated collapsed densities rapidly become very cumbersome and we have not continued the computation for higher iterates.●

7. Summary

Here we have shown that it is possible to obtain sufficient conditions to ensure that many coupled map lattices with variable or constant coupling are expanding and thus exhibit the property of asymptotic stability. These conditions provide usable estimates of the parameter ranges for which a unique invariant measure exists for the coupled map lattice and imply that certain measurable quantities calculated using these invariant measures evolve to a maximum with time. Numerical studies support the validity of our sufficient conditions and also indicate that unique stationary densities probably exist when the sufficient conditions are violated. The parameter ranges for which asymptotic stability occurs in the coupled map lattice include the ranges for which the weaker properties of ergodicity and mixing occur [1,11].

Our results further show that it is possible to estimate the collapsed limiting density generated in a L-dimensional coupled map lattice by examining the evolution of densities of a one-dimensional dynamical

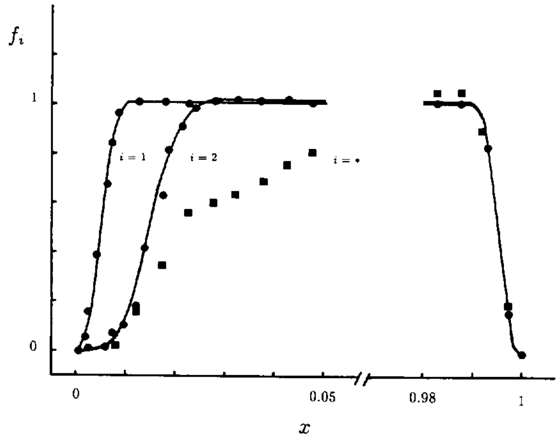

Fig. 3. Comparison of the one-dimensional collapsed densities at the first (f_1) and second (f_2) iteration computed from (23) (solid lines) to those computed numerically (dots) using (13) when S is the tent map with $a = 2$. In both cases the initial density was uniform on $[0, 1]$. The values of the computed collapsed density represent the average of 100 trials for a lattice containing 10^5 elements. The total number of bins was 100. The numerically computed limiting density is denoted by $i = \infty$.

system (21). The approximation of the density g of η by $\mathcal{G}$ is quite accurate when ϵ is small. Consequently it is possible, in principle, to obtain the entire sequence $\{\mathcal{P}_\epsilon^t f\}_{t=0}^{\infty}$. Although we illustrated this procedure for the tent map, an identical approximation can be made for any coupled map lattice when the conditions of ergodicity and small ϵ are satisfied.

Other interesting statistical properties may occur in coupled map lattices–namely the situation in which the sequence of densities displays a periodicity rather than smoothly approaching a unique stationary density as in asymptotic stability. This property of asymptotic periodicity of the densities in coupled map lattices has been treated in Refs. [17,18].

Acknowledgments

We thank L. Bunimovich, G. Keller, K. Kaneko, J. Losson and P. Kamthan for their helpful comments. This work was supported by grants from the National Institutes of Mental Health (JGM), North Atlantic Treaty Organization (JGM, MCM), the Natural Science and Engineering Research Council of Canada, the Royal Society of London, and the Alexander von Humboldt Stiftung (MCM).

Appendix A

To derive condition (3) for the asymptotic stability of lattices like (2) with variable coupling, set $S_j = S(x^j)$, $S'_j = dS(x^j)/dx^j$, $\epsilon_j = \epsilon(S(x^j))$, and $\epsilon'_j = \partial\epsilon(S(x^j))/\partial S(x^j)$. It is clear that $d\mathcal{W}(m)$ maps the jth component of the vector $\xi = (\xi^1, \dots, \xi^L)$ into

$$\Gamma_j = (A_j + B_j)\xi^j + \frac{\epsilon_j}{2}\left[S'_{j-1}\xi^{j-1} + S'_{j+1}\xi^{j+1}\right],$$

wherein $A_j = (1 - \epsilon_j)S'_j$, and $B_j = S'_j \frac{\epsilon'_j}{2}\left[S_{j-1} - 2S_j + S_{j+1}\right]$. Consequently

$$\Gamma_j^2 = (A_j + B_j)^2(\xi^j)^2 + \left(\frac{\epsilon_j}{2}\right)^2 \left[S'_{j-1}\xi^{j-1} + S'_{j+1}\xi^{j+1}\right]^2 + \epsilon_j(A_j + B_j)\left[S'_{j-1}\xi^{j-1}\xi^j + S'_{j+1}\xi^j\xi^{j+1}\right]$$
$$\geq (A_j^2 + 2A_jB_j)(\xi^j)^2 + \epsilon_j(A_j + B_j)\left[S'_{j-1}\xi^{j-1}\xi^j + S'_{j+1}\xi^j\xi^{j+1}\right].$$

