Jump clustering, Shlesinger-Hughes stochastic renormalization, and interacting Lévy flights

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A continuous-time approach for Shlesinger-Hughes stochastic renormalization is used for investigating the cooperative behavior of interacting Lévy flights. We consider a set of m interacting random flights and assume that each walker can exist either in an active state for which the clustering of jumps can occur or in a passive state for which the jump clustering is impossible. We assme that the system has a very strong cooperative behavior; the clustering of jumps occurs only if all walkers are in the active state. The probability distribution $\xi(n_1,\ldots,n_m)$ of the numbers $n_1,\ldots,n_m$ of jumps from a cluster corresponding to the different walkers depends only on two parameters: the number m of walkers and a fractal exponent $H=\overline{\tau}_r/\overline{\tau}_c$ given by the ratio between the mean relaxation time $\overline{\tau}_r$ corresponding to an individual clustering event and the mean time $\overline{\tau}_c$ of the clustering process as a whole. The very strong cooperative behavior of the jump clustering is displayed by the asymptotic behavior of the probability $\xi(n_1,\ldots,n_m)$ for large $n_1,\ldots,n_m$: for $n^*=n_1=n_2=\cdots=n_m$, $\xi(n_1,\ldots,n_m)$ has a maximum value that corresponds to a long tail of the inverse power law type $\xi(n^*,\ldots,n^*)\sim n^{*-(1+mH)}$ as $n^*\to\infty$. If the n_l are different from each other $\xi(n_1,\ldots,n_m)$ decreases exponentially to zero as $n_l\to\infty$, $l=1,\ldots,m$. We identify a critical point that corresponds to $mH=1$; for $H>1/m$ each walker performs a Gaussian flight, whereas for $H<1/m$ each walker performs a Lévy flight with a fractal parameter $\beta=2mH$. The interaction among the walkers decreases the efficiency of jump clustering: for a set of interacting flights the Lévy behavior occurs only if $\overline{\tau}_c>m\overline{\tau}_r$; this threshold value for the average total clustering time is m times larger than in the case of noninteracting flights.

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I. INTRODUCTION

Random flights of Lévy type have been used in connection with the study of critical phenomena, dielectric relaxation, anomalous diffusion, turbulence, the interaction of sound or light with a rough surface [1–4], etc. The Lévy flights are intimately related to the theory of fractal random processes and stochastic renormalization; their mechanism of generation is relatively well understood if the moving particles are noninteracting. In contrast, very little is known about the interacting Lévy flights.

A useful method for the generation and analysis of Lévy flights is the Shlesinger-Hughes stochastic renormalization (SHSR) [5]. By starting out from a classical Gaussian random flight SHSR leads to a Lévy flight by assuming that the jumps are clustering into self-similar blocks of random size. This method has been successfully applied to a broad class of problems from condensed matter physics, hydrodynamics, nonlinear optics, and even economics (see [2–5] and references therein).

The purpose of this work is to investigate a particular class of Lévy flights for which a fairly detailed analytical study is possible. Our analysis is based on a continuous-time description of jump clustering for noninteracting Lévy flights within the framework of SHSR [6]. First we summarize the main results of the continuous-time approach to jump clustering for noninteracting Lévy flights and then proceed to the study of interacting flights.

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II. NONINTERACTING LÉVY FLIGHTS

We consider a symmetric Markovian random walk in d_s-dimensional Euclidean space and denote by $p(\mathbf{r})d\mathbf{r}$ the probability that the displacement of a jump is between $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$. Denoting by $\xi(n)$ the probability that a cluster consists of n jumps, the renormalized jump probability density $\tilde{p}(\mathbf{r})$ is given by

$$\tilde{p}(\mathbf{r})=\sum\xi(n)[p(\mathbf{r})\otimes]^n , \quad (1)$$

where $[p(\mathbf{r})\otimes]^n$ is the n-fold convolution product of $p(\mathbf{r})$. To compute the probability $\xi(n)$ we make three assumptions [6].

(a) The jump clustering is a collection of independent events.

