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Statistical Dynamics of Random Maps

PANKAJ K. KAMTHAN
Scientific Computing Laboratory, Department of Computer Science; Department of Mathematics and Statistics, Concordia University, Montreal, Quebec, Canada H3G 1M8 and Center for Nonlinear Dynamics, McGill University, Montreal, Quebec, Canada H3G 1Y6
and
MICHAEL C. MACKEY
Departments of Physiology, Physics and Mathematics and Center for Nonlinear Dynamics, McGill University, Montreal, Quebec, Canada H3G 1Y6

Abstract

The dynamics of many physical systems are often governed by a randomly changing environment and can thus be described by a random map R whose evolution is represented by choosing a transformation from a given set of transformations and applying it with a given probability. We can describe the asymptotic behavior of densities f of such systems using a Markov operator P_R. We give results regarding existence and uniqueness of a stationary density f_R for P_R. For random maps generated by a large class of piecewise monotonic maps on $[0,1]^N \equiv I^N$, we prove that the sequence of densities $\{P_R^n f\}$ is asymptotically periodic. An upper bound for the period is given. A relation between the spectral representation of P_R and the ergodic decomposition of R is derived. For random maps composed of a large class of piecewise expanding transformations on I^N, $\{P_R^n f\}$ is shown to converge strongly to a unique fixed point f_R, and hence is asymptotically stable. We present sufficient conditions for f_R to be the density of a Sinai-Ruelle-Bowen or SRB-measure. The properties of asymptotic periodicity and asymptotic stability are examined under perturbations of the initial density and under perturbation of R. Stability of P_R and f_R under perturbation of R is studied. Two methods for approximation of f_R are presented. The asymptotic behavior of the conditional entropy, a generalization of Boltzmann-Gibbs entropy, and that of correlation functions are studied for both asymptotically periodic and asymptotically stable systems. The inverse Frobenius-Perron problem in the random map context is discussed briefly.

Contents

1 Introduction

The dynamical behavior of various natural processes can often be described by a random map which is a discrete time Markov process in which one of a given number of transformations is selected with a given probability and applied. For example, such maps arise naturally in a variety of contexts: in the study of stochastically forced oscillators [33]; in the problem of the distribution of particles floating on the surface of a fluid [57]; generation of fractals [4]; modelling interference effects such as those that occur in the two-slit experiment of quantum physics [11]; stochastic learning automata and neural networks [19]; construction of orthonormal wavelets [24] and the development of plant models [77].

The study of asymptotic statistical behavior of random dynamical systems initiated in the spirit of ergodic theory [83], has recently been the focus of intense research. Various issues such as calculation of Lyapunov exponents [1]; dimension spectra [32]; and large deviation theory [51], have been investigated.

Our interest lies in the statistical study of attractor(s) which result from the asymptotic motion of a chaotic system. A useful tool in this study is ergodic theory where often the unpredictable behavior of trajectory evolution in systems can be better comprehended if one examines their behavior in terms of density evolution [58]. In this case, instead of studying the evolution of a single point in the phase space by the transformation, we study the evolution of a density f of a collection (or ensemble) of points in the phase space by a linear integral operator P known as the *Markov operator*. One of the main advantages of studying P rather than the transformation is that while the transformation may be nonlinear (and often discontinuous) on the phase space, P is a bounded linear operator on L^1. Thus, in examining the behavior of $\{P^n f\}$ we can apply the powerful tools of linear operator theory.

Given the density f_0 of the initial preparation of a system, the nature of the dynamic approach to equilibrium states displayed by the density evolution of $\{P^n f_0\}$ may be *ergodicity, mixing or exactness*. In all these three cases the system possesses a stationary density f^* and their respective occurrence depends on whether the convergence of the sequence $\{P^n f_0\}$ to f^* is Cesàro, weak or strong. The types of density evolution we are particularly interested here are that of *asymptotic periodicity* (Definition 4.1) and *asymptotic stability* (Definition 5.1), which are also related [58, §5.5] to the equilibrium states mentioned above.

The alternative viewpoint (density evolution vs. trajectory evolution) has a particular appeal when applying the ergodic theory results on the evolution of thermodynamic states characterized by densities to give a unified treatment of the origin of classical thermodynamic behavior [65]. As a result, all of these dynamic evolution behaviors have thermodynamic analogs. For example, the existence of a stationary density may be associated with a state of thermodynamic equilibrium of a dynamical system. The property of asymptotic periodicity is particularly significant from a thermodynamic point of view: it ensures the existence of at least one state of thermodynamic equilibrium with stationary density and allows us to prove a weak form of the Second Law of thermodynamics in which the conditional entropy, a generalization of Boltzmann-Gibbs entropy, (of the sequence of densities) increases to (at least) a local maximum [65, Theorem 6.5]. Also, for an asymptotically periodic system, analytical expressions for statistical quantifiers characterizing the dynamics such as conditional entropy and correlation functions simplify, and can be readily calculated [76]. The property of asymptotic stability implies not only the existence of unique state of thermodynamic equilibrium, but also of an approach to equilibrium from all initial preparations of a system.

In this paper, we examine the existence and consequences of these equilibrium states in the density evolution of random maps R (Definition 2.2) composed of a large class of piecewise monotonic transformations τ on I^N which are defined on rectangular partitions of I^N and on each element of such a partition, each component of τ depends only on one variable known as Jabłoński transformations (Definition 2.1). Our ultimate goal is to provide a statistical mechanical formalism for random maps under consideration. In Section 2, we present the background required to establish our main results, which includes an introduction to Jabłoński transformations, functions of bounded variation on I^N, and random maps. In Section 3, issue of the existence and uniqueness of a fixed point f_R for P_R, the Markov operator corresponding to R (Remark 3.1), is dealt with. In Section 4, using a spectral decomposition theorem, we prove the asymptotic periodicity of a sequence of densities $\{P_R^n f\}$ and give an upper bound on its period. We derive a connection between the number of densities in the spectral representation of P_R and that of the ergodic decomposition of R for special classes of Jabłoński transformations. In Section 5, asymptotic stability of $\{P_R^n f\}$ is shown by proving that $\{P_R^n f\}$ converges strongly in L^1 to a unique invariant density f_R of R. For some specific cases of random maps composed of exact Jabłoński transformations, asymptotic stability is exhibited. We emphasize the physical relevance of the density f_R and show that it is the density of an SRB measure. In Section 6, we consider the issue of stability of $\{P_R^n f\}$ under perturbations of the initial density with which the system is prepared and perturbation of R and in this context the properties of asymptotic periodicity and asymptotic stability are examined. We also examine stability of f_R and P_R under the perturbations of R. In Section 7, we present two methods of approximation of f_R by constructing a sequence of finite-rank operators in each case (one on the space of piecewise constant functions and the other on that of continuous piecewise linear functions) which converge strongly to P_R and whose fixed points converge to f_R. In Section 8, thermodynamic connections are drawn using the results of asymptotic periodicity and asymptotic stability obtained in Sections 4 and 5, to give estimates of conditional entropy, auto-correlation and time-correlation functions of $\{P_R^n\}$. In Section 9, the inverse problem of finding a dynamical system, which represents the dynamics of a random map whose invariant density is given, is considered. Finally, in Section 10, we briefly discuss some problems for future research.

2 Preliminaries

Let m_j denote the Lebesgue measure on I^j; for $j = N$, let $m = m_N$. We let L^1 denote the space of all Lebesgue integrable functions on I^N. The transformation $\tau : I^N \to I^N$ is written as

$$\tau(x) = (\varphi_1(x), \ldots, \varphi_N(x)),$$

where $x = (x_1, \ldots, x_N)$ and for any $i = 1, \ldots, N$, $\varphi_i(x)$ is a functional from I^N into [0,1].

We say that a measure μ is *τ-invariant* if $\mu(\tau^{-1}A) = \mu(A)$, for each measurable subset A of I^N, i.e, for one application of the transformation, the amount of mass that leaves a set is equal to the mass that enters, so the transformation is in a state of equilibrium with respect to the measure. μ is said to be *τ-ergodic* if $\tau^{-1}A = A$ implies $\mu(A) = 0$ or $\mu(I^N/A) = 0$ for each measurable subset A of I^N. μ is said to be *τ-mixing* if $\lim_{n\to\infty} \mu(\tau^{-n}(A) \cap B) = \mu(A)\mu(B)$ for all measurable sets A, B of I^N. We say that μ is *τ-exact* if $\lim_{n\to\infty} \mu(\tau^n(A)) = 1$ for each measurable set A of I^N with $\mu(A) > 0$. In these cases, we also say that the corresponding transformation τ is invariant, ergodic, mixing and exact, respectively.

A function $f \in L^1$ is called a *density* if $f \geq 0$ and $\|f\|_1 = 1$, where $\|.\|_1$ denotes the L^1 norm. Let $\mathcal{D}$ denote the space of all densities on I^N. The *support* of a density is the set $\{x \in I^N : f(x) > 0\}$.

We say that a measurable transformation τ is *nonsingular*, if $\mu(A) = 0$ implies that $\mu(\tau^{-1}(A)) = 0$. For a nonsingular transformation $\tau : I^N \to I^N$ and for any $f \in L^1$ we define P_τ, the *Frobenius-Perron operator* corresponding to τ, by

$$\int_A P_\tau f dm = \int_{\tau^{-1}(A)} f dm.$$

where $A \subset I^N$ is a measurable set.

Then from [58],

$$P_\tau f(x) = \frac{\partial^N}{\partial x_1 \cdots \partial x_N} \int_{\tau^{-1}(\prod_{i=1}^N [0,x_i])} f(y)dy.$$

It is well known that the operator P_τ is linear, positive, preserves integrals, and $P_{\tau^n} = P_\tau^n$. An important property of a Frobenius-Perron operator P_τ is that its stationary density f is the density function of an τ-invariant measure μ which is an absolutely continuous with respect to Lebesgue measure (ACIM), i.e., $P_\tau f = f$ if and only if $d\mu = f dm$.

Let $\mathcal{M}(I^N)$ denote the space of measures on I^N and let $\tau : I^N \to I^N$ be a measurable transformation. τ induces a transformation τ_* on $\mathcal{M}(I^N)$ defined by $(\tau_*(m))(A) = m(\tau^{-1}A)$, for each measurable subset A of I^N. The operators $\tau_* : \mathcal{M}(I^N) \to \mathcal{M}(I^N)$ and $P_\tau : \mathcal{D}(I^N, m) \to \mathcal{D}(I^N, \tau_* m)$ are equivalent, though P_τ is often easier to work with.

2.1 Piecewise Monotonic Transformations and Functions of Bounded Variation on I^N

Let $\mathcal{P} = \{D_1, \ldots, D_M\}$ be a finite ($M < \infty$) partition of I^N, i.e., $\cup_{j=1}^M = I^N$ and $D_j \cap D_k = \phi$ for $j \neq k$. A partition $\mathcal{P}$ of I^N is called *rectangular* if for any $1, \ldots, M$, D_j is an N-dimensional rectangle.

We now define the class of transformations which form the component transformations of the random maps we consider here.

Definition 2.1 (Jabłoński Transformation) A transformation $\tau : I^N \to I^N$ is called a *Jabłoński transformation*, if it is defined on a rectangular partition $\mathcal{P} = \{D_1, \ldots, D_M\}$ and is given by the expression

$$\tau(x) = (\varphi_{1j}(x_1), \ldots, \varphi_{Nj}(x_N)),$$

where $x \equiv (x_1, \ldots, x_N) \in D_j$, $1, \ldots, M$, $D_j = \Pi_{i=1}^N [a_{ij}, b_{ij})$, and $\varphi_{ij} : [a_{ij}, b_{ij}] \to [0,1]$. We write $[a_{ij}, b_{ij})$ as $[a_{ij}, b_{ij})$ if $b_{ij} < 1$ and as $[a_{ij}, b_{ij}]$ if $b_{ij} = 1$.

A Jabłoński transformation is said to be *piecewise* C^2 (respectively *linear*) if $\varphi_{ij} : [a_{ij}, b_{ij}] \to [0,1]$ are C^2 (respectively linear) functions and *expanding* if $\inf_{i,j} \mid \varphi'_{ij} \mid > 1$.

Jabłoński transformations are nontrivial higher dimensional generalizations of Lasota-Yorke transformations on the unit interval [59]. They form a large class of transformations from $I^N \to I^N$, and are in fact dense (in L^1) in the class of all piecewise expanding transformations on I^N [64]. The existence of an ACIM for piecewise C^2 and expanding Jabłoński transformations was established in [46]. They have been used in variety of applications, for example, in modelling the pattern formation of cellular automata on the space of configurations [40] and approximating fractal measures [8], [16].

To define the variation of N variables, we use the Tonelli definition [23]. Denote by $\Pi_{i=1}^N A_i$ the Cartesian product of the sets A_i and by proj_i the projection of $\mathbf{R}^N$ onto $\mathbf{R}^{N-1}$ given by

$$\text{proj}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N).$$

Let $g : A \to \mathbf{R}$ be a function of the N-dimensional interval $A = \Pi_{i=1}^N [a_i, b_i]$. Fixing i, define a function $\bigsqcup_i^A g$ of the $N-1$ variables $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ by the expression

$$\begin{aligned} \bigsqcup_i^A g = \bigsqcup_i g &= \sup\{\sum_{k=1}^n \mid (g(x_1, \ldots, x_i^k, \ldots, x_N) - (g(x_1, \ldots, x_i^{k-1}, \ldots, x_N) \mid : \\ &\quad a_i = x_i^0 < x_i^1 < \cdots < x_i^n = b_i, n \in \mathbf{N}\}. \end{aligned}$$

For $f : A \to \mathbf{R}$, where $A = \Pi_{i=1}^N [a_i, b_i]$, let

$$\bigvee_i^A f = \inf\{\int_{\text{proj}_i(A)} \bigsqcup_i g dm_{N-1} : g = f \text{ almost everywhere}, \bigsqcup_i g \text{ measurable}\}$$

and let

$$\bigvee^A f = \sup_{1 \le i \le N} \bigvee_i^A f.$$

If $\bigvee^A f < \infty$, we say that f is a *function of bounded variation* on A and its total variation is $\bigvee^A f$. Let BV denote the space of functions of bounded variation in the above sense.