Now $A_j^2 = (1-\epsilon_j)^2(S'_j)^2 \geq (1-\epsilon_{\max})^2\alpha^2$ and $2A_jB_j = \epsilon'_j(1-\epsilon_j)(S'_j)^2[S_{j-1} - 2S_j + S_{j+1}] \geq -2\epsilon'_{\max}M^2$. Furthermore, for $k = j-1$ or $j+1$ we have $\epsilon_jA_jS'_k = \epsilon_j(1-\epsilon_j)S'_jS'_k \geq -\gamma M^2\Delta$ and $\epsilon_jB_jS'_k = \frac{1}{2}\epsilon'_j\epsilon_jS'_jS'_k[S_{j-1} - 2S_j + S_{j+1}] \geq -\epsilon'_{\max}\epsilon_{\max}M^2$. As a result,

$$\Gamma_j^2 \geq \left[(1-\epsilon_{\max})^2\alpha^2 - 2\epsilon'_{\max}M^2\right](\xi^j)^2 - \left[\epsilon'_{\max}\epsilon_{\max}M^2 - 2\gamma M^2\Delta\right]\left[\xi^{j-1}\xi^j + \xi^j\xi^{j+1}\right],$$

and since $\sum_{j=1}^{L}\xi^j\xi^{j+k} \leq ||\xi||^2$ we have

$$||d\mathcal{W}(m)\xi||^2 = \sum_{j=1}^{L}\Gamma_j^2 \geq \left\{(1-\epsilon_{\max})^2\alpha^2 - 2\epsilon'_{\max}(1+\epsilon_{\max})M^2 - 2\gamma M^2\Delta\right\}||\xi||^2.$$

Thus, if $\lambda = \left\{(1-\epsilon_{\max})^2\alpha^2 - 2\epsilon'_{\max}(1+\epsilon_{\max})M^2 - 2\gamma M^2\Delta\right\}^{1/2} > 1$, then $\mathcal{W}$ is expanding. An application of Theorem 6.8.1 of [15] completes the proof.

Appendix B

Here we give the expressions for f_1 and f_2 calculated for (21) using (23) when S is the tent map with $a = 2$. The initial density, f_0, for these calculations was the uniform density on $[0,1]$.

$$f_1(x) = \frac{2x^2}{\epsilon^2(1-\epsilon)}1_{[0,\frac{\epsilon}{2})}(x) + \frac{\epsilon^2 - 4\epsilon x + 2x^2}{\epsilon^2(\epsilon-1)}1_{[\frac{\epsilon}{2},\epsilon)}(x) + \frac{1}{1-\epsilon}1_{[\epsilon,1-\epsilon)}(x)$$
$$+ \frac{2 - 4\epsilon + \epsilon^2 - 4x + 4\epsilon x + 2x^2}{\epsilon^2(\epsilon-1)}1_{[1-\epsilon,1-\frac{\epsilon}{2})}(x) + \frac{2(x-1)^2}{\epsilon^2(1-\epsilon)}1_{[1-\frac{\epsilon}{2},1]}(x),$$

$$f_2(x) = \frac{x^4}{6\epsilon^4(\epsilon-1)^4}1_{[0,\frac{\epsilon}{2})}(x) + \frac{-\epsilon^4 + 8\epsilon^3x^2 - 24\epsilon^2x^2 + 32\epsilon x^3 - 8x^4}{48\epsilon^4(\epsilon^4-1)^4}1_{[\frac{\epsilon}{2},\epsilon(1-\epsilon))}(x)$$
$$+ \left\{\frac{-17\epsilon^4 + 64\epsilon^5 - 96\epsilon^6 + 64\epsilon^7 - 16\epsilon^8 + 72\epsilon^3x - 192\epsilon^4x^2}{48\epsilon^4(\epsilon-1)^4}\right.$$
$$+ \frac{192\epsilon^5x - 64\epsilon^6x - 120\epsilon^2x^2 + 192\epsilon^3x^2 - 96\epsilon^4x^2}{48\epsilon^4(\epsilon-1)^4}$$
$$\left. + \frac{96\epsilon x^3 - 64\epsilon^2x^3 - 24x^4}{48\epsilon^4(\epsilon-1)^4}\right\}1_{[\epsilon(1-\epsilon),\epsilon)}(x)$$
$$+ \left\{\frac{-9\epsilon^4 + 64\epsilon^5 - 96\epsilon^6 + 64\epsilon^7 - 16\epsilon^8 + 40\epsilon^3x - 192\epsilon^4x}{48\epsilon^4(\epsilon-1)^4}\right.$$
$$+ \frac{192\epsilon^5x - 64\epsilon^6x - 72\epsilon^2x^2 + 192\epsilon^3x^2 - 96\epsilon^4x^2 + 64\epsilon x^3}{48\epsilon^4(\epsilon-1)^4}$$
$$\left. + \frac{-64\epsilon^2x^3 - 16x^4}{48\epsilon^4(\epsilon-1)^4}\right\}1_{[\epsilon,\epsilon(1-\epsilon)+\frac{\epsilon}{2})}(x)$$