(b) The jump clustering is described in terms of two characteristic time scales: the time scale $\overline{\tau}_c$ in which the clustering as a whole occurs (the characteristic time necessary for the occurrence of a cluster of clusters) and the time scale $\overline{\tau}_r$ characteristic for the occurrence of an individual clustering event (the characteristic time scale necessary for the occurrence of a single cluster). The clustering of jumps can be considered a stochastic renormalization procedure; through renormalization a cluster of steps of random size is replaced by a single renormalized (overall) step. The characteristic time scales $\overline{\tau}_r, \overline{\tau}_c$ for the clustering process and the characteristic time scale attached to the random flight itself are well separated; in the characteristic time scale of the random flight the clustering process is practically instantaneous.

(c) The time evolution of the clustering process is scale invariant, i.e., the distribution functions of the two characteristic times introduced above have the same form for any initial time.

From these hypotheses we can express $\xi(n)$ as [6]:

$$\xi(n)=\int_0^\infty \eta_c(\tau)[\lambda(\tau)]^{n-1}[1-\lambda(\tau)]d\tau \; , \tag{2}$$

where $\eta_c(\tau)d\tau$ is the probability density of the total clustering time,

$$\lambda(\tau)=\int_0^\tau \eta_r(\tau)d\tau \tag{3}$$

is the probability that the walker is in an "active" state, i.e., the probability that a clustering event takes place in the time interval of length τ, and $\eta_r(\tau)d\tau$ is the probability density of the time required for the occurrence of an individual clustering event. In [6] it is shown that the assumption (c) of scale invariance of the clustering process requires that both $\eta_c(\tau)$ and $\eta_r(\tau)$ be exponential functions of time:

$$\eta_{c,r}(\tau)=\overline{\tau}_{c,r}^{-1}\exp(-\tau/\overline{\tau}_{c,r}) \; , \tag{4}$$

where $\overline{\tau}_{c,r}=\int_0^\infty \tau\eta_{c,r}(\tau)d\tau$ are average times. By combining Eqs. (2)–(4) we get the following expression for $\xi(n)$:

$$\xi(n)=H\Gamma(1+H)(n-1)!/\Gamma(H+n+1) \; , \tag{5}$$

where $\Gamma(x)=\int_0^\infty t^{x-1}\exp(-t)dt$ is the complete gamma function and H is the ratio between the relaxation time $\overline{\tau}_r$ and the clustering time $\overline{\tau}_c$,

$$H=\overline{\tau}_r/\overline{\tau}_c \; . \tag{6}$$

We introduce the structure functions

$$\overline{p}(\mathbf{k})=\int e^{i\mathbf{k}\cdot\mathbf{r}}p(\mathbf{r})d\mathbf{r}, \quad \overline{\overline{p}}(\mathbf{k})=\int e^{i\mathbf{k}\cdot\mathbf{r}}\overline{p}(\mathbf{r})d\mathbf{r} \tag{7}$$

for the initial and renormalized flights, respectively. For a symmetric nonrenormalized flight with a finite square displacement $\langle r_0^2\rangle=\langle|\mathbf{r}|^2\rangle$ the nonrenormalized structure function is analytic near $\mathbf{k}=0$:

$$\overline{p}(\mathbf{r})=1-\langle r_0^2\rangle|\mathbf{k}|^2/2d_s+o(|\mathbf{k}|^4) \; . \tag{8}$$

By taking the Fourier transform of Eq. (1) and using Eqs. (5) and (7) we obtain [6,7]

$$\overline{\overline{p}}(\mathbf{k})=1-H\langle r_0^2\rangle|\mathbf{k}|^2/[2d_s(H-1)]+o(|\mathbf{k}|^4) \; , \quad H>1 \tag{9}$$

$$\overline{\overline{p}}(\mathbf{k})=1-|\mathbf{k}|^{2H}\left[\frac{\langle r_0^2\rangle}{2d_s}\right]^H\frac{\pi H}{\sin(\pi H)}+o(|\mathbf{k}|^2) \; , \quad H<1 \; . \tag{10}$$

By considering a succession of a large number of renormalized steps $N\gg1$, from Eqs. (9) and (10) it follows that the probability density for the position of a walker obeys a Gaussian law for $H>1$ and a Lévy law for $H<1$.

III. INTERACTING LÉVY FLIGHTS

Now we can proceed to the analysis of the interacting flights. In addition to the three hypotheses (a)–(c) introduced above we have the following.