Denote by $\mathcal{C}$, the set of functions of the form

$$g = \sum_{j=1}^M g_j \chi_{A_j},$$

where χ_{A_j} is the characteristic function of the set

$$A_j = \Pi_{i=1}^{N}[\alpha_{ij}, \beta_{ij}] \subset I^N,$$

(we do not assume that $\alpha_{ij} < \beta_{ij}$; the interval $[\alpha_{ij}, \beta_{ij}]$ can be degenerate), and $g_j : I^N \to \mathbf{R}$ is a C^1 function on A_j. Then from [46], we have the following:

Lemma 2.1 *If $f \in \mathcal{C}$, then*

$$\bigvee^{I^N} f < \infty.$$

Lemma 2.2 *The set $\mathcal{C}$ is a dense subspace of the space L^1.*

The next result is a higher dimensional generalization of Helley's Theorem and follows from Lemma 3 of [46].

Lemma 2.3 *Let S be a set of functions $f : I^N \to \mathbf{R}$ such that $f \geq 0, \bigvee^{I^N} f < M$ and $\| f \|_1 \leq 1$. Then S is weakly relatively compact on L^1.*

We now introduce some preliminary concepts related to random maps.

2.2 Random Maps on I^N

We begin with stating

Definition 2.2 (Random Map) Let $\tau_k : I^N \to I^N$, $k = 1, \ldots, l$ be given transformations. We define a *random map* R by choosing τ_k with probability p_k, $p_k > 0$, $\sum_{k=1}^{l} p_k = 1$.

We shall henceforth assume that $l \geq 2$ (so that R can not be reduced to a single deterministic map), unless stated otherwise.

A measure μ is called *invariant* under R if $\mu(A) = \sum_{i=1}^{l} p_i \mu(\tau_i^{-1} A)$, for each measurable set A of I^N.

R can be viewed as a stochastic process. Let $\tau_k : I^N \to I^N$, $k = 1, \ldots, l$, be nonsingular transformations and let $\alpha_1, \ldots, \alpha_l$ be random variables such that $\text{prob}(\alpha_k) = p_k$. Consider the stationary stochastic process defined by,

$$X_{n+1} = \sum_{k=1}^{l} \alpha_k \tau_k(X_n).$$

Then,

$$\begin{aligned} \text{prob}\{X_{n+1} \leq x\} &= \text{prob}\{\sum_{k=1}^{l} \alpha_k \tau_k(X_n)\} \\ &= \text{prob}\{\sum_{k=1}^{l} \alpha_k \tau_k(X_n)\} \sum_{k=1}^{l} \text{prob}(\alpha_k) \end{aligned}$$

i.e.,

$$\text{prob}\{X_{n+1} \leq x\} = \sum_{k=1}^{l} p_k \text{prob}\{\tau_k(X_n) \leq x\}. \tag{1}$$

The dynamics of R can be described in terms of a skew-product transformation [75]. Let $\Omega = \{\omega = (\omega_i)_{i=0}^{\infty} : \omega_i \in \{1,\ldots,l\}\}$ be the set of all infinite one-sided sequences of symbols in $S = \{1,\ldots,l\}$. The *left shift* $\sigma : \Omega \to \Omega$ is defined by

$$(\sigma(\omega_i))_j = \omega_{j+1}, \;\; j = 0, 1, \ldots.$$

The topology on Ω is the product of the discrete topology on S. The Borel measure μ_Ω on Ω is defined as the product of the distribution on S given by $\text{prob}(j) = p_j$. Then there is a one-one correspondence between the ways in which the maps τ_{k_j}, $j = 1, 2, \ldots$ can be chosen and points in Ω. Therefore, R can be treated from a dynamical systems point of view by a *skew-product transformation* $\tau_{SP} : I^N \times \Omega \to I^N \times \Omega$ defined by

$$\tau_{SP}(x,\omega) = (R_{\omega_0}x, \sigma(\omega)), \;\; (x,\omega) \in I^N \times \Omega.$$

3 Stationary Points of P_R

Let $\tau_k : I^N \to I^N$, $k = 1,\ldots,l$, be a C^2 Jabłoński transformation on the partition $\mathcal{P}_k = \{D_{k_1},\ldots,D_{k_M}\}$. Then

$$\tau_k(x) = (\tau_{k,1}(x),\ldots,\tau_{k,N}(x)),$$

and for any $i = 1,\ldots,N$ and $k = 1,\ldots,l$,

$$\tau_{k,i}(x) = \varphi_{k,ij}(x_i), \;\; x \in D_{k_j}.$$

The following result implying the existence of a fixed point f_R for P_R, i.e., an ACIM for random maps composed of Jabłoński transformations, was established in [15].

Theorem 3.1 *Let $R(x) = \{\tau_k(x), p_k, k = 1,\ldots,l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a (not necessarily expanding) Jabłoński transformation on the partition $\mathcal{P}_k = \{D_{k_1},\ldots,D_{k_M}\}$. Assume $\tau_{k,i}(x) = \varphi_{k,ij}(x_i)$ is C^2 and monotonic for $x \in \bar{D}_{k_j}$. If for $i = 1,\ldots,N$*

$$\sum_{k=1}^{l} \sup_j \frac{p_k}{|\varphi'_{k,ij}(x_i)|} \le \gamma < 1, \tag{2}$$

for some constant γ, then for all $f \in L^1$ we have:

(a)
$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n-1} P_R^t f = f_R, \;\; f \in L^1 \tag{3}$$

where
$$P_R = \sum_{k=1}^{l} p_k P_{\tau_k}. \tag{4}$$

(b) $P_R f_R = f_R$.

(c) $\bigvee^{I^N} f_R \le C\|f\|_1$, *for some constant C, independent of f.*

The following inquality was established in the proof of Theorem 3.1.

Proposition 3.1 *Let R be the random map satisfying the conditions of Theorem 3.1. Then for $f \in \mathcal{C}$*

$$\bigvee^{I^N} P_R f \leq \alpha \bigvee^{I^N} f + K \parallel f \parallel_1, \tag{5}$$

for some constants $0 < \alpha < 1$ and $K > 0$, both independent of f.

Remark 3.1 Condition (2) is an *expanding-on-average* condition. For a random map $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$, where each map $\tau_k : I^N \to I^N$ is nonsingular, we say that P_R, given by (4), is the *Frobenius-Perron operator* corresponding to R. P_R has the following properties [75]: P_R is linear; $\|P_R\|_1 \leq 1$; P_R is nonnegative; $P_R f = f$ if and only if $d\mu = f dm$ is invariant under R and; $P_R^n f = P_{R^n} f$. Existence of ACIMs for random maps in one dimension has been shown in [75], [69], [68].

By Theorem 3.1(b) and Remark 3.1, we have the existence of an ACIM for a large class of random maps R on I^N, which is closely related to the observability of chaos [53]. The support of f_R, which is a set of positive Lebesgue measure, indicates that part of the phase space on which the chaos resides.

Now, let the random variable X_n in (1) have a density $f \in L^1$. Then by the Radon-Nikodym Theorem, the random variable $\tau_k(X_n)$, $k = 1, \ldots, l$, has a density given by $P_{\tau_k} f$. Differentiating both sides of (1), we obtain equation (4), where $P_R f$ denotes the density of X_{n+1}. A stationary point of P_R can therefore be interpreted as the density of the stationary measure for the stochastic process (1).

3.1 Uniqueness of f_R

In this section, we present results which give sufficient conditions for the uniqueness of f_R. This is significant from the point of view that it gives us the exact location of the region of chaos. Also, during calculation of Lyapunov exponents of random maps, such as in [1], uniqueness of an invariant density is *assumed.* For random maps in one-dimension, results implying uniqueness have been given in [75] using symbolic dynamical methods, in [8] using matrix methods and in [9] using graph-theoretic methods. We generalize some of these results to higher dimensions. We first consider the problem in a general context and then treat some special cases.

Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 (not necessarily expanding) Jabłoński transformation satisfying condition (2).

We shall need the following theorem from [82].

Theorem 3.2 (Ionescu-Tulcea and Marinescu Theorem) *Let $P : L^1 \to L^1$ and let it satisfy the following properties:*

(a) $P \geq 0$, $\int Pf = \int f dm$, *for $f \in L^1$, which implies that $\|P\|_1 = 1$.*

(b) *there exist constants $0 < \alpha < 1$, $K > 0$ such that*

$$\|Pf\|_V \leq \alpha \|f\|_V + K\|f\|_1,$$

for $f \in BV$.

(c) *the image of any bounded subset of* BV *under* P *is relatively compact in* L^1.
Then, P *is quasi-compact operator on* $(BV, \|.\|_V)$. *Thus,* P *has only finitely many eigenvalues* $\{\lambda_1, \ldots, \lambda_h\}$ *of modulus 1. The corresponding eigenspaces* E_i *are finite-dimensional subspaces of* BV. *Furthermore,* P *has thc following representation:*

$$P = \sum_{i=1}^{h} \lambda_i \Phi_i + Q,$$

where Φ_i *are projections onto the* E_i, $\|\Phi_i\|_1 \leq 1$, $\Phi_i \circ \Phi_j = 0$, *for* $i \neq j$ *and* $Q : L^1 \to L^1$ *is a linear subspace with* $\sup_n \|Q^n\|_1 \leq h+1$, $Q(BV) \subset BV$, $\|Q^n\|_V \leq Hq^n$ *for some* $0 < q < 1$ *and* $H > 0$, *and* $Q \circ \Phi_i = \Phi_i \circ Q = 0$ *for all* $i = 1, \ldots, h$.

For each $\lambda \in \mathbf{R}$ with $|\lambda| = 1$ and $f \in L^1$, the limit

$$\Phi(\lambda, P)(f) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (\lambda P)^j (f). \tag{6}$$

exists in L^1, and

$$\Phi(\lambda, P) = \begin{cases} \Phi_i & \text{if } \lambda = \lambda_i \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.1 P_R *has finitely many eigenvalues* $f_1^*, \ldots, f_h^*$ *of modulus 1.*

Proof. The proof is an immediate consequence of definition of P_R, Proposition 3.1 and Theorem 3.2. □

By the Ionescu-Tulcea and Marinescu Theorem 3.2 and Lemma 3.1, it follows that the number of ergodic ACIMs of R is finite. Furthermore, S_i's, the support of f_i's, respectively are disjoint, i.e., each S_i is a support of an ergodic ACIM $\mu_{f_i^*}$ for R, where $\mu_{f_i^*}$ denotes the measure such that $\frac{d\mu_{f_i^*}}{dm} = f_i^*$ and every other ACIM for R is a linear combination of the measures $\mu_{f_i^*}$. This existence of ergodic components of R can be made more precise if we have information regarding the ACIMs of the individual maps τ_k, $k = 1, \ldots, l$. To draw that connection, we first establish the following:

Lemma 3.2 *The measure* μ *is R-invariant if and only if* $\mu \times \mu_\Omega$ *is* τ_{SP}*-invariant.*

Proof. For any measurable set A of I^N, we have

$$\mu \times \mu_\Omega(A \times (\omega_0, \ldots)) = \mu(A) p_(\omega_0).$$

Therefore,

$$\begin{aligned} \mu \times \mu_\Omega(\tau_{SP}^{-1}(A \times (\omega_0, \ldots))) &= \mu \times \mu_\Omega(\cup_{k=1}^{l} [\tau_k^{-1} A \times (k, \omega_0, \ldots)]) \\ &= \sum_{k=1}^{l} \mu \times \mu_\Omega(\tau_k^{-1} A \times (k, \omega_0, \ldots)) \\ &= \sum_{k=1}^{l} \mu(\tau_k^{-1} A) p_k p_(\omega_0) \\ &= p_{\omega_0} \sum_{k=1}^{l} p_k \mu(\tau_k^{-1} A) \\ &= p_{\omega_0} \mu(A) \end{aligned}$$

$$= \mu \times \mu_\Omega(A \times (\omega_0, \ldots)),$$

and the result follows. □

By Lemma 3.2, we say that the measure μ is *R-ergodic* if and only if $\mu \times \mu_\Omega$ is τ_{SP}*-ergodic*. Then, R is ergodic with respect to $\mu_{f_i^*}$ if τ_{SP} is ergodic with respect to the measure $\mu_{f_i^*} \times \mu_\Omega$. Therefore, it is sufficient to show that τ_{SP} is ergodic with respect to the measure $\mu_{f_i^*} \times \mu_\Omega$, for each $i = 1, \ldots, h$ and to discuss its ergodic components. This can be done using the method of [75].

Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ and let A be a measurable set in I^N. We say that A is *R-invariant*, if

$$m(A \bigtriangleup \cup_{k=1}^{l} \tau_k(A)) = 0,$$

where $\bigtriangleup$ denotes the symmetric difference.

Lemma 3.3 *S_i's are R-invariant for each $i = 1, \ldots, h$.*

As in [75], we have following characterization of sets invariant under τ_{SP}.

Lemma 3.4 *Let $\nu = \mu \times \mu_\Omega$ be R-invariant and let $A \subset I^N \times \Omega$, $\nu(A) > 0$, be a set τ_{SP}-invariant. Then*

$$\nu(A \bigtriangleup (B \times \Omega)) = 0$$

for some R-invariant set $B \subset I^N$.

From (6) it follows that the projection

$$\Phi(1, P_R)(\chi_{I^N}) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} P_R^k(\chi_{I^N}) := 1^*$$

is nonzero onto each basis vector f_i^*. Hence the number of ergodic componenets of τ_{SP} with respect to $\mu_{1^*} \times \mu_\Omega$ is the maximum possible for any product measure whose first factor is an ACIM. Therefore, Lemmas 3.3 and 3.4 establish a one-one correspondence between the sets S_i and the ergodic components of R with respect to $\mu_{1^*} \times \mu_\Omega$.

Now as in [75], we have the following connection between the number of ergodic components of R and that of τ_{SP}.

Lemma 3.5 *Let any one of the maps $\tau_1, \ldots, \tau_l$ have an ACIM with ergodic components $Q_1, \ldots, Q_h$. Define a relation:*

$$Q_i \sim Q_j \text{ if } m(\tau_{l_n} \circ \tau_{l_{n-1}} \circ \cdots \tau_{l_1}(Q_i) \cap Q_j) > 0,$$

for some $l_1, \ldots, l_n$. Then $\sim$ is an equivalence relation and the number of ergodic components of R with respect to $\mu_{1^*} \times \mu_\Omega$ is the number of equivalence classes.

Remark 3.2 Lemma 3.5 gives an upper bound on the number of ACIMs for R. Let τ_1 satisfy the hypothesis of the lemma. Then the number of ergodic ACIMs of R is *atmost* equal to the number of ergodic ACIMs of τ_1.