$$
\begin{aligned}
&+\left\{\frac{153\epsilon^4 - 368\epsilon^5 + 336\epsilon^6 - 128\epsilon^7 + 16\epsilon^8 - 392\epsilon^3 x + 672\epsilon^4 x}{48\epsilon^4(\epsilon-1)^4}\right.\\
&\quad + \frac{-384\epsilon^5 x + 64\epsilon^6 x + 360\epsilon^2 x^2 - 384\epsilon^3 x^2 + 96\epsilon^4 x^2 - 128\epsilon x^3}{48\epsilon^4(\epsilon-1)^4}\\
&\quad\quad \left. + \frac{64\epsilon^2 x^3 + 16x^4}{48\epsilon^4(\epsilon-1)}\right\} 1_{[\epsilon(1-\epsilon)+\frac{\epsilon}{2},2\epsilon(1-\epsilon))}(x)\\
&+\left\{\frac{281\epsilon^4 - 880\epsilon^5 + 1104\epsilon^6 - 640\epsilon^7 + 144\epsilon^8 - 648\epsilon^3 x + 1440\epsilon^4 x}{48\epsilon^4(\epsilon-1)^4}\right.\\
&\quad + \frac{-1152\epsilon^5 x + 320\epsilon^6 x + 552\epsilon^2 x^2 - 768\epsilon^3 x^2 + 288\epsilon^4 x^2 - 192\epsilon x^3}{48\epsilon^4(\epsilon-1)^4}\\
&\quad\quad \left. + \frac{128\epsilon^2 x^3 + 24x^4}{48\epsilon^4(\epsilon-1)^4}\right\} 1_{[2\epsilon(1-\epsilon),\epsilon(1-\epsilon)+\epsilon)}(x)\\
&+\left\{\frac{25\epsilon^5 - 368\epsilon^5 + 720\epsilon^6 - 512\epsilon^7 + 128\epsilon^8 - 136\epsilon^3 x - 672\epsilon^4 x}{48\epsilon^4(1-\epsilon)^4}\right.\\
&\quad + \frac{-768\epsilon^5 x + 256\epsilon^6 x + 168\epsilon^2 x^2 - 384\epsilon^3 x^2 + 192\epsilon^4 x^2 - 64\epsilon x^3}{48\epsilon^4(1-\epsilon)^4}\\
&\quad\quad \left. + \frac{64\epsilon^2 x^3 + 8x^4}{48\epsilon^4(1-\epsilon)^4}\right\} 1_{[\epsilon(1-\epsilon)+\epsilon,2\epsilon(1-\epsilon)+\frac{\epsilon}{2})}(x)\\
&+\left\{\frac{-75\epsilon^4 + 204\epsilon^5 - 210\epsilon^6 + 96\epsilon^7 - 18\epsilon^8 + 108\epsilon^3 x - 216\epsilon^4 x}{6\epsilon^4(1-\epsilon)^4}\right.\\
&\quad + \frac{144\epsilon^5 x - 32\epsilon^6 x - 54\epsilon^2 x^2 + 72\epsilon^3 x^2 - 24\epsilon^4 x^2 + 12\epsilon x^3}{6\epsilon^4(1-\epsilon)^4}\\
&\quad\quad \left. + \frac{-8\epsilon^2 x^2 - x^4}{6\epsilon^4(1-\epsilon)^4}\right\} 1_{[2\epsilon(1-\epsilon)+\frac{\epsilon}{2},2\epsilon(1-\epsilon)+\epsilon)}(x)\\
&+ \frac{1}{(\epsilon-1)^2} 1_{[2\epsilon(1-\epsilon)+\epsilon,1-\epsilon)}(x)\\
&+ \frac{-2 + 4\epsilon - \epsilon^2 + 4x - 4\epsilon x - 2x^2}{\epsilon^2(\epsilon-1)^2} 1_{[1-\epsilon,1-\frac{\epsilon}{2})}(x)\\
&+ \frac{2(x-1)^2}{\epsilon^2(\epsilon-1)^2} 1_{[1-\frac{\epsilon}{2},1]}(x).
\end{aligned}
$$

The expression for f_3 divides $[0,1]$ into 29 subintervals!