(d) If there are many walkers the jump clustering is not an independent process that occurs for each of the walkers separately; it is a cooperative effect that occurs only if all the walkers are in their active state. Since all walkers are equivalent, the jumps from a cluster are randomly and uniformly distributed among the m walkers.

As the probability that all walkers are in an active state is the product of the individual probabilities $\lambda(\tau_1),\ldots,\lambda(\tau_m)$, the probability that a cluster is made up of $n_1,\ldots,n_m$ jumps corresponding to the m walkers, respectively, is given by

$$\xi(\mathbf{n})=\frac{\left[\sum n_l-m\right]!}{\prod(n_l-1)!}\left[\frac{1}{m}\right]^{\sum n_l-m}\int\cdots\int_0^\infty[1-\lambda(\tau_1)\cdots\lambda(\tau_m)][\lambda(\tau_1)\cdots\lambda(\tau_m)]^{\sum n_l-m-1}\times\eta_c(\tau_1)\cdots\eta_c(\tau_m)d\tau_1\cdots d\tau_m \; , \tag{11}$$

with

$$\mathbf{n}=(n_1,\ldots,n_m) \; . \tag{12}$$

In Eq. (11) the factor in front of the integral is given by a uniform multinomial distribution of the jumps among the m walkers, whereas the multiple integral expresses the probability of occurrence of a total number of jumps equal to $n_1+\cdots+n_m$.

For evaluating the integrals in Eq. (11) we use Eqs. (3) and (4) to express $\lambda(\tau_l)$ and $\eta_c(\tau_l)$, $l=1,\ldots,m$, as functions of time and introduce the integration variables

$$x_l=\exp(-\tau_l/\overline{\tau}_r), \quad l=1,\ldots,m \; . \tag{13}$$

The expression (11) for $\xi(\mathbf{n})$ reduces to the difference of the products of two groups of identical definite integrals multiplied by a multinomial probability law: by expressing the definite integrals in terms of the complete Γ function we obtain

$$\xi(\mathbf{n})=\frac{\left[\sum n_l-m\right]!}{\prod(n_l-1)!}m^{m-\sum n_l}H^m\left\{\prod_j\left[\int_0^1(1-x_j)^{\sum n_l-m}x_j^{H-1}dx_j\right]-\prod_j\left[\int_0^1(1-x_j)^{\sum n_l-m+1}x_j^{H-1}dx_j\right]\right\}$$

$$=H^m m^{m-\sum n_l}\frac{\left[\sum n_l-m\right]!}{\prod(n_l-1)!}\left[\frac{\left[\sum n_l-m\right]!\Gamma(H)}{\Gamma\left[\sum n_l-m+1+H\right]}\right]^m\left[1-\left[\frac{\sum n_l-m+1}{\sum n_l-m+1+H}\right]^m\right]. \quad (14)$$

The Lévy flights are generated by very large clusters, which is why we investigate the asymptotic behavior of $\xi(\mathbf{n})$ as $n_l\to\infty$, $l=1,\ldots,m$. By using the Stirling approximation for the Γ function we get

$$\xi(\mathbf{n})\cong\frac{mH[\Gamma(H+1)]^m}{(\sqrt{2\pi})^{m-1}}(n-m)^{-(mH+1)}\times\exp\left[-\frac{n-m}{m}\sum(\varphi_l+1)\ln(\varphi_l+1)\right],$$
$$n_l\to\infty,\ l=1,\ldots,m, \quad (15)$$

where

$$\varphi_l=[n_l-(n/m)]/[(n/m)-1] \quad (16)$$

and

$$n=n_1+\cdots+n_m \quad (17)$$

is the total number of jumps from a cluster. From Eq. (15) it follows that for a constant total number of jumps $n\gg1$ the probability $\xi(\mathbf{n})$ has a very sharp maximum for

$$n_1=n_2=\cdots=n_m=n^*=n/m, \quad (18)$$

a situation that corresponds to

$$\varphi_l=0,\quad l=1,\ldots,m. \quad (19)$$

This shows that the interacting flights have a very strong cooperative behavior. For the direction in the $\mathbf{n}$ space defined by the conditions (18), $\xi(\mathbf{n})$ has a long tail of the inverse power law type that corresponds to a statistical fractal behavior:

$$\xi[\mathbf{n}=(n^*,\ldots,n^*)]\sim(n-m)^{-(mH+1)}\sim n^{-(mH+1)},$$
$$n\to\infty,\ n\gg m. \quad (20)$$

If at least two of the numbers of jumps $n_1,\ldots,n_m$ are different then the asymptotic behavior of $\xi(\mathbf{n})$ changes: it decreases exponentially to zero; this decrease is much faster than that in the case of the inverse power law given by Eq. (20).