Theorem 3.3 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \dots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 (not necessarily expanding) Jabłoński transformation, satisfying condition (2). Then, if any one of the maps τ_k, $k = 1, \dots, l$, has a unique ACIM, R has a unique ACIM.*

Proof. The proof of the theorem follows immediately from Lemmas 3.1-3.5. □

Remark 3.3 Condition (2) can be satisfied even if only one of the component maps τ_k, $k = 1, \dots, l$, is piecewise expanding and others are contracting (and hence do not have an ACIM). Therefore, Corollary 2 of [36], (where uniqueness is obtained under the assumption that each map τ_k has an ACIM) does not yield uniqueness.

The following corollary is an obvious consequence of Theorem 3.3.

Corollary 3.1 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \dots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and expanding Jabłoński transformation. Then, if any one of the maps τ_k, $k = 1, \dots, l$, has a unique ACIM, R has a unique ACIM.*

Remark 3.4 The converse of Corollary 3.1 is not true in general, i.e., if R has a unique ACIM, it does not necessarily imply that any of the maps τ_k, $k = 1, \dots, l$, have a unique ACIM. We shall see this in Example 3.2.

3.1.1 Uniqueness under Special Cases

We now proceed to establish two special results of uniqueness: random maps composed of Jabłoński transformations with strong expansive property and that with communication property.

Let $\tau_k : I^N \to I^N$ be a Jabłoński transformation on a partition $\mathcal{P}_k = \{D_{k_1}, \dots, D_{k_M}\}$. Let $J(\tau_k, x)$ be the absolute value of the Jacobian of τ_k at x. Let $\mathcal{V}_k$ be the set of vertices of elements of $\mathcal{P}_k$ which lie in the interior of I^N. We now define a number $M_N^{(k)}$ related to the geometry of the partition $\mathcal{P}_k$. For any fixed $z \in \mathbf{R}$, let $H_{N-1}^{(j)}(z)$ denote the $(N-1)$-dimensional hyperplane given by the equation $x_j = z, j = 1, \dots, N$. Let

$$M_N^{(k)} = \max_{z\in\mathbf{R}} \max_{1\le j\le N} \#\{D_{k_i} : H_{N-1}^{(j)}(z) \cap \mathrm{int}(D_{k_i}) \neq \phi\}.$$

The following result is contained in Theorem 4 of [73]:

Lemma 3.6 *Let $\tau_k : I^N \to I^N$ be a piecewise C^2 and expanding Jabłoński transformation and let*

$$\Lambda_k = \inf\{J(\tau_k, x) : x \in I^N\}.$$

If

$$\frac{\Lambda_k}{M_N^{(k)}} > 1, \tag{7}$$

then the number of ergodic ACIMs for τ_k *is at most equal to* $\#\mathcal{V}_k$.

We call condition (7) the *strong expansive property*.

We conclude our first result from Lemma 3.6 and Corollary 3.1:

Proposition 3.2 *Let* $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ *be a random map, where each* $\tau_k : I^N \to I^N$ *is a piecewise* C^2 *(not necessarily expanding) Jabłoński transformation, satisfying condition (2). Let* τ_1 *be expanding on partition* $\mathcal{P}_1 = \{D_{1_1}, \ldots, D_{1_M}\}$ *related to which the number* $M_N^{(1)}$ *is defined. Let* $\mathcal{V}_1$ *be the set of vertices of elements of* $\mathcal{P}_1$ *which lie in the interior of* I^N *such that* $\#\mathcal{V}_1 = 1$. *If* τ_1 *has the strong expansive property, then* R *has a unique ACIM.*

Let $\tau : I^N \to I^N$ be a Jabłoński transformation. Without loss of generality assume that there exist $0 = a_{i,0} < a_{i,1} < \cdots < a_{i,r_i} = 1$, $i = 1, \ldots, N$, for some positive integers $r_1, \ldots, r_N$ such that the partition $\mathcal{P}$ is composed of sets $D_{s_1,\ldots,s_N} = \Pi_{i=1}^N D_{s_i}$, where $D_{s_i} = [a_{i,s_i-1}, a_{i,s_i})$, $s_i = 1, \ldots, r_i - 1$, $D_{r_i} = [a_{i,r_i-1}, a_{i,r_i}]$, and τ is given by the formula

$$\tau(x) = (\varphi_{1,s_1,\ldots,s_N}(x_1), \ldots, \varphi_{N,s_1,\ldots,s_N}(x_N)),\ x \in D_{s_1,\ldots,s_N},$$

where $\varphi_{i,s_1,\ldots,s_N}(x_1) : \bar{D}_{s_i} \to [0,1]$ are C^2 functions.

We say that the partition $\mathcal{P}$ has the *communication property* under the transformation $\tau : I^N \to I^N$ if for any elements $D'_{s_1,\ldots,s_N}$ and $D''_{s_1,\ldots,s_N}$ of $\mathcal{P}$ there exist integers u and v such that $D'_{s_1,\ldots,s_N} \subset \tau^u(D''_{s_1,\ldots,s_N})$ and $D''_{s_1,\ldots,s_N} \subset \tau^v(D'_{s_1,\ldots,s_N})$.

Proposition 3.3 *Let* $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ *be a random map, where each* $\tau_k : I^N \to I^N$ *is a piecewise* C^2 *(not necessarily expanding) Jabłoński transformation, satisfying condition (2). If* τ_1 *be a Jabłoński transformation on a partition* $\mathcal{P}$ *satisfying the following conditions:*

(a) $\inf |\varphi_i'| > 0$ *and* $\inf |(\varphi_i^w)'| > 1$ *for some integer* w,

(b) τ *is piecewise* C^2, *and*

(c) *the partition* $\mathcal{P}$ *has the communication property.*

Then R *has a unique ACIM.*

Proof. By Theorem 5 of [14], τ_1 has a unique ACIM and so the result follows from Corollary 3.1. □

3.1.2 Uniqueness for Random Maps composed of Markov transformations

Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and expanding Jabłoński transformation. If any one of the τ_k's has a unique ACIM, Theorem 3.2 implies that R has a unique invariant density, say f_R, but does not reflect any of its properties. The result of this section shows that for a class of random maps composed of Markov transformations, f_R is a piecewise constant function on the common partition where each of the component transformations is defined.

Let $\mathcal{P} = \{D_1, \ldots, D_M\}$, $M < \infty$ be a partition of I^N. A transformation $\tau : I^N \to I^N$ is called a *nonsingular piecewise C^2 transformation* with respect to $\mathcal{P}$ if for any $j = 1, \ldots, M$

$$\tau(x_1, ..., x_n) = \tau_j(x_1, ..., x_n)$$

on D_j, τ_j is a C^2 function on $\bar{D}_j$ and the Jacobian matrix $A_j = \frac{\partial d\tau_j}{\partial x}$ satisfies $\det A_j \neq 0$.

Definition 3.1 (Markov Transformation) A transformation $\tau : I^N \to I^N$ is called a *Markov transformation* with respect to $\mathcal{P}$ if, for any $j = 1, \ldots, m$, the image $\bar{D}_j$ is a union of $\bar{D}_k$'s, i.e.,

$$\tau(\bar{D}_j) = \cup_{i=1}^{L} \bar{D}_{j_i}, \tag{8}$$

for some $D_{j_1}, \ldots, D_{j_L}$.

We say that a function f is *piecewise constant* on $\mathcal{P}$ if $f = \sum_{j=1}^{M} C_j \chi_{D_j}$, for some constants $C_1, \ldots, C_M$. Let $\mathcal{S}$ denote the class of all functions which are piecewise constant on $\mathcal{P}$.

We need the following result contained in Theorem 3.1 of [64].

Lemma 3.7 *Let $\tau : I^N \to I^N$ be a nonsingular piecewise C^2 Markov transformation with respect to the partition $\mathcal{P}$. If for any $j = 1, \ldots, M$, τ_j is a homeomorphism from $\bar{D}_j$ onto $\bar{D}_{j_1} \cup \ldots \cup \bar{D}_{j_L}$ and has $\tau_j^{-1}(x_1, \ldots, x_N)$ as its inverse, and $\det A_j$ is a constant on $\bar{D}_j$, where $A_j = \frac{\partial d\tau_j}{\partial x}$, then there exists an $M \times M$ matrix $\mathcal{M}_\tau$ such that $P_\tau f = \mathcal{M}_\tau f \in \mathcal{S}$ for every $f \in \mathcal{S}$.*

$\mathcal{M}_\tau$ is known as the *matrix induced by* τ. It is nonnegative and for each $j = 1, \ldots, M$ the nonzero entries in the j^{th} column are equal to $|\det(A_j^{-1})|$. If the partition $\mathcal{P}$ is equal, $\mathcal{M}_\tau$ is stochastic.

We now present the main result of this section.

Theorem 3.4 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and expanding Jabłoński transformation with respect to a common partition $\mathcal{P} = \{D_1, \ldots, D_M\}$ satisfying condition (8). Assume for each $k = 1, \ldots, l$,*

(A) *τ_{kj} is a homeomorphism from $\bar{D}_j$ onto $\bar{D}_{j_1} \cup \ldots \cup \bar{D}_{j_L}$ and has $\tau_{kj}^{-1}(x_1, \ldots, x_N)$ as its inverse, and*

(B) *$\det(A_{kj})$ is a constant on $\bar{D}_j$, where $A_{kj} = \frac{\partial d\tau_{kj}}{\partial x}$.*

Then,

(a) *if $P_R f_R = f_R$, then $f_R \in \mathcal{S}$, i.e., every invariant density of R is piecewise constant,*

(b) *there exists an $M \times M$ matrix $\mathcal{M}_R = \sum_{k=1}^{l} p_k \mathcal{M}_k$, where $\mathcal{M}_k$ is the matrix induced by τ_k, such that $\mathcal{M}_R f = P_R f \in \mathcal{S}$ for every $f \in \mathcal{S}$, and $f_R \equiv (f_{R1}, \ldots, f_{RM})$, $f_{Rj} = f_R |_{D_j}$ is a right-eigenvector of $\mathcal{M}_R$,*

(c) *$\mathcal{M}_R$ has 1 as the eigenvalue of maximum modulus,*

(d) *if $\mathcal{M}_R$ is irreducible, f_R is unique (up to constant multiples).*

Proof. By Theorem 3.1, R has an invariant density f_R.

(a) The proof is along the lines of Lemma 2 of [8]. For notational convenience we shall restrict ourselves to two-dimensions and to random maps composed of two transformations.

The proof is the same for N-dimensions and for random maps composed of arbitrary number l of component transformations.

By a simple computation, the Frobenius-Perron operator for τ_k, $k=1,2$ is given by the expression

$$P_{\tau_k} f(x) = \sum_{j=1}^{M} f(\tau_{kj}^{-1}(x)) \mid \det(A_{kj}^{-1}) \mid \chi_{\tau_{kj}(D_j)}(x),$$

where $x=(x_1,x_2)$. For a fixed point f_R of P_R, we then have

$$\begin{aligned} P_R f_R(x) &= p_1 \sum_{j=1}^{M} f_R(\tau_{1j}^{-1}(x)) \mid \det(A_{1j}^{-1}) \mid \chi_{\tau_{1j}(D_j)}(x) \\ &\quad + p_2 \sum_{j=1}^{M} f_R(\tau_{2j}^{-1}(x)) \mid \det(A_{2j}^{-1}) \mid \chi_{\tau_{2j}(D_j)}(x) \\ &= f_R(x). \end{aligned}$$

By (B) and since each τ_k, $k=1,2$ is expanding, $\sup_j \mid \det(A_{kj}^{-1}) \mid = a_k < 1$. Now let D_j be any element of the partition $\mathcal{P}$ and let $x, y \in D_j$ be distinct and fixed. Then $\chi_{\tau_{kj}(D_j)}(x) = \chi_{\tau_{kj}(D_j)}(y)$, for $k=1,2$ and for all $j=1,\ldots,M$. Thus,

$$\begin{aligned} f_R(x) - f_R(y) &= p_1 \sum_{j=1}^{M} [f_R(\tau_{1j}^{-1}(x)) - f_R(\tau_{1j}^{-1}(y))] \mid \det(A_{1j}^{-1}) \mid \chi_{\tau_{1j}(D_j)}(x) \\ &\quad + p_2 \sum_{j=1}^{M} [f_R(\tau_{2j}^{-1}(x)) - f_R(\tau_{2j}^{-1}(y))] \mid \det(A_{2j}^{-1}) \mid \chi_{\tau_{2j}(D_j)}(x). \end{aligned}$$

Let j_1 vary over all integers $j \in \{1,\ldots,M\}$ such that $x \in \tau_{1j}(D_j)$ and let z_1 vary over all integers $j \in \{1,\ldots,M\}$ such that $x \in \tau_{2j}(D_j)$. Then

$$\begin{aligned} f_R(x) - f_R(y) &= p_1 \sum_{j_1} [f_R(\tau_{1j_1}^{-1}(x)) - f_R(\tau_{1j_1}^{-1}(y))] \mid \det(A_{1j_1}^{-1}) \mid \\ &\quad + p_2 \sum_{z_1} [f_R(\tau_{2z_1}^{-1}(x)) - f_R(\tau_{2z_1}^{-1}(y))] \mid \det(A_{2z_1}^{-1}) \mid . \end{aligned}$$

Similarly, for each j_1 and z_1 we have,

$$\begin{aligned} f_R(\tau_{1j_1}^{-1})(x) - f_R(\tau_{1j_1}^{-1})(y) &= p_1 \sum_{j_2} [f_R((\tau_{1j_2}^{-1})\tau_{1j_1}^{-1}(x)) - f_R((\tau_{1j_2}^{-1})\tau_{1j_1}^{-1}(y))] \mid \det(A_{1j_2}^{-1}) \mid \\ &\quad + p_2 \sum_{j_2} [f_R((\tau_{2z_2}^{-1})\tau_{1j_1}^{-1}(x)) - f_R((\tau_{2z_2}^{-1})\tau_{1j_1}^{-1}(y))] \mid \det(A_{2z_2}^{-1}) \mid, \end{aligned}$$

where j_2 varies over all integers $j \in \{1,\ldots,M\}$ such that $\tau_{1j_1}(x) \in \tau_{1j}(D_j)$ and z_2 varies over all integers $j \in \{1,\ldots,M\}$ such that $\tau_{1j_1}(x) \in \tau_{2j}(D_j)$. Repeating the same argument, we obtain

$$\begin{aligned} f_R(\tau_{2z_1}^{-1})(x) - f_R(\tau_{2z_1}^{-1})(y) &= p_1 \sum_{s_2} [f_R((\tau_{1s_2}^{-1})\tau_{2z_1}^{-1}(x)) - f_R((\tau_{1s_2}^{-1})\tau_{2z_1}^{-1}(y))] \mid \det(A_{1s_2}^{-1}) \mid \\ &\quad + p_2 \sum_{t_2} [f_R((\tau_{2t_2}^{-1})\tau_{2z_1}^{-1}(x)) - f_R((\tau_{2t_2}^{-1})\tau_{2z_1}^{-1}(y))] \mid \det(A_{2t_2}^{-1}) \mid \end{aligned}$$