References

[1] V.I. Arnold and A. Avez. Ergodic Problems of Classical Mechanics (Benjamin, New York, 1968).

[2] L.A. Bunimovich and Ya G. Sinai, Space-time chaos in coupled map lattices, Nonlinearity 1 (1988) 491–516.

[3] J.P. Crutchfield and K. Kaneko, Phenomenology of spatio-temporal chaos, in: Directions in Chaos, ed. H. Bai-Lin (World Scientific, Singapore, 1986) pp. 272–353.

[4] S. Ito, S. Tanaka and H. Nakada, On unimodal linear transformations and chaos II, Tokyo J. Math. 2 (1979) 240–259.

[5] K. Kaneko, Pattern dynamics in spatio-temporal chaos, Physica D 43 (1989) 1–41.

[6] K. Kaneko, Self-consistent Perron-Frobenius operator for spatio-temporal chaos, Phys. Lett. A 139 (1989) 47–52.

[7] K. Kaneko, Clustering, coding, switching, hierarchical ordering and control in a network of chaotic elements, Physica D 41 (1990) 137–172.
[8] K. Kaneko, Mean field fluctuation of a network of chaotic elements, Physica D55 (1992) 368–384.
[9] R. Kapral, Pattern formation in two-dimensional arrays of coupled, discrete time oscillators, Phys. Rev 31 A (1985) 3868–3879.
[10] J.D. Keeler and J.D. Farmer, Robust space-time intermittency and $1/f$ noise, Physica D 32 (1986) 413–435.
[11] G. Keller and M. Künzle, Transfer operators for coupled map lattices, Ergod. Th. & Dynam. Sys 12 (1992) 297–318.
[12] K. Krzyżewski, Some results on expanding mappings, Soc. Math. France Astérique 50 (1977) 205–218.
[13] K. Krzyżewski and W. Szlenk, On invariant measures for expanding differential mappings, Stud. Math. 33 (1969) 83–92.
[14] S.P. Kuznetsov and A.S. Pikovsky, Universality and scaling of period-doubling bifurcations in a dissipative distributed medium, Physica 19 D (1986) 384–396.
[15] A. Lasota and M.C. Mackey, Probabilistic Properties of Deterministic Systems (Cambridge University Press, New York, 1985); Chaos, Fractals and Noise: Stochastic Aspects of Dynamics (Springer, New York, 1994, second edition).
[16] A. Lasota and M.C. Mackey, Noise and statistical periodicity, Physica D 22 (1987) 143–154.
[17] J. Losson and M.C. Mackey, Coupling induced statistical cycling in two diffusively coupled maps, Physica D 72 (1994) 324–342.
[18] J. Losson and M.C. Mackey, Statistical cycling in coupled map lattices, Phys. Rev. E 50 (1994) 843–856.
[19] M.C. Mackey, Time's Arrow: The Origins of Thermodynamic Behavior (Springer, New York, 1992).
[20] P. Manneville, Intermittency, self-similarity and $1/f$ spectrum in dissipative dynamical systems, J. Physique 41 (1980) 1235–1243.
[21] J.G. Milton, P.H. Chu and M.C. Mackey, Statistical properties of networks of coupled neural elements, Proc. IEEE Eng. Med. Biol 13 (1991) 2194–5.
[22] H. Mori, B.-C. So and O. Tomoaki, Time correlation functions of one-dimensional transformations, Prog. Theor. Phys. 66 (1981) 1266–83.
[23] S. Puri, R.C. Desai and R. Kapral, Coupled map model with a conserved order parameter, Physica D 50 (1991) 207–230.
[24] A. Rényi, Representation for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hung 8 (1957) 477–493.
[25] V.A. Rochlin, Exact endomorphisms of Lebesgue spaces, Am. Math. Soc. Transl 39 (1964) 1–36.
[26] D.A. Sholl, The Organization of the Cerebral Cortex (Methuen, London, 1956).
[27] R.V. Solé and J. Valls, On structural stability and chaos in biological systems, J. theor. Biol 155 (1992) 87–102.
[28] V.L. Volevich, Kinetics of coupled map lattices, Nonlinearity 4 (1991) 37–48.