Although on the hypersurface $\xi=\xi(\mathbf{n})$ there is only one direction for which the statistical fractal behavior occurs, it is enough to generate renormalized flights of the Lévy type. By analogy to Eq. (1) we introduce a renormalized joint jump probability density $\tilde{p}_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ given by

$$\tilde{p}_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)=\sum_{n_1}\cdots\sum_{n_m}\xi(\mathbf{n})[p(\mathbf{r}_1)\otimes]^{n_1}\cdots[p(\mathbf{r}_m)\otimes]^{n_m}. \quad (21)$$

The Fourier transform of $\tilde{p}_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)$,

$$\bar{\tilde{p}}_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)=\int_{-\infty}^{+\infty}\cdots\int\exp\left[i\sum\mathbf{k}_l\cdot\mathbf{r}_l\right]\times\tilde{p}_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)\times d\mathbf{r}_1\cdots d\mathbf{r}_m, \quad (22)$$

is a renormalized m-particle structure function. From Eqs. (21) and (22) we have

$$\bar{\tilde{p}}_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)=\sum_{n_1}\cdots\sum_{n_m}\xi(\mathbf{n})[\bar{p}(\mathbf{k}_1)]^{n_1}\cdots[\bar{p}(\mathbf{k}_m)]^{n_m}. \quad (23)$$

By expressing in Eq. (23) the probability $\xi(\mathbf{n})$ in terms of multiple integrals over $x_1,\ldots,x_m$ [Eq. (14)] and evaluating the sums over $n_1,\ldots,n_m$ we obtain

$$\bar{\tilde{p}}_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)=\left[mH^m\int_0^1\cdots\int_0^1\frac{\left[1-\prod(1-x_l)\right]\left[\prod x_l\right]^{H-1}}{1+u(\mathbf{k}_1,\ldots,\mathbf{k}_m)-\prod(1-x_l)}\times dx_1\cdots dx_m\right]\frac{\prod\bar{p}(\mathbf{k}_l)}{\sum\bar{p}(\mathbf{k}_l)}, \quad (24)$$

where

$$u(\mathbf{k}_1,\ldots,\mathbf{k}_m)=m\Big/\left[\sum\bar{p}(\mathbf{k}_l)\right]-1. \quad (25)$$

Now we use the integral identities

$$(u+y)^{-1}=\int_0^\infty e^{-ut}e^{-yt}dt, \quad (26)$$

$$H^m\int_0^1\cdots\int_0^1\left[\prod x_l\right]^{H-1}dx_1\cdots dx_m=1 \quad (27)$$

and rewrite Eq. (24) in the form

$$\tilde{\bar{p}}_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)=\frac{m\prod\bar{p}(\mathbf{k}_l)}{\sum\bar{p}(\mathbf{k}_l)}\left[1-u(\mathbf{k}_1,\ldots,\mathbf{k}_m)H^m\times\int_0^\infty\exp[-tu(\mathbf{k}_1,\ldots,\mathbf{k}_m)]\times g(t)dt\right], \quad (28)$$

where

$$g(t)=\int_0^1\cdots\int_0^1\left[\prod x_l\right]^{H-1}\times\exp\left\{-t\left[1-\prod(1-x_l)\right]\right\}\times dx_1\cdots dx_m . \quad (29)$$