Let $\eta = \max\{a_1, a_2\}$. Then if $\mid . \mid$ denotes the Euclidean norm on I^2, we have

$$\mid f_R(x) - f_R(y) \mid \;\; \leq \;\; p_1 \eta \sum_{j_1} \mid f_R(\tau_{1j_1}^{-1}(x)) - f_R(\tau_{1j_1}^{-1}(y)) \mid$$

$$+p_2\eta\sum_{z_1} \mid f_R(\tau_{2z_1}^{-1}(x)) - f_R(\tau_{2z_1}^{-1}(y)) \mid$$

$$\begin{aligned} \leq\ & p_1^2\eta^2\sum_{j_1}\sum_{j_2} \mid f_R((\tau_{1j_2}^{-1})\tau_{1j_1}^{-1}(x)) - f_R((\tau_{1j_2}^{-1})\tau_{1j_1}^{-1}(y)) \mid \\ & +p_1p_2\eta^2\sum_{j_1}\sum_{z_2} \mid f_R((\tau_{2z_2}^{-1})\tau_{1j_1}^{-1}(x)) - f_R((\tau_{2z_2}^{-1})\tau_{1j_1}^{-1}(y)) \mid \\ & +p_2p_1\eta^2\sum_{z_1}\sum_{s_2} \mid f_R((\tau_{1s_2}^{-1})\tau_{2z_1}^{-1}(x)) - f_R((\tau_{1s_2}^{-1})\tau_{2z_1}^{-1}(y)) \mid \\ & +p_2^2\eta^2\sum_{z_1}\sum_{t_2} \mid f_R((\tau_{2t_2}^{-1})\tau_{2z_1}^{-1}(x)) - f_R((\tau_{2t_2}^{-1})\tau_{2z_1}^{-1}(y)) \mid . \end{aligned}$$

Continuing this process U times, we obtain using $p_2 = 1 - p_1$ that

$$\mid f_R(x) - f_R(y) \mid \leq \sum_{u=0}^{U} \binom{U}{u} p_1^u \eta^U S_U$$

where each S_U is of the form:

$$\sum_{\xi_1}\cdots\sum_{\xi_U} \mid f_R(\Gamma_{\xi_U}^{-1}\cdots\Gamma_{\xi_1}^{-1}(x)) f_R(\Gamma_{\xi_U}^{-1}\cdots\Gamma_{\xi_1}^{-1}(y)) \mid$$

with $\Gamma_{\xi_v} = \tau_{1\xi_v}$ or $\tau_{2\xi_v}$. Now for each $u = 0, 1, \ldots, U$,

$$\{\Gamma_{\xi_U}^{-1}\cdots\Gamma_{\xi_1}^{-1}(x), \Gamma_{\xi_U}^{-1}\cdots\Gamma_{\xi_1}^{-1}(y)\}_{\xi_1,\ldots,\xi_U}$$

is a collection of atmost m^U nonintersecting regions in I^N. So for all u,

$$S_U \leq \bigvee^{I^N} f_R.$$

Therefore, for any U,

$$\mid f_R(x) - f_R(y) \mid \leq \eta^U \bigvee^{I^N} f_R \tag{9}$$

Since $\eta < 1$ and by Theorem 3.1(c), f_R is of bounded variation, (9) implies that $f_R(x) = f_R(y)$, i.e., f_R is constant on D_j's.

(b) For any $f \in \mathcal{S}$ we have by (4) and Lemma 3.7 that

$$P_R = \sum_{k=1}^{l} p_k P_{\tau_k} = \sum_{k=1}^{l} p_k \mathcal{M}_k := \mathcal{M}_R.$$

Therefore $\mathcal{M}_R f = P_R f$, for every $f \in \mathcal{S}$. By Theorem 3.1(b), P_R has a fixed point f_R, which by (a) belongs to $\mathcal{S}$. So f_R is a right-eigenvector of $\mathcal{M}_R$.

(c) It is shown in course of the proof of Theorem 3.1 of [64], that $\mathcal{M}_1, \ldots, \mathcal{M}_l$ are similar to stochastic matrices via simple diagonal matrices. Therefore, $r(\mathcal{M}_1), \ldots, r(\mathcal{M}_l)$ are all equal to 1, where r denotes the spectral radius. It is then easy to show that also $r(\mathcal{M}_R) = 1$. Hence, $\mathcal{M}_R$ has 1 as the eigenvalue of maximum modulus.

(d) Since by (c) $\mathcal{M}_R$ has 1 as the eigenvalue of maximum modulus and is irreducible, the Perron-Frobenius Theorem implies that the algebraic and geometric multiplicities of the eigenvalue 1 are also 1 and f_R is unique (up to constant multiples). □

Remark 3.5 Assumptions (A) and (B) of Theorem 3.4 are automatically satisfied if each τ_k, $k = 1, \ldots, l$ is *piecewise linear.* (The converse, however, is not true in general.) Piecewise linear Markov transformations have found applications in approximation of various statistical measures of chaos and the construction of switched capacitor circuits [47].

From Theorem 3.4 and Remark 3.5, we obtain the following corollary, which is a higher dimensional generalization of Theorem 1 of [12].

Corollary 3.2 *Let $\tau : I^N \to I^N$ be a piecewise linear and expanding Jabłoński transformation with respect to partition $\mathcal{P}$ satisfying condition (8). Then every invariant density of τ is piecewise constant with respect to $\mathcal{P}$.*

3.1.3 Examples

In this section, we present examples of random maps composed of Jabłoński transformations on the unit square I^2 which illustrate our uniqueness results.

Example 3.1 Consider a random map $R(x) = \{\tau_k(x), p_k, k = 1, 2\}$, where $\tau_1(x_1, x_2) = (\frac{1}{3}x_1, \frac{1}{3}x_2)$ and $\tau_2 : I^2 \to I^2$ is a piecewise linear and expanding Jabłoński transformation on the partition $\mathcal{P} = \{D_1, \ldots, D_9\}$, shown in Fig. 1, and defined by

$$\tau_2(x_1, x_2) = (\varphi_{2,1j}(x_1), \varphi_{2,2j}(x_2)),$$

where

$$\varphi_{2,ij}(x) = \begin{cases} 3x & \text{if } 0 \le x < \frac{1}{3} \\ 3x - 1 & \text{if } \frac{1}{3} \le x < \frac{2}{3} \\ 3x - 2 & \text{if } \frac{2}{3} \le x \le 1 \end{cases},$$

$i = 1, 2$ and $j = 1, \ldots, 9$. $\varphi_{2,ij}$ is shown in Fig. 2.

Let $p_1 = \frac{1}{5}$ and $p_2 = \frac{4}{5}$. R has the essential ingredients for transition to chaos for random maps [57]. Since τ_1 does not have an ACIM, Corollary 2 of [36] does not yield uniqueness. Now, condition (2) is satisfied and therefore R has an ACIM μ_R. By Proposition 3.3, μ_R is unique.

Example 3.2 Let $X = [-2, 2] \times [-3, 3]$ and let $\mathcal{P} = \{D_1, D_2, D_3, D_4\}$, where D_i is the intersection of X with the i^{th} quadrant of the plane (see Fig. 3). Define $\tau_1 : X \to X$ by,

$$\tau_1^1 \equiv \tau_1 |_{D_1} (x_1, x_2) = (-\frac{4}{3}x_2 + 2, \frac{3}{2}x_1), \quad \tau_1^2 \equiv \tau_1 |_{D_2} (x_1, x_2) = (-\tfrac{4}{3}x_2 + 2, -\tfrac{3}{2}x_1),$$
$$\tau_1^3 \equiv \tau_1 |_{D_3} (x_1, x_2) = (\frac{4}{3}x_2 + 2, \frac{3}{2}x_1), \quad \tau_1^4 \equiv \tau_1 |_{D_4} (x_1, x_2) = (\tfrac{4}{3}x_2 + 2, -\tfrac{3}{2}x_1).$$

Then τ_1^1 maps D_1 onto $D_1 \cup D_2$, τ_1^2 maps D_2 onto $D_1 \cup D_2$, τ_1^3 maps D_3 onto $D_3 \cup D_4$, and τ_1^4 maps D_4 onto $D_3 \cup D_4$. The map τ_1 is based on Example 1 of [73]. Similarly, we can define $\tau_2 : X \to X$ by,

$$\tau_2^1 \equiv \tau_2 |_{D_1} (x_1, x_2) = (\frac{4}{3}x_2 - 2, \frac{3}{2}x_1), \quad \tau_2^2 \equiv \tau_2 |_{D_2} (x_1, x_2) = (\tfrac{4}{3}x_2 - 2, -\tfrac{3}{2}x_1),$$
$$\tau_2^3 \equiv \tau_2 |_{D_3} (x_1, x_2) = (-\frac{4}{3}x_2 - 2, \frac{3}{2}x_1), \quad \tau_2^4 \equiv \tau_1 |_{D_4} (x_1, x_2) = (-\tfrac{4}{3}x_2 - 2, -\tfrac{3}{2}x_1).$$

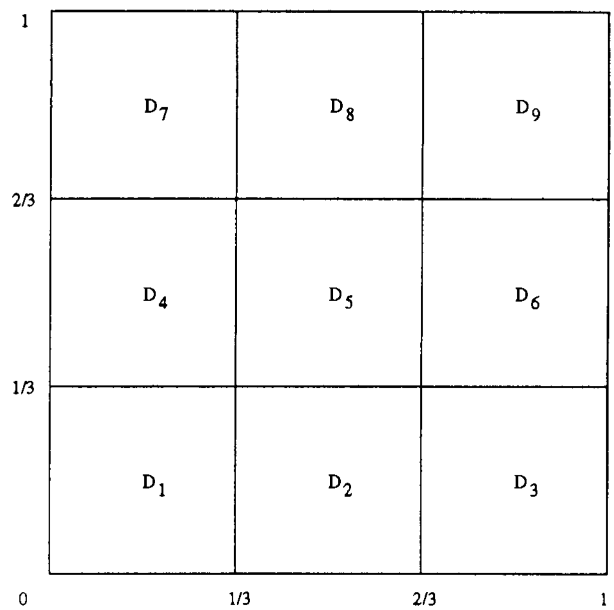

Figure 1: The partition $\mathcal{P} = \{D_1, \ldots, D_9\}$

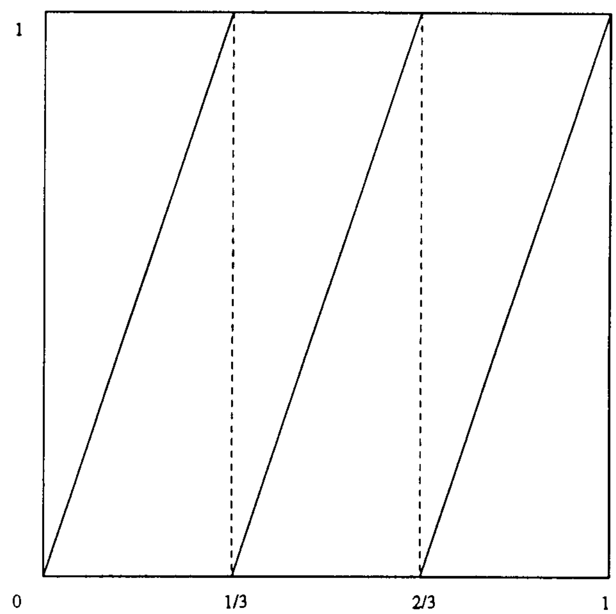

Figure 2: The map $\varphi_{2,ij}$

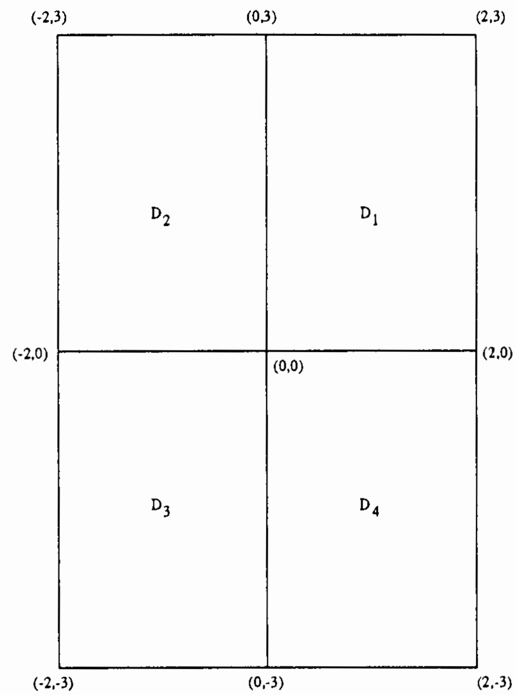

Figure 3: The partition $\mathcal{P} = \{D_1, D_2, D_3, D_4\}$

Then, both τ_1 and τ_2 are piecewise linear and expanding Jabłoński transformations (on I^2, by an appropriate scaling) and Markov on the common partition $\mathcal{P}$. It is obvious that both τ_1 and τ_2 have two ergodic ACIMs. Consider a random map $R(x) = \{\tau_1(x), \tau_2(x), p_1, p_2\}$, where $p_1 = \frac{1}{3}$ and $p_2 = \frac{2}{3}$. Then by Theorem 3.1, R has an ACIM, say μ_R. Since both τ_1 and τ_2 have more than one ACIM, uniqueness of μ_R does not follow either from Theorem 3.3 or from Corollary 2 of [36]. But since the Markov matrix induced by R is irreducible, the hypothesis of Theorem 3.4 is satisfied and μ_R is unique.

4 Asymptotic Periodicity of $\{P_R^n f\}$

Let X be a compact metric space and let $\tau : X \to X$ be a non-singular transformation. An interesting property that the sequence $\{P_\tau^n f\}$ can display is that of asymptotic periodicity, which has been investigated and applied extensively before in various contexts such as: deterministic and noise-induced piecewise monotonic maps [58], time-summating binary neural networks [74] and coupled map lattices (CMLs) [63]. In this section, we consider this property in our context of random maps.