We are interested in the behavior of the renormalized jump probability density $\tilde{p}_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ for large values of $|\mathbf{r}_l|\to\infty$, $l=1,\ldots,m$, which in $\mathbf{k}$ space corresponds to $\mathbf{k}_1,\ldots,\mathbf{k}_m\to\mathbf{0}$, $u\to 0$ and in terms of the t variable to $t\to\infty$. As $t\to\infty$ the multiple integral in Eq. (29) may be approximated by

$$g(t)=\int_0^1\cdots\int_0^1\left[\prod x_l\right]^{H-1}\exp\left\{-t\left[\sum x_l-\sum\sum x_l x_{l'}+\cdots\right]\right\}dx_1\cdots dx_m$$
$$\cong\left[\int_0^1 x^{H-1}\exp(-tx)dx\right]^m\cong t^{-mH}[\Gamma(H)]^m, \quad t\gg 0 , \quad (30)$$

which leads to the following approximate expression for $\tilde{\bar{p}}(\mathbf{k}_1,\ldots,\mathbf{k}_m)$:

$$\tilde{\bar{p}}_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)\cong\frac{m\prod\bar{p}(\mathbf{k}_l)}{\sum\bar{p}(\mathbf{k}_l)}\{1-[u(\mathbf{k}_1,\ldots,\mathbf{k}_m)]^{mH}\Gamma(1-mH)[\Gamma(H+1)]^m\} , \quad 1>mH,\ \mathbf{k}_1,\ldots,\mathbf{k}_m\to\mathbf{0} . \quad (31)$$

This approximation is valid only for $H<1/m$; by using expression (8) for the asymptotic behavior of the nonrenormalized structure function $\bar{p}(\mathbf{k})$ as $\mathbf{k}\to\mathbf{0}$ Eqs. (25) and (31) lead to

$$\tilde{\bar{p}}_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)\cong 1-\left[\frac{\langle r_0^2\rangle}{2md_s}\right]^{mH}\Gamma(1-mH)[\Gamma(H+1)]^m\times\left[\sum|\mathbf{k}_l|^2\right]^{mH},$$
$$\mathbf{k}_1,\ldots,\mathbf{k}_m\to 0,\ 1>mH . \quad (32)$$

For $1>mH$ the renormalized joint jump structure function $\tilde{\bar{p}}_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)$ has a nonanalytic behavior as $\mathbf{k}_1,\ldots,\mathbf{k}_m\to 0$; it is easy to check that in this case the average values of the numbers of jumps from a cluster corresponding to the different walkers are all infinite

$$\langle n_l\rangle=\sum_{n_1}\cdots\sum_{n_m}n_l\xi(\mathbf{n})=\infty . \quad (33)$$

This divergence of $\langle n_1\rangle,\ldots,\langle n_m\rangle$ is due to the very slow decay of $\xi(\mathbf{n})$ in the direction given by Eq. (18).

For $1<mH$ the probability $\xi(\mathbf{n})$ decreases sufficiently fast as $n_1,\ldots,n_m\to\infty$ and thus the average values of the numbers of jumps are finite

$$\langle n_1\rangle=\cdots=\langle n_m\rangle=\langle n_l\rangle(m) . \quad (34)$$

Due to the symmetry of $\xi(\mathbf{n})$ with respect to $n_1,\ldots,n_m$ all average values $\langle n_1\rangle,\ldots,\langle n_m\rangle$ are equal to each other; the cooperative character of the clustering process leads to a dependence of the average numbers of jumps on the total number of walkers. In this case the m-particle structure function has an analytic behavior as $\mathbf{k}_1,\ldots,\mathbf{k}_m\to 0$:

$$\tilde{\bar{p}}_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)\cong 1-\langle r_0^2\rangle\langle n_l\rangle(m)\left[\sum|\mathbf{k}_l|^2\right]\Big/2d_s ,$$
$$\mathbf{k}_1,\ldots,\mathbf{k}_m\to 0,\ mH>1 . \quad (35)$$