A linear operator $P : L^1 \to L^1$ is called *Markov* if $P(\mathcal{D}) \subset \mathcal{D}$. The Frobenius-Perron operator is an important example of a Markov operator. An operator $P : L^1 \to L^1$ is *weakly (strongly) constrictive*, if there exists a weakly (strongly) relatively compact set $A \subset L^1$ such that,

$$\lim_{n\to\infty} \inf_{g\in A} \|P^n f - g\|_1 = 0,$$

for each $f \in \mathcal{D}$. The set A is referred to as a *constrictor*. The above condition can be weakened even further. An operator $P : L^1 \to L^1$ is *quasi-constrictive*, if there exists a weakly relatively compact set $A \subset L^1$ and $\theta < 1$ such that,

$$\lim_{n\to\infty} \sup_{g\in A} \|P^n f - g\|_1 \leq \theta, \quad f \in \mathcal{D}.$$

The importance of quasi-constrictiveness comes from the realization that it can be considered as a generalization of the Doeblin condition [29] in the theory of Markov processes. From [55], we have:

Lemma 4.1 *In the class of Markov operators, the notions of weakly, strongly and quasi-constrictive operators are equivalent.*

The significance of Lemma 4.1 comes from the fact that for many operators appearing in applied problems, weak constrictiveness is relatively easy to verify. In the sequel, we delete the adjectives strong, weak and quasi for constrictive Markov operators.

An important property of constrictive Markov operators is reflected in the following result [58]:

Theorem 4.1 (Spectral Decomposition Theorem) *Let $P : L^1 \to L^1$ be a constrictive Markov operator. Then there is an integer r, two sequences of nonnegative functions $g_i \in \mathcal{D}$ and $k_i \in L_\infty$, $i = 1, \ldots, r$ and an operator $Q : L^1 \to L^1$ such that for all $f \in L^1$,*

$$Pf = \sum_{i=1}^{r} \lambda_i(f) g_i + Qf \tag{10}$$

where

$$\lambda_i(f) = \int_{I^N} f k_i dm. \tag{11}$$

The functions g_i and the operator Q have the following properties:

(a) $g_i(x)g_j(x) = 0$, for all $i \neq j$, $i, j = 1, \ldots, r$, so that the functions g_i have disjoint supports;

(b) for each integer i, there exists a unique integer $\alpha(i)$ such that $Pg_i = g_{\alpha(i)}$, where $\alpha(i)$ is a permutation on the numbers $\{1, \ldots, r\}$. Furthermore, $\alpha(i) \neq \alpha(j)$ for $i \neq j$, and thus the operator P just serves to permute the functions g_i;

(c) $\|P^n Qf\|_1 \to 0$, as $n \to \infty$, for each $f \in L^1$.

From equation (10) it immediately follows that,

$$P^n f = \sum_{i=1}^{r} \lambda_i(f) g_{\alpha^n(i)} + Q_n f, \tag{12}$$

where $Q_n = P^{n-1}Q$, and $\alpha^n(i) = \alpha(\alpha^{n-1}(i))\ldots$. Furthermore, $\|Q_n f\|_1 \to 0$, as $n \to \infty$, so the terms in the summation in equation (12) are just permuted with each application of P. Since $r < \infty$ and $\|Q_n f\|_1 \to 0$, as $n \to \infty$, the sequence

$$P^n f \approx \sum_{i=1}^{r} \lambda_i(f) g_{\alpha^n(i)}, \tag{13}$$

must be periodic with a period $\leq r!$. Since $\{\alpha^n(1), \ldots, \alpha^n(r)\}$ is just a permutation of $\{1, \ldots, r\}$, there is a unique i corresponding to each $\alpha^n(i)$. Thus the summation in (12) can be written as

$$\sum_{i=1}^{r} \lambda_{\alpha^{-n}(i)}(f) g_i, \tag{14}$$

where $\{\alpha^{-n}(i)\}$ denotes the inverse permutation of $\{\alpha^n(i)\}$.

A useful consequence [58, Proposition 5.4.1] of Theorem 4.1 is:

Proposition 4.1 *A constrictive Markov operator $P : L^1 \to L^1$ has a stationary density f^* given by*

$$f^* = \frac{1}{r} \sum_{i=1}^{r} g_i.$$

Theorem 4.1 leads to the following definition [58]:

Definition 4.1 (Asymptotic Periodicity) If a Markov operator $P : L^1 \to L^1$ has the representation given by (9), we say that the sequence of densities $\{P^n f\}$ is *asymptotically periodic.*

Thus, Lemma 4.1 and the Spectral Decomposition Theorem 4.1 say that if P is constrictive Markov operator then $\{P^n f\}$ is asymptotically periodic. By (9) it follows that, asymptotically, $P^n f$ is either equal to one of the pure states g_i or to a mixture of these states, each having a weight λ_i, which implies that the system is *quantized* from a statistical point of view [65]. Physically, we can associate the densities in (9) with a set of *metastable* states, each transformed into another of the sequence under the action of P. Since $P^n f$ may not necessarily evolve to a stationary density, even if there is one, asymptotic periodic systems can be treated as a nonequilibrium systems, periodically alternating among the metastable states with some characteristic period [65].

We can now prove the asymptotic periodicity of $\{P_R^n f\}$.

Theorem 4.2 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 (not necessarily expanding) Jabłoński transformation satisfying condition (2). Then, the sequence of densities $\{P_R^n f\}$ is asymptotically periodic.*

Proof. It is easy to see that P_R is a Markov operator and so by Lemma 4.1, it is sufficient to prove that P_R is constrictive. Using induction on inequality (5) in Proposition 3.1 we obtain

$$\bigvee^{I^N} P_R^M f \leq \alpha^M \bigvee^{I^N} f + K \left[\frac{1 - \alpha^M}{1 - \alpha}\right] \|f\|_1.$$

Therefore, if $f \in \mathcal{D} \cap \mathcal{C}$, we have from Lemma 2.1 that

$$\limsup_{M\to\infty} \bigvee^{I^N} P_R^M f \leq K', \tag{15}$$

where $K' \geq \frac{K}{1-\alpha}$. Denote

$$A = \{g \in \mathcal{D} : \bigvee^{I^N} g \leq K'\}.$$

Now, inequality (15) implies that $P_R^M f \in A$ for M large enough, and so $\{P_R^M f\}$ converges to the set A in the sense that $\liminf_{n\to\infty}{}_{g\in A}\|P_R^n f - g\|_1 = 0$. The set A satisfies the conditions of Lemma 2.3 and so is weakly relatively compact in L^1. Furthermore, by Lemma 2.2, the set $\mathcal{D} \cap \mathcal{C}$ is dense in $\mathcal{D}$. Hence, it follows that P_R is constrictive. Therefore, by the Spectral Decomposition Theorem 4.1, $\{P_R^M f\}$ is asymptotically periodic. □

Remark 4.1 Let $\tau : I^N \to I^N$ be a piecewise C^2 and expanding Jabłoński transformation and let P_τ be its corresponding Frobenius-Perron operator. Constrictiveness of P_τ follows by an argument similar to that used in the proof of Theorem 4.2.

Remark 4.2 Asymptotic periodicity of $\{P_R^n f\}$ also follows as a consequence of Ionescu-Tulcea and Marinescu Theorem 3.2 (though under stronger assumptions than those of Theorem 4.1) and it also implies [43] that stationary processes governed by the system satisfy a central limit theorem and weak invariance principles.

4.1 An Upper Bound on the Period

By the Spectral Decomposition Theorem 4.1, the period of $\{P_R^n\}$ is bounded above by $r!$. We can make this bound more precise by the next result.

Theorem 4.3 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and expanding Jabłoński transformation. Let P_{τ_k} be the Frobenius-Perron operator corresponding to τ_k, having r_k densities in its spectral representation. Then P_R has r densities in its spectral representation, where $r \leq \min(r_1, \ldots, r_l)$.*

Proof. Since each τ_k is expanding, condition (2) is automatically satisfied. By Remark 3.1, the operators P_{τ_k}, $k = 1, \ldots, l$, are constrictive. Therefore, the proof follows immediately from Theorem 5 of [13]. □

Thus, by Theorem 4.3, the upper bound on the period of an asymptotic periodic sequence of densities $\{P_R^n f\}$ of a random map R composed of piecewise C^2 and expanding Jabłoński transformations $\tau_1, \ldots, \tau_l$ is the minimum of the upper bound of the period of asymptotic periodic sequences of densities $\{P_{\tau_1}^n f\}, \ldots, \{P_{\tau_l}^n f\}$, i.e., $\min(r_1, \ldots, r_l)!$.

4.2 Spectral Representation of P_R and Ergodic Decomposition of R

By Theorem 4.2, P_R is constrictive and so by Theorem 4.1 it has a spectral representation. Also, by Proposition 4.1 P_R has a stationary density which is an invariant density of R. Then the question that arises is: *what is the relation between the number of independent invariant densities of R and the number of densities in the spectral representation of P_R?* We draw this connection in our next result.

Theorem 4.4 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 (not necessarily expanding) Jabłoński transformation satisfying condition (2). Then the number n of independent invariant densities in the ergodic decomposition of R is bounded above by the number r of densities in the spectral representation of P_R, i.e., $n \le r$.*

Proof. By Theorem 4.1, we know that P_R is constrictive. So by the Spectral Decomposition Theorem 4.1, it can be expressed as

$$P_R(f) = \sum_{i=1}^{r} \lambda_i(f) g_i + Q(f).$$

Let T be the period of $\sum_{i=1}^{r} \lambda_i(f) g_i$. By the Theorem 3.1(b), P_R has a stationary density, say f_R. Therefore, it follows by (3) that, for any k,

$$P_R^{kT}(f_R) = f_R = \sum_{i=1}^{r} \lambda_i(f_R) g_{\alpha^{kT}(i)} + Q_{kT}(f_R).$$

Now, since $\alpha^{kT}(i)$ is a permutation of i, f_R and $\sum_{i=1}^{r} \lambda_i(f) g_i$ are independent of k and $\|Q_{kT}(f)\|_1 \to 0$, as $k \to \infty$, we have

$$f_R = \sum_{i=1}^{r} \lambda_i(f_R) g_i.$$

This implies that $\text{supp} f_R \subset \bigcup_{i=1}^{r} \text{supp } g_i$. Hence, the number of (independent) stationary densities of P_R must be less than or equal to r, i.e., $n \le r$. □

Remark 4.3 By Proposition 4.1, a constrictive Markov operator *always* has a stationary density. Using this fact, it is easy to see that the conclusion of Theorem 4.4 holds, in general, for *any* constrictive Markov operator P, i.e., the number of stationary densities of P is at most equal to the number of densities in its spectral representation.

We now make the above relation between the number of invariant densities of R and the number of densities in the spectral representation of P_R more precise by giving an explicit upper bound on the number of independent invariant densities of the component transformations in our next result.

Theorem 4.5 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and expanding Jabłoński transformation on the partition $\mathcal{P}_k = \{D_{k_1}, \ldots, D_{k_m}\}$ related to which the number $M_N^{(k)}$ is defined. Let $\mathcal{V}_k$ be the set of vertices of elements of $\mathcal{P}_k$ which lie in the interior of I^N. Let P_{τ_k} be the Frobenius-Perron operator corresponding to τ_k, having r_k densities in its spectral representation such that the permutation $\{\alpha_k(1), \ldots, \alpha_k(r_k)\}$ of $\{1, \ldots, r_k\}$ is cyclical[2]. If $\min_{1\le k\le l} \frac{\Lambda_k}{M_N^{(k)}} > 1$, then*

[2]By Theorem 5.5.1 of [58] this is equivalent to each of the P_{τ_k}'s being ergodic

$$n \le r \le \min\{r_1, \dots, r_l\} = \min\{\#\mathcal{V}_1, \dots, \#\mathcal{V}_l\}.$$

Proof. By Theorem 4.4 $n \le r$ and by Theorem 4.3 $r \le \min\{r_1, \dots, r_l\}$. We now prove that $r_k = \#\mathcal{V}_k$, for each $k = 1, \dots, l$.

Fix one of the maps τ_k, $k = 1, \dots, l$, say τ_1. Since α_1 is cyclical, $P_{\tau_1}^{r_1} g_i = g_i$, where $\{g_1, \dots, g_{r_1}\}$ are the densities in the spectral representation of P_{τ_1}. Since $\frac{\Lambda_1}{M_N^{(1)}} > 1$, by Lemma 3.6, there can be at most $\#\mathcal{V}_1$ such densities. Hence, $r_1 \le \#\mathcal{V}_1$. But by Remark 4.3, $r_1 \ge \#\mathcal{V}_1$. Thus, $r_1 = \#\mathcal{V}_1$. Repeating the argument for $\tau_2, \dots, \tau_l$, we obtain $r_k = \#\mathcal{V}_k$, for each k. Therefore, we have

$$n \le r \le \min\{r_1, \dots, r_l\} = \min(\#\mathcal{V}_1, \dots, \#\mathcal{V}_l).$$

□

Remark 4.4 For $N = 1$, i.e., in one dimension, $M_N^{(k)} = 1$ and $\#\mathcal{V}_k$ is just the number of discontinuity points of the transformation τ_k. Using this fact, we obtain a result similar to (but stronger than) Theorem 6 of [13], given for random maps composed of one-dimensional piecewise expanding transformations, even though our Theorem 4.5 is not an obvious generalization. The bounded variation techniques used in one dimension do not carry over easily to higher dimensions [41] due to the much more complex setting and obtaining an upper bound for the number of ergodic ACIMs is still a nontrivial task.

We also have an analog of Theorem 4.5 for piecewise linear Markov transformations, similar to Theorem 7 of [13], where we *do not* require that the permutation $\{\alpha_k(1), \dots, \alpha_k(r_k)\}$ of $\{1, \dots, r_k\}$ be cyclical.