Now we introduce the renormalized marginal jump probability density for one walker from a set of m walkers,

$$\tilde{p}^{(m)}(\mathbf{r})=\int\cdots\int\tilde{p}_m(\mathbf{r},\mathbf{r}_2,\ldots,\mathbf{r}_m)d\mathbf{r}_2\cdots d\mathbf{r}_m , \quad (36)$$

and the corresponding renormalized marginal structure function

$$\tilde{\bar{p}}^{(m)}(\mathbf{k})=\int\exp(i\mathbf{k}\cdot\mathbf{r})\tilde{p}^{(m)}(\mathbf{r})d\mathbf{r}=\tilde{\bar{p}}_m(\mathbf{k},0,\ldots,0) ; \quad (37)$$

we have

$$\tilde{\bar{p}}^{(m)}(\mathbf{k})\cong 1-\left[\frac{\langle r_0^2\rangle}{2md_s}\right]^{mH}\Gamma(1-mH)[\Gamma(H+1)]^m|\mathbf{k}|^{2mH} ,$$
$$\mathbf{k}\to 0,\quad H<1/m , \quad (38)$$

and

$$\tilde{\bar{p}}^{(m)}(\mathbf{k})\cong 1-\left[\frac{\langle r_0^2\rangle}{2d_s}\right]\langle n_l\rangle(m)|\mathbf{k}|^2 ,$$
$$\mathbf{k}\to 0,\ H>1/m . \quad (39)$$

Note that due to the cooperative behavior of the jump clustering the marginal structure function $\tilde{\tilde{p}}^{(m)}(\mathbf{k})$ depends on the total number m of walkers. We consider a succession N of renormalized steps and denote by

$$\tilde{P}_N^{(m)}(\mathbf{r})d\mathbf{r} \quad \text{with} \int \tilde{P}_N^{(m)}(\mathbf{r})d\mathbf{r}=1 \tag{40}$$

the probability density of the position of one of the m walkers after N steps, thus giving

$$\tilde{P}_N^{(m)}(\mathbf{r})d\mathbf{r}=[\tilde{p}^{(m)}(\mathbf{r})\otimes]^N d\mathbf{r} . \tag{41}$$

To evaluate the asymptotic behavior of $\tilde{P}_N^{(m)}(\mathbf{r})$ as $N\to\infty$ we take the Fourier transform of Eq. (41), insert into the resulting equation expressions (38) and (39) for the marginal structure function, and return to the real space variables by means of an inverse Fourier transformation. After some algebra we arrive at

$$P_N^{(m)}(\mathbf{r})\sim(b_m)^{-d_s}\mathcal{L}_\beta^{(d_s)}(|\mathbf{r}|/b_m) , \quad H<1/m, \ N\to\infty , \tag{42}$$

and

$$P_N^{(m)}(\mathbf{r})\sim\left[\frac{d_s}{2N\pi\langle r_0^2\rangle\langle n_l\rangle(m)}\right]^{d_s/2} \times\exp\left[-\frac{d_s|r|^2}{2N\langle r_0^2\rangle\langle n_l\rangle(m)}\right] , \quad H>1/m, \ N\to\infty , \tag{43}$$

where

$$\beta=2mH , \tag{44}$$

$$b_m=[N\Gamma(1-mH)]^{1/(2mH)}[\Gamma(H+1)]^{1/2H}\left[\frac{\langle r_0^2\rangle}{2md_s}\right]^{1/2} , \tag{45}$$

and

$$\mathcal{L}_\beta^{(d_s)}=(2\pi)^{-d_s}\int\exp(-i\mathbf{k}\cdot\mathbf{r}-|\mathbf{k}|^\beta)d\mathbf{k} \tag{46}$$

is the d_s-dimensional symmetric stable Lévy law with a fractal exponent $2>\beta>0$.

IV. DISCUSSION

In order to clarify the notions of "active state," "clustering," and "time scales" for noninteracting and interacting flights we make a comparison between the two renormalized approaches. Table I shows the main features of the renormalized approaches for noninteracting and interacting flights, respectively. By examining it we notice that the interaction among walkers decreases the efficiency of the clustering process. For interacting flights the passage from the Gaussian to the Lévy behavior corresponds to a critical point given by $mH=1$. Expressed in terms of the average clustering and relaxation times $\overline{\tau}_c,\overline{\tau}_r$ this critical condition is given by

TABLE I. The probability of occurrence of the active state, the distribution of the cluster size, the critical relation between the characteristic time scales, and the fractal Lévy exponents for noninteracting and interacting Lévy flights, respectively.