Theorem 4.6 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \dots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise linear and expanding Jabłoński transformation, with Markov property with respect to a common partition $\mathcal{P}$. Let P_{τ_k} and P_R be the Frobenius-Perron operators corresponding to τ_k and R having r_k and r densities in their spectral representations, respectively. If n_k and n are the number of independent invariant densities in the ergodic decomposition of τ_k and R respectively, then we have*

$$n \le r \le \min\{r_1, \dots, r_l\} = \min\{n_1, \dots, n_l).$$

Proof. Let $\mathcal{E}_k$, $k = 1, \dots, l$, and $\mathcal{E}$ denote the eigenspaces of $\mathcal{M}_k$ and $\mathcal{M}_R$ respectively, associated with eigenvalue 1. If $\mathcal{I}_1$ is an invertible matrix such that $\mathcal{M}_1 = \mathcal{I}_1^{-1}\mathcal{M}_1\mathcal{I}_1$, then

$$R^{-1}\mathcal{M}_R\mathcal{I}_1 = p_1\mathcal{M}_1 + \sum_{k=2}^{l} p_k \mathcal{I}_1^{-1}\mathcal{M}_k\mathcal{I}_1.$$

Since $\mathcal{M}_1$ is similar to a stochastic matrix, it can be put into the Perron-Frobenius normal form [34, Chapter XIII, Theorem 10]:

$$\mathcal{M}_1 = \begin{bmatrix} \mathcal{M}_1^1 & & 0 & & \cdots & 0 & 0 & 0 \\ 0 & \mathcal{M}_1^2 & & & & & & \vdots \\ \vdots & & \ddots & & & 0 & 0 & 0 \\ 0 & & & \mathcal{M}_1^q & & & & \vdots \\ \mathcal{M}_1^{q+1,1} & \cdots & \mathcal{M}_1^{q+1,q} & & \mathcal{M}_1^{q+1} & 0 & 0 & 0 \\ \vdots & & & & & & \ddots & \vdots \\ \mathcal{M}_1^{s,1} & \cdots & \mathcal{M}_1^{s,q} & & \cdots & & & \mathcal{M}_1^s \end{bmatrix},$$

where $\mathcal{M}_1^1, \dots, \mathcal{M}_1^q$ are irreducible stochastic matrices. Then for each $\mathcal{M}_1^i$, $i = 1, \dots, q$, 1 is an eigenvalue of algebraic and geometric multiplicity 1. The remaining matrices $\mathcal{M}_1^{q+1}, \dots, \mathcal{M}_1^s$ on the diagonal have maximal eigenvalues less than 1. Hence, $\dim\mathcal{E}_1 = q$. Now, $p_1\mathcal{M}_1 + \sum_{k=2}^{l} p_k \mathcal{I}_1^{-1}\mathcal{M}_k\mathcal{I}_1$ has q or fewer blocks along the diagonal. Therefore, $\dim\mathcal{M}_R \leq q \leq \dim\mathcal{M}_1$. Repeating the same argument for $\mathcal{M}_2, \dots, \mathcal{M}_l$, we obtain:

$$\dim\mathcal{M}_R \leq \min_{1\leq k\leq l} \dim\mathcal{M}_k.$$

Therefore, by Theorems 3.4(b), 4.2-4.4 and the foregoing we have the desired result. □

5 Asymptotic Stability of $\{P_R^n f\}$

In this section, we consider another asymptotic property of that the sequence $\{P_R^n f\}$ can exhibit: strong convergence to a unique stationary density.

By Proposition 4.1, a constrictive Markov operator always has a stationary density. Even though the system is asymptotically periodic, it may not necessarily evolve to a stationary density. But if $r = 1$ in the representation of $\{P^n f\}$ given in equation (12), then the summation is reduced to a single term and for every $f \in \mathcal{D}$, the sequence $\{P^n f\}$ converges strongly to a unique stationary density, independent of f. We can make this more precise by the following [58]:

Definition 5.1 (Asymptotic Stability) Let $P : L^1 \to L^1$ be a Markov operator. Then the sequence of densities $\{P^n f\}$ is said to be *asymptotically stable* if
(i) there exists a $f^* \in \mathcal{D}$ such that $Pf^* = f^*$ and
(ii) $\lim_{n\to\infty} \|P^n f - f^*\|_1 = 0$, for every $f \in \mathcal{D}$.

Clearly an f^* satisfying Definition 5.1(ii) is unique. In general, condition (ii) does not imply that f^* is stationary under the action of P.

The property of asymptotic stability has been studied in various cases: deterministic and noise-induced piecewise monotonic maps [58], CMLs [66] and random maps composed of affine transformations such as iterated function systems (IFS) [60].

Remark 5.1 We say that a Markov operator $P : L^1 \to L^1$ is *exact* if it has a stationary density $f^* \in \mathcal{D}$ such that $\lim_{n\to\infty} \|P^n f - f^*\|_1 = 0$, for every $f \in \mathcal{D}$. Therefore, exactness of P is equivalent to asymptotic stability of $\{P^n f\}$.

We can now prove the asymptotically stability of the sequence of densites $\{P_R^n f\}$ in our case of a random map R.

Theorem 5.1 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \dots, l\}$ be a random map having a unique ACIM μ_R with density f_R, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and expanding Jabłoński transformation. For each $k = 1, \dots, l$, let the Frobenius-Perron operator P_{τ_k} corresponding to τ_k, have r_k densities in its spectral representation. Then, if any one of the maps τ_k, $k = 1, \dots, l$, is exact, the sequence of densities $\{P_R^n f\}$ is asymptotically stable.*

Proof. By Remark 5.1, it is sufficient to prove that P_R is exact and $\{P_R^n f\}$ converges strongly to f_R. Let, without loss of generality, τ_1 be exact. Since P_{τ_1} is constrictive, by Theorem 5.5.3 of [58][3], this implies that $r_1 = 1$. Therefore, by Theorem 4.3

$$r = \min(r_1, \dots, r_l) = 1. \tag{16}$$

Furthermore, since each τ_k is expanding, condition (2) is satisfied and so by Theorem 4.2, P_R is constrictive. This fact together with (16), implies by Theorem 5.5.2 of [58] that P_R is exact, i.e., there exists an f^*, $P_R f^* = f^*$ such that

$$\lim_{n \to \infty} \|P_R^n f - f^*\|_1 = 0,$$

uniformly for every $f \in \mathcal{D}$. Since μ_R is unique, $f^* = f_R$. Therefore, $\{P_R^n f\}$ is asymptotically stable. □

5.1 Asymptotic Stability under Special Cases

In this section, we consider some specific cases where due to existence of a unique ACIM of R and exactness of any one of the component transformations, we obtain asymptotic stability.

Theorem 5.2 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \dots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and expanding Jabłoński transformation. If any one of the maps τ_k, $k = 1, \dots, l$, has a unique ACIM with respect to which it is exact, then the sequence of densities $\{P_R^n f\}$ is asymptotically stable.*

Proof. By Corollary 3.1, R has a unique ACIM and so the result follows immediately from Theorem 5.1. □

Remark 5.2 By Theorem 4.4.1 of [58], it follows that R is exact if and only if P_R is exact. Then by Theorem 5.2, we conclude that if for any one of the maps τ_k, $k = 1, \dots, l$, $\{P_{\tau_k}^n f\}$ is asymptotically stable then $\{P_R^n f\}$ also is asymptotically stable. The converse, however, is not true in general, i.e., $\{P_R^n f\}$ can be asymptotically stable even if none of the $\{P_{\tau_k}^n f\}$'s are since R can have a unique ACIM even if none of the τ_k's have a unique ACIM (see Remark 3.4).

[3]Even though Theorems 5.5.2 and 5.5.3 of [58] are given for the case when a Markov operator has a *uniform* stationary density, they can be applied to the case of a nonuniform stationary density by a suitable normalization

For random maps composed of piecewise linear Markov transformations, asymptotic stability of the sequence of densities $\{P_R^n f\}$ follows from the following result.

Proposition 5.1 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise linear and expanding Jabłoński transformation with Markov property with respect to a common partition $\mathcal{P}$. Suppose for any one of the maps τ_k, $k = 1, \ldots, l$, there exists a $t > 0$ such that for every pair of elements $D_1, D_2 \in \mathcal{P}$, we have*

$$m(\tau_k^t(D_1) \cap D_2) \neq 0. \tag{17}$$

Then the sequence of densities $\{P_R^n f\}$ is asymptotically stable.

Proof. Let, without loss of generality, τ_1 satisfy the property (17). Then from the results of Theorem 1.3 of [78, Chapter III] and [64, §3.4], it follows that τ_1 has a unique ACIM with respect to which it is exact. Therefore, by Theorem 5.2, $\{P_R^n f\}$ is asymptotically stable. □

The following lemma is contained in Theorem 2 of [81] which has been adapted here to Jabłoński transformations.

Lemma 5.1 *Let $\tau : I^N \to I^N$ be a Jabłoński transformation on a partition $\mathcal{P} = \{D_1, \ldots, D_M\}$ defined by*

$$\tau(x) = \tau_i(x) \equiv \tau\mid_{D_i}: D_i \to I^N \text{ for } x \in D_i, i = 1, \ldots, M,$$

where τ_i satisfies the following conditions:

(a) *τ_i is a C^2-diffeomorphism onto its image.*

(b) *$\frac{\partial T_{i,j}}{\partial x_l} \geq 0$, $\det DT_i > 0$ for $j, l = 1, \ldots, N$, where $T_i = (\tau_i)^{-1}$ and $T_i = (T_{i,1}, \ldots, T_{i,N})$.*

(c) *there exists a $\lambda > 1$ such that*

$$\inf\{\mid DT_i(x) \cdot y \mid : \mid y \mid = 1\} \geq \lambda,$$

where $D\tau_i(x)$ is the Jacobian matrix of τ_i.

(d) *if $x \in D_i$ and $x_j \neq a_{i,j}$ for a given $1 \leq j \leq N$, then $\tau_{i,j}(x) \neq 0$, where the $a_{i,j}$ are given in Definition 2.1.*

(e) *if $x \in \tau_i(D_i)$ and $y_j \leq x_j$, for $j = 1, \ldots, N$, then $y \in \tau_i(D_i)$.*

Then τ has a unique ACIM with respect to which it is exact.

Then by Theorem 5.2 and Lemma 5.1, we immediately obtain the next result.

Proposition 5.2 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and expanding Jabłoński transformation. If any one of the maps τ_k satisfies the hypotheses of Lemma 5.1, then the sequence of densities $\{P_R^n f\}$ is asymptotically stable.*

In the next result, we *do not* assume that any of the map τ_k's have a unique ACIM but can still obtain asymptotic stability.

Proposition 5.3 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise linear and expanding Jabłoński transformation with Markov property with respect to a common partition $\mathcal{P}$. If any one of the maps τ_k, $k = 1, \ldots, l$, is exact and $\mathcal{M}_R$ is irreducible, then the sequence of densities $\{P_R^n f\}$ is asymptotically stable.*

Proof. By Theorem 3.4, f_R is unique (up to constant multiples). Therefore by Theorem 5.1, $\{P_R^n f\}$ is asymptotically stable. □

5.2 Asymptotic Stability and Physical Measures

Asymptotic stability implies a global stability of the system R in a statistical sense: the iterates $\{P_R^n f\}$, $n = 0, 1, \ldots$, converge strongly to the unique density f_R. Therefore, it is of significance to know whether f_R has any physical relevance. In this section, we show that f_R is indeed the density of a *physical* measure.

In many physical systems it appears that the observed orbits have well-defined time-averages whose histograms are approximately those defined by ergodic invariant measures of the system. Often, a dynamical system can have many ergodic invariant measures, though not all of these are physically relevant, i.e., they are not observed during physical experiments. For example, if x is an unstable fixed point of a dynamical system, then the Dirac measure at a point is an invariant measure but it not observed. In general, criteria for selecting a physical measure are: (a) *that it describes physical time-averages* and (b) *it is stable under small perturbations.* A physical measure from this point of view is the SRB-measure [79]. For a random map composed of contracting (or even contracting-on-average) transformations on a locally compact metric space in [30] and for random diffeomorphisms of a compact manifold in [61], the existence of an SRB-measure was established.

By Theorem 3.1, for a random map R we have the existence of an ACIM μ_R. Since an ACIM is an invariant measure which often appears in numerical and computer experiments, the obvious question is: *when is μ_R an SRB-measure*? The existence of an SRB-measure for a random map is also of interest from the point of view that it is this measure which appears when the dynamics of the two-slit experiment of quantum physics is modelled by a random map [11].

For a transformation $\tau : I^N \to I^N$, the SRB-measure is defined as the vague limit of $\{\tau_*^n(m)\}$, if it exists. Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$. Analogously, we can define the SRB-measure corresponding to the random map R as the vague limit of $\{R_*^n(m)\}$, if it exists, where

$$R_*^n(m) = \sum_{k=1}^{l} p_k(\tau_k)_*(m).$$

Since each τ_k is measurable, $R_*(m)$ is well-defined. Now, if each τ_k is a piecewise C^2 and expanding Jabłoński transformation, then by Theorem 3.1, R has an ACIM μ_R and $dR_*^n(m) = (P_R^n 1)dm$. Since $\{P_R^n 1\}$ is a weakly compact set in L^1, $\{P_R^n 1\}$ converges strongly to the density of μ_R if and only if R is exact [58, Theorem 4.4.1]. Therefore, it follows that if R has a unique ACIM μ_R, then μ_R is the SRB-measure if and only if R is exact with respect to μ_R. By Corollary 3.1, μ_R is unique, if any one of the maps τ_k, $k = 1, \ldots, l$, say τ_1 has a unique ACIM and by Theorem 5.2 R is exact if τ_1 is exact. We therefore arrive at the following result:

Theorem 5.3 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map having a unique ACIM μ_R, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and expanding Jabłoński transformation. If any one of the maps τ_k, $k = 1, \ldots, l$, has a unique ACIM with respect to which it is exact, then μ_R is an SRB-measure.*

Thus, Theorems 4.2 and 4.3 together imply a *physical* asymptotic stability, i.e., strong convergence of $\{P_R^n f\}$ to a physical density.

We now present an example which shows the importance of condition (2) in Theorems 4.2 and 5.1.

Example 5.1 Consider a random map $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, 4\}$, where $\tau_k : I^2 \to I^2$ are defined as follows: $\tau_1(x_1, x_2) = (\frac{1}{3}x_1, \frac{1}{3}x_2)$, $\tau_2(x_1, x_2) = (\frac{1}{3}x_1 + \frac{2}{3}, \frac{1}{3}x_2)$, $\tau_3(x_1, x_2) = (\frac{1}{3}x_1, \frac{1}{3}x_2 + \frac{2}{3})$, $\tau_4(x_1, x_2) = (\frac{1}{3}x_1 + \frac{2}{3}, \frac{1}{3}x_2 + \frac{2}{3})$, and $p_k = \frac{1}{4}$ for $k = 1, \ldots, 4$. Then $R(x)$ comprises an IFS [4] whose unique attractor A is a two-dimensional Cantor set of Lebesgue measure 0. We claim that the sequence of densities $\{P_R^n f\}$ is neither asymptotically periodic nor asymptotically stable. Suppose $\{P_R^n f\}$ is asymptotically periodic. Then, by definition, P_R is constrictive. Therefore, by Proposition 4.1, P_R has a stationary density, say μ_R. Now, the unique attractor A of R, is the support of the unique invariant measure of R [4, §9.6, Theorem 2]. This implies that support of μ_R would be of 0 Lebesgue measure, which is a contradiction. From this, it is obvious that $\{P_R^n f\}$ is also not asymptotically stable.

6 Stability of Asymptotic Properties of Densities

Physical systems are usually affected by a number of small external fluctuations (e.g. due to external noise or due to roundoff/truncation errors in computation). We are thus concerned with the question of stability of quantitators/properties characterizing the dynamics of the system in presence of these fluctuations. The properties which recover from the fluctuations can be considered as stable under perturbations and thus have physical sense. The quantitators/properties we are interested here are Frobenius-Perron operator P_R, its stationary density f_R, asymptotic periodicity and asymptotic stability. In spite of the fact that both the properties of asymptotic periodicity and asymptotic stability have been extensively studied both numerically and analytically, the issue of their stability or persistence under perturbations has not been emphasized. This is significant since in various cases [63], [76] these properties have been numerically illustrated and conclusions have been drawn on their basis.

In this section, we examine the above question in the context of random maps from two different perspectives: (a) *stability under perturbation of the initial density*, and (b) *stability under perturbation of the map R.*

6.1 Perturbation of the Initial Density

Let a perturbation be such that a density f_0 be transformed to a density f_0' (and does not perturb the map R itself). Then, due to equations (3) and (4), f_R and P_R respectively are independent of the initial density, and hence are stable. Let a sequence of densities $\{P_R^n f_0\}$ have the properties of asymptotic periodicity and asymptotic stability. We are then led to the question whether $\{P_R^n f_0'\}$ has the properties of asymptotic periodicity and asymptotic stability. Now, the property of asymptotic stability is stable, since by definition it is independent of the initial density. This however is not necessarily the case with asymptotic periodicity, which depends on the initial density, as the scaling coefficient λ_i's in (12) depend on f. For piecewise monotonic maps of an interval, this has also been indicated in the illustrations of [63]. Therefore, it is of interest to obtain conditions under which $\{P_R^n f_0'\}$ has asymptotic periodicity; we leave this problem for future research.

6.1.1 Shadowing Property for P_R

To deal with the stability of the orbit $\{P_R^n f\}$ itself, we employ the (pseudo-orbit) shadowing property, which has been widely used in testing the reliability of computer-generated orbits of chaotic systems in presence of unavoidable roundoff/truncation errors and the system's sensitive dependence to initial conditions [48]. The shadowing property for random maps composed of hyperbolic transformations has been analysed in [85], [52], [6]. We shall show that, regardless of whether a random map $R : I^N \to I^N$ has the shadowing property, its corresponding Frobenius-Perron operator $P_R : \mathcal{D}_c \to \mathcal{D}_c$ can still have the shadowing property, where $\mathcal{D}_c$ is any compact subset of $\mathcal{D}$.

Let $\{\phi_n\}$ be a countable dense subset of $C(I^N)$, the space of real-valued continuous functions on I^N in the sup-norm topology. Let $\beta_n = \sup_{x\in I^N} \mid \phi_n \mid > 0$ and let $\{\alpha_n\}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n\beta_n \leq 1$. Define the norm $\|.\|_W$ on $L^1(I^N)$ by

$$\|f\|_W = \sum_{n=1}^{\infty} \alpha_n \left| \int_{I^N} \phi_n f(x) dm \right|.$$

Then if $\mathcal{D}_c \subset \mathcal{D}$ is a weakly compact set, $\|.\|_W$ defines the weak topology of L^1 restricted to $\mathcal{D}_c$.

Let $T : L^1 \to L^1$ be a linear operator and a $\delta > 0$ be given. A *δ-pseudo-orbit* for $T : L^1 \to L^1$ is a sequence $\{f_n\}_{n=0}^{\infty}$, $f_n \in L^1$ such that $\|Tf_n - f_{n+1}\|_1 \leq \delta$, for each $n = 0, 1, \ldots$. Given an $\epsilon > 0$, a δ-pseudo-orbit for T is *ϵ-shadowed* by a point $f \in L^1$, if $\|T^n f - f_n\|_W \leq \epsilon$, for each $n = 0, 1, \ldots$.

Definition 6.1 (Shadowing Property) We say that a linear operator $T : (L^1, \|.\|_1) \to (L^1, \|.\|_W)$ has the *shadowing property* if for each $\epsilon > 0$ there exists a $\delta > 0$ such that every δ-pseudo-orbit for T can be ϵ-shadowed by some point of L^1.

We can interpret the existence of the shadowing property for T as implying stability of $\{T^n f_0\}$ with respect to small perturbations of the initial density f_0 in the following sense: given a pseudo-orbit of T, there will always exist a true orbit of T (starting from a different initial density) close to it, for arbitrary long times. A shadowing property for constrictive Markov operators was given in [10]; the following lemma is contained in Theorem 1 of [10].

Lemma 6.1 *Let $P : L^1 \to L^1$ be a constrictive Markov operator with the constrictor A consisting of a single element in $\mathcal{D}_c$ and let $P(\mathcal{D}_c) \subset \mathcal{D}_c$. If*

$$\lim_{n\to\infty} \|P^n f - A\|_1 = 0,$$

uniformly for all $f \in \mathcal{D}_c$, then $P : (\mathcal{D}_c, \|.\|_1) \to (\mathcal{D}_c, \|.\|_W)$ has the shadowing property.

Applying Lemma 6.1 to our setting of random maps, we have the following result which can be interpreted as stability of $\{P_R^n f_0\}$ with respect to small perturbations of the initial density f_0.

Theorem 6.1 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \dots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is piecewise C^2 and expanding Jabłoński transformation. If any one of the maps τ_k, $k = 1, \dots, l$, has a unique ACIM with respect to which it is exact, then $P_R : (\mathcal{D}_c, \|.\|_1) \to (\mathcal{D}_c, \|.\|_W)$ has the shadowing property.*

Proof. Let

$$\mathcal{D}_c = \{g \in \mathcal{D} : \bigvee^{I^N} g \le K, K < \infty\}.$$

Then, $\mathcal{D}_c$ is a weakly compact subset of L^1 and by (11), $P_R(\mathcal{D}_c) \subset \mathcal{D}_c$. From Theorem 4.2, P_R is constrictive and by Corollary 3.1, R has a unique ACIM with density say f_R. By Theorem 2.1(c), $f_R \in \mathcal{D}_c$. Therefore, from the proof of Theorem 5.1, it follows that $\lim_{n\to\infty} \|P_R^n f - A\|_1 = 0$, uniformly for all $f \in \mathcal{D}_c$, where $A = \{f_R\}$. Hence, by Lemma 6.1, $P_R : (\mathcal{D}_c, \|.\|_1) \to (\mathcal{D}_c, \|.\|_W)$ has the shadowing property. □

6.2 Perturbation of R

In this section, we deal with the question of stability of different asymptotic properties of densities of R studied in previous sections with respect to the perturbation of the map.

6.2.1 Stability of Asymptotic Periodicity and Asymptotic Stability

In this section, we examine the stability of asymptotic periodicity and asymptotic stability under perturbations of the map R. To do this, we invoke the theory of stability developed in [71] to study the behavior of ACIMs under small stochastic perturbations, which can also be applied to our case.

Let $R(x) = \{\tau_k(x), p_k, k = 1, \dots, l\}$ be a random map, where each each $k = 1, \dots, l$, $\tau_k : I^N \to I^N$ be a piecewise C^2 (not necessarily expanding) Jabłoński transformation. For $n = 1, 2, \dots$, we consider a family of densities $t_n(x, .)$, $x \in I^N$ with respect to m, the Lebesgue measure on I^N. The densities t_n are bounded and measurable functions of $2N$ variables.

Definition 6.2 For each $k = 1, \dots, l$, respectively the family of transition densities

$$s_{k,n}(x, .) = t_n(\tau_k(x), .), \;\; n = 1, 2, \dots,$$

with respect to m, is called a *stochastic perturbation* of the map τ_k. It is *small* if for any $u > 0$, we have

$$\inf_{x\in I^N} \int_{\{y:|x-y|<u\}} t_n(x,y)dm(y) \to 1, \; as \;\; n \to \infty. \tag{18}$$

Define operators

$$(Q_n f)(y) = \int_{I^N} t_n(x,y) f(x) dm(x), \;\; y \in I^N$$

and

$$(P_n f)(y) = \sum_{k=1}^{l} \int_{I^N} p_k s_{k,n}(x,y) f(x) dm(x), \;\; y \in I^N. \tag{19}$$

Since t_n and $\sum_{k=1}^{l} p_k s_{k,n}$ are stochastic kernels, it is easy to see [58, §5.7] that Q_n and P_n are Markov operators on L^1.

Perturbations considered in the sequel are small as they are *local*, i.e., for $n = 1, 2, \ldots$, there exists $u_n > 0$ such that $t_n(x,y) = 0$ for $|x - y| > u_n$, and $u_n \to 0$ as $n \to \infty$.

Specifically, let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where for each $k = 1, \ldots, l$, $\tau_k : I^N \to I^N$ is a piecewise C^2 (not necessarily expanding) Jabłoński transformation, satisfying the condition (2). Let $\Pi_n = \{D_{n,1}, \ldots, D_{n,a(n)}\}$ be a partition of I^N into rectangles such that $\max_{1\le j\le a(n)} m(D_{n,j}) \to 0, n \to \infty$. Define,

$$t_n(x,y) = t(\Pi_n)(x,y) = \begin{cases} [m(D_{n,j})]^{-1} & \text{for } x, y \in D_{n,j} \\ 0 & \text{otherwise} \end{cases}, \tag{20}$$

and set[4]

$$t(\Pi_n) = t_n, \;\; Q(\Pi_n) = Q_n, \;\; P(\Pi_n) = P_n,$$

for $n = 1, 2, \ldots$. Since Q_n is an *operator of conditional expectation*, the perturbations generated by t_n's are known as *average-like* perturbations.

The proofs of the following two lemmas are analogous to that of Lemmas 4 and 7 of [16] respectively.

Lemma 6.2 *$P_n = Q_n \circ P_R$, for each $n = 1, 2, \ldots$.*

Lemma 6.3 *For $f \in L^1$,*

$$\bigvee^{I^N} Q_n f \le \bigvee^{I^N} f,$$

for all $n = 1, 2, \ldots$.

Lemma 6.4 *For $n = 1, 2, \ldots$, P_n is constrictive.*

Proof. Using Lemmas 6.2 and 6.3, we have

$$\bigvee^{I^N} P_n f = \bigvee^{I^N} Q_n \circ P_R f \le \bigvee^{I^N} P_R f.$$

Now, using (5), the result follows as in Theorem 4.2. □

[4]Neither of the operators $P(\Pi_n)$ and $Q(\Pi_n)$ are related to the operators P and Q in the Spectral Decomposition Theorem 3.1; we keep the notation for historical reasons

We then have:

Theorem 6.2 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ be a piecewise C^2 (not necessarily expanding) Jabłoński transformation, satisfying condition (2). Then $\{P_R^n f\}$ is asymptotically periodic. Let R be subjected to perturbations given by Definition 6.2 and (20) giving rise to a sequence of densities governed by the operator P_n defined by (19). Then $\{P_n^i f\}$ is also asymptotically periodic.*

Proof. By Lemma 4.1, it is sufficient to prove that P_n is constrictive, which follows by Lemma 6.4. Therefore, by the Spectral Decomposition Theorem 4.1 $\{P_n^i f\}$ is asymptotically periodic for each $n = 1, 2, \ldots$. □

The proof of Theorem 6.2 also follows from Corollary 5.7.2 of [58]. We now show the asymptotic stability of $\{P_n^i f\}$.

Theorem 6.3 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map having a unique ACIM μ_R with density f_R, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and expanding Jabłoński transformation. If any one of the maps τ_k, $k = 1, \ldots, l$, is exact, then $\{P_R^n f\}$ is asymptotically stable. Let R be subjected to perturbations given by Definition 6.2 giving rise to the sequence of densities governed by the operator P_n defined by (19). Then $\{P_n^i f\}$ is also asymptotically stable.*

Proof. Let $K(x, y) = \sum_{k=1}^{l} p_k s_{k,n}(x, y)$. By definition of $s_{k,n}$ and (19) it follows that, K_i, the kernel corresponding to P_n^i satisfies

$$\inf \int_{x \in I^N} K_i(x, y) dm(y) > 0.$$

Then, by Corollary 5.7.1 of [58], $\{P_n^i f\}$ is asymptotically stable. □

6.2.2 Stability of P_R and f_R

In this section, we examine the stability of P_R and f_R under perturbations given by (20). In this context, we can invoke a theory of stability for quasi-compact operators developed in [50].

By $\mathcal{T}$ we denote the class of operators $P : L^1 \to L^1$ which satisfies conditions (a)-(c) of Ionescu-Tulcea and Marinescu Theorem 3.2. The subclass of $\mathcal{T}$ satisfying condition (b) for a given α and K is denoted by $\mathcal{T}(\alpha, K)$. For operators $P : BV \to L^1$ define the norm

$$|||\,.\,||| := \sup\{\|Pf\|_1 : f \in BV, \|f\|_V \leq 1\}.$$

A sequence $\{P_n\}_{n \in \mathbf{N}}$ of operators is called *$\mathcal{T}$-bounded* if there are $0 < \alpha < 1$ and $K > 0$ such that $P_n \in \mathcal{T}(\alpha, K)$ for all $n \in \mathbf{N}$. Since Φ defined in (6) is continuous, we can introduce the following:

Definition 6.3 $P \in \mathcal{T}$ is *stable* if $|||\, P_n - P \,||| \to 0$, as $n \to \infty$ implies that $\|\Phi(1, P_n) - \Phi(1, P)\|_1 \to 0$, as $n \to \infty$ for each $\mathcal{T}$-bounded sequence $\{P_n\}_{n \in \mathbf{N}}$.

The sequence $\{P_n\}_{n\in\mathbf{N}}$ can be interpreted as having resulted from external perturbations of the operator P. The following lemma is contained in Theorem 2 of [50][5].

Lemma 6.5 *$P \in \mathcal{T}$ is stable if and only if* $\dim\{f : Pf = f\} = 1$.