Property	Noninteracting Lévy flights	Interacting Lévy flights	Comparison
probability of occurrence of the active state of a walker	$\lambda(\tau)=\int_0^\tau\eta_r(\tau)d\tau$	$\lambda(\tau)=\int_0^\tau\eta_r(\tau)d\tau$	the probability of activation of an individual walker is the same in both cases
distribution of the cluster size n	for each walker there is a probability $\xi(n)$ of the cluster size n given by Eq. (2)	for a set of m walkers there is a single joint probability $\xi(n_1,\ldots,n_m)$ of the sizes $n_1,\ldots,n_m$ of the clusters attached to the different walkers [Eq. (11)]	for noninteracting flights the clustering of jumps for a walker is independent of the clustering of other walkers for interacting flights the clustering is strongly cooperative—it occurs if all walkers are in their active state
renormalization equations	$\tilde{p}(\mathbf{r})=\sum\xi(n)[p(\mathbf{r})\otimes]^n$	$\tilde{p}_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)=\sum\cdots\sum\xi(\mathbf{n})\times[p(\mathbf{r}_1)\otimes]^{n_1}\cdots[p(\mathbf{r}_m)\otimes]^{n_m}$	
critical condition between the time scales $\overline{\tau}_r$ and $\overline{\tau}_c$ characteristic for the transition from the Gaussian to the Lévy regime	$\overline{\tau}_c=\overline{\tau}_r$	$\overline{\tau}_c=m\overline{\tau}_r$	for interacting flights the total clustering time should be m times larger than for noninteracting flights this time is necessary for the activation of all walkers the interaction decreases the efficiency of clustering.
the fractal Lévy exponent	$\beta=2H=2\overline{\tau}_r/\overline{\tau}_c$	$\beta=2mH=2m\overline{\tau}_r/\overline{\tau}_c$	

$$\bar{\tau}_c = m\bar{\tau}_r \; ; \tag{47}$$

for $\bar{\tau}_c < m\bar{\tau}_r$ the random flights are Gaussian, whereas for $\bar{\tau}_c > m\bar{\tau}_r$ they are of the Lévy type. The threshold value for the clustering time $\bar{\tau}_c$ is m times larger than the threshold value $\bar{\tau}_c = \bar{\tau}_r$ characteristic for noninteracting random flights [6]. The physical significance of this result is clear. For m interacting flights the clustering process leads to infinite averages $\langle n_l \rangle = \infty$, $l = 1, \ldots, m$, only if the clustering time is at least equal to the sum $\bar{\tau}_r + \cdots + \bar{\tau}_r = m\bar{\tau}_r$ of the individual average times necessary for the activation of all walkers; if $\bar{\tau}_c < m\bar{\tau}_r$ the average clustering time is too small and the average values of the numbers of jumps $n_1, \ldots, n_m$ are finite.

The above analysis outlines another feature of our renormalized approach. Unlike the case of the classical Shlesinger-Hughes renormalization approach [5], in this paper the basic random variable is the number of jumps from a cluster rather than the corresponding displacement vector. The divergence of the mean square displacement for a Lévy flight is due to the divergence of the mean number of jumps from a cluster.

The model presented in this paper is a system with interacting Lévy flights. The type of interaction discussed here is related to the mechanism of jump clustering. It is not clear whether this type of interaction is related in a simple way to the interactions analyzed in the literature for non-Lévy flights, for instance, to the problem of self-avoiding random walks from polymer physics [8] or to the "true" self-avoiding random walks [9].

Our approach is of interest in connection with the theory of fractal random processes. Furthermore, there are general possibilities of application of the model introduced in this paper, for instance, in the study of space-dependent supercritical branching processes [10] or in the description of multifragmentation processes [11]. In the latter case we have in mind a fragmentation mechanism in which particles of different compositions are broken into pieces as a result of interparticle collisions. For this process the jump clustering has a simple physical significance: it corresponds to an individual fragmentation event. A similar physical interpretation is possible for space-dependent branched chain processes. The detailed mathematical models for these two concrete problems are more complicated than the general abstract scheme considered here; however, they all share the same basic features. Work on the above mentioned problems is in progress and the main results will be presented elsewhere.

Note added in proof. Recently, we have learned that Fogedby, Bohr, and Jensen [12] have discussed the density fluctuations of an ideal Browian gas of noninteracting Lévy flights. It would be interesting to generalize their approach to the case of interacting Lévy flights considered here.

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