Lemma 6.6 *Let $\{P_n\}$ be the sequence of operators defined by (17). Then $\{P_n\}_{n\in\mathbf{N}}$ are $\mathcal{T}$-bounded.*

Proof. The proof immediately follows from Lemma 6.4. □

Lemma 6.7 $P_R \in \mathcal{T}$.

Proof. It is sufficient to show that P_R satisfies conditions (a)-(c) of Ionescu-Tulcea and Marinescu Theorem 3.2 from which the result follows.

Condition (a) follows immediately since P_R is a Markov operator. Condition (b) follows from Lemmas 2.1, 2.2 and Proposition 3.1 and the fact that,

$$\begin{aligned}\|P_R f\|_V &= \bigvee^{I^N} P_R f + \|P_R f\|_1 \\ &\leq \alpha \bigvee^{I^N} f + K\|f\|_1 + \|f\|_1 \\ &\leq \alpha\|f\|_V + K'\|f\|_1,\end{aligned}$$

with $K' \geq K + 1$. Finally, condition (c) follows from the definition of P_R, Remark 3.1 and Lemma 2.3. □

Now, since P_n is constrictive, by Proposition 4.1, it has a stationary density, say f_n, for each $n = 1, 2, \ldots$. Proposition 3.1 and Lemmas 6.2 and 6.3 imply

Lemma 6.8 *The sequence $\{\bigvee^{I^N} f_n\}$ is bounded.*

From [14], we have:

Lemma 6.9 *For $f \in L^1$, the sequence $Q_n f \to f$ in L^1 as $n \to \infty$.*

From Lemmas 6.2 and 6.9, we have

Lemma 6.10 *For $f \in L^1$, the sequence $P_n f \to P_R f$ in L^1 as $n \to \infty$.*

The following lemma is a special case of a result due to Kolmogorov [70, Chapter IV].

Lemma 6.11 *Let $\mathcal{F} \subset L^1(\mathbf{R}^N)$ be a norm bounded set of functions and assume the following limits are attained uniformly over $f \in \mathcal{F}$:*

(a)
$$\lim_{\Delta x \to 0} \int_{\mathbf{R}^N} | f(x + \Delta x) - f(x) | \, dx = 0,$$

(b)
$$\lim_{K \to \infty} \int_{\mathbf{R}^N \setminus \{-K, K\}} | f(x) | \, dx = 0,$$

[5] Although Theorem 2 of [50] deals only with functions of bounded variation of one variable, the proof given there carries over without changes to our setting since it uses only the spectral representation of P

where $\Delta x \in \mathbf{R}^N, |\Delta x| < 1$, $\Delta x \neq 0$. *Then* $\mathcal{F}$ *is a (strongly) precompact subset of* $L^1(\mathbf{R}^N)$.

Proof. The proof is based on Theorem 3.5 of [7]. For $f : I^N \to \mathbf{R}$ define $\bar{f} : \mathbf{R}^N \to \mathbf{R}$ by

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in I^N \\ 0 & \text{if } x \in \mathbf{R}^N \setminus I^N \end{cases}$$

Let $\mathcal{F} = \{\bar{f}_n\}_{n=1}^{\infty}$, where $\bar{f}_n$ are extensions of the fixed points f_n of P_n. Then condition (b) is trivially satisfied. Since by Lemma 6.8, $\{\bigvee^{I^N} f_n\}$ is bounded, condition (a) follows from an argument similar to the proof of Lemma 3.7 of [7]. □

Lemma 6.12 *Let* $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ *be a random map, where each* $\tau_k : I^N \to I^N$ *is a nonsingular Jabłoński transformation. Let there be a sequence of bounded linear operators* $P_n : L^1 \to L^1$ *satisfying the following conditions:*

(a) $P_n \to P_R$ *strongly in* L^1.

(b) *For each* n, *there exists* $f_n \in L^1$, $f_n \geq 0$, $\| f_n \|_1$ *such that* $P_n f_n = f_n$.

Then any limit point f *of the sequence* $\{f_n\}_{n=1}^{\infty}$ *is the density of an ACIM* μ_R *of* R.

Proof. The proof is an obvious extension of Theorem 1.1 of [7] to random maps and follows immediately by noting that: (1) any limit point of f_n, say f, obviously is a density and (2) by the Uniform Bounded Principle, $\sup_n \| P_n \|_1 < \infty$, so that

$$\begin{aligned} \| P_R - f \|_1 &\leq \| P_R f - P_n f \|_1 + \| P_n f - P_n f_k \|_1 \\ &\quad + \| P_n f_k - P_n f_k \|_1 + \| f_k - f \|_1, \\ &\quad \to 0 \text{ as } n \to \infty. \end{aligned}$$

□

Remark 6.1 Lemma 6.11 and a deterministic version of Lemma 6.12 were used in [7] for the approximation of invariant densities of Jabłoński transformations on I^N. Lemma 6.12 is important from the point of view that it *does not* assume the existence of an ACIM for R. Rather, it implies the existence of one, which is in agreement with the fact that condition (2) of Theorem 3.1 is sufficient but not necessary for the existence of an ACIM for R.

By Lemma 6.5-6.12, we therefore arrive at the following result:

Theorem 6.4 *Let* $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ *be a random map, where each* $\tau_k : I^N \to I^N$ *be a piecewise* C^2 *(not necessarily expanding) Jabłoński transformation, satisfying condition (2). Let* R *be subjected to perturbations given by (20) giving rise to the sequence of densities governed by the operator* P_n *defined by (19). Then*

(a) *for* $f \in L^1$, *the sequence* $P_n f \to P_R f$ *in* L^1 *as* $n \to \infty$.

(b) *If* R *has a unique ACIM with density* f_R, *then* $f_n \to f_R$ *in* L^1.

Now, by Lemma 6.6 $\{P_n\}_{n \in \mathbf{N}}$ are $\mathcal{T}$-bounded and by Lemma 6.7, $P_R \in \mathcal{T}$. From Lemma 6.5, P_R is stable in sense of Definition 6.3 if and only if it has a unique invariant density. We therefore conclude, using Theorem 6.4, the following:

Theorem 6.5 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map having a unique ACIM, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and (not necessarily expanding) Jabłoński transformation. Then P_R is stable under perturbations given by (20).*

7 Approximation of f_R

Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ is a random map, where each $\tau_k : I^N \to I^N$ be a piecewise C^2 (not necessarily expanding) Jabłoński transformation satisfying condition (2). Even though Theorem 3.1 guarantees the existence of an invariant density f_R for R, solving the resulting functional equation $P_R f_R = f_R$ is usually impossible except in the most trivial cases. From (3), we know that any invariant density f_R of R is approximated by the sequence

$$\left\{\frac{1}{n}\sum_{t=0}^{n-1} P_R^t f\right\}, \quad f \in L^1,$$

and we can have an explicit formula for P_R using the expression

$$P_R f(x) = \sum_{k=1}^{l} p_k \sum_{j=1}^{M} f(\tau_{kj}^{-1}(x)) \mid \det(A_{kj}^{-1}) \mid \chi_{\tau_{kj}(D_{kj})}(x)$$

where D_{kj} is an element of the partition $\mathcal{P}_k = \{D_{k1}, \ldots, D_{kM}\}$, $k = 1, \ldots, l$, $j = 1, \ldots, M$ on which τ_k is defined. Even then it is a very difficult to use the sequence of partial sums. If R satisfies the conditions of Theorem 5.1, we have

$$\|P_R^n f - f_R\|_1 \to 0, \quad \text{as } n \to \infty,$$

uniformly for every $f \in \mathcal{D}$. But again the iteration of P_R is prohibitively complex. Furthermore, analytic expressions of various statistical quantifiers of chaos such as entropy, correlation functions (see Section 8) and Lyapunov exponents [1]), involve an expression containing f_R explicitly. Therefore, it is of importance to have a *numerical* means of approximating f_R.

Let X a compact metric space and $\tau : X \to X$ be a non-singular transformation with Frobenius-Perron operator P_τ. In [84, Chapter 6, §4], Ulam conjectured that it was possible to construct a numerically computable sequence of finite-dimensional operators which approximate P_τ and whose fixed points approximate the fixed points of P_τ. Using a method based on von Neumann Ergodic Theorem, the conjecture was first proven in [62] for one-dimensional piecewise expanding maps by reducing the original infinite dimensional fixed point problem to the fixed point problem of stochastic matrices. Since then the conjecture has been proven in various cases: for one-dimensional piecewise expanding maps of an interval in [54], [26], [25], [27], [44]; for one-dimensional non-expanding maps of an interval in [67]; for maps on the real line in [20]; for higher dimensional maps in [5], [14], [7], [45], [28]; and for random maps in [16], employing different techniques such as Markov approximation methods [26], Monte-Carlo methods [44] and Galerkin projection methods [27]. Convergence rate analysis and error estimates of projection methods [31] and for a Markov approximation method [21] have also been carried out. In this section, we extend two of these methods to our case of random maps.

7.1 Approximation in the Space of Piecewise Constant Functions

When a Markov operator P is restricted to a finite dimensional subspace of L^1, it is said to be of *finite rank* [25]. In this situation the matrix representation of P with respect to the basis consisting of densities is a stochastic matrix.

In [16] it was shown that there exists a sequence of finite rank operators P_n whose fixed points f_n converge *weakly* to the fixed points of P_R, i.e., $f_n \to f_R$ weakly in L^1. In this section, we show that $f_n \to f_R$ *strongly* in L^1. As pointed out in [71, Example I.A.1], [50, §4] these approximating sequence of operators can be interpreted as having resulted from stochastic perturbations of the map (like the ones we have considered in Section 6). Therefore, from this point of view, results of this section can be considered as a special case of those of Section 6. We now proceed to construct the desired sequence of operators P_n.

For any integer n, let I^N be divided into n^N equal subsets $I_1, \ldots, I_{n^N}$ with

$$I_j = \left[\frac{r_1}{n}, \frac{r_1+1}{n}\right) \times \cdots \times \left[\frac{r_N}{n}, \frac{r_N+1}{n}\right)$$

for some $r_1, \ldots, r_N \in \{0, 1, \ldots, n-1\}$ and $m(I_k) = \frac{1}{n^N}$, $k = 1, \ldots, n^N$. Define P^k_{st} as the fraction I_s which is mapped into I_t by τ_k, i.e.,

$$P^k_{st} = \frac{m(I_s \cap \tau_k^{-1} I_t)}{m(I_s)},$$

and

$$P_{st} = \sum_{k=1}^{l} p_k P^k_{st}.$$

Let Δ_n be the n^N-dimensional linear subspace of L^1 which is the finite space generated by $\{\chi_j\}_{j=1}^{n^N}$, where χ_j denotes the characteristic function of I_j, i.e., $f \in \Delta_n$ if and only if $f = \sum_{j=1}^{n^N} a_j \chi_j$ for some constants $a_1, \ldots, a_{n^N}$.

Define a linear operator $P_n'(R) \equiv P_n' : \Delta_n \to \Delta_n$

$$P_n' \chi_r = \sum_{s=1}^{n^N} P_{rs} \chi_s = \sum_{k=1}^{l} p_k P^k_n \chi_r,$$

where

$$P^k_n \chi_r = \sum_{s=1}^{n^N} P^k_{rs} \chi_s.$$

We then have the following lemma [16].

Lemma 7.1 *Let* $\Delta'_n = \{\sum_{r=1}^{n^N} a_r \chi_r : a_r \geq 0$ and $\sum_{r=1}^{n^N} a_r = 1\}$. *Then* P_n' *maps* Δ'_n *to a subset of* Δ'_n.

Since $P_n'(\Delta'_n) \subset \Delta'_n$ is a compact convex set, the Brouwer Fixed Point Theorem implies that there exists a function $g_n \in \Delta'_m$ for which $P_n' g_n = g_n$. Let $f_n = n^N g_n$. Then $f_n \in \Delta_n$ and $\| f_n \|_1 = 1$ for each n.

For $f \in L^1$ and for every positive integer n, define an operator of conditional expectation $Q_n : L^1 \to \Delta_n$ by

$$Q_n f = \sum_{r=1}^{n^N} c_r \chi_r, \text{ where } c_r = \frac{1}{m(I_r)} \int_{I_r} f(x) dx.$$

Then P_n' and Q_n satisfy Lemmas 6.2-6.4,6.8 and 6.9. We extend P_n' to an operator on L^1 in the following natural way

$$P_n f = P_n' Q_n f.$$

We then have:

Lemma 7.2 *For $f \in L^1$, $P_n f \to P_R f$ in L^1 as $n \to \infty$.*

Proof. The proof follows from Lemma 6.9 and from the fact that

$$\begin{aligned} \| P_n f - P_R f \|_1 &= \| Q_n P_R Q_n f - P_R f \|_1 \\ &\leq \| Q_n P_R Q_n f - Q_n P_R f \|_1 + \| Q_n P_R f - P_R f \|_1 \\ &\leq \| Q_n f - f \|_1 + \| Q_n P_R f - P_R f \|_1 . \end{aligned}$$

□

We therefore finally have :

Theorem 7.1 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ be a piecewise C^2 (not necessarily expanding) Jabłoński transformation, satisfying condition (2). Then*

(a) *there exists a sequence of finite rank operators P_n with stationary densities f_n on the space of piecewise constant functions such that $P_n \to P_R$ in L^1.*

(b) *If R has a unique ACIM μ_R with density f_R, then $f_n \to f_R$ in L^1.*

Proof. (a) follows from the definition of P_n and Lemma 7.2. Now, by Lemma 6.8, the sequence $\{f_c\}$ is bounded in $\text{BV}(I^N)$. Therefore, by Lemma 2.3, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges to a g in $L^1(I^N)$. Now

$$\begin{aligned} \| P_R g - g \|_1 \leq\ & \| g - f_{n_k} \|_1 + \| f_{n_k} - Q_{n_k} \circ P_R f_{n_k} \|_1 \\ & + \| Q_{n_k} \circ P_R f_{n_k} - Q_{n_k} \circ P_R g \|_1 \\ & + \| Q_{n_k} \circ P_R g - P_R g \|_1 . \end{aligned}$$

Let $n_k \to \infty$. Then obviously the first term is zero. Since $Q_{n_k} \circ P_R f_{n_k} = f_{n_k}$ the second term is zero and since

$$\| Q_{n_k} \circ P_R \|_1 \leq \| Q_{n_k} \|_1 \| P_R g \|_1 = 1,$$

the first and third terms are zero. Fourth term is zero by Lemma 6.9. Hence, $P_R g = g$. It is obvious that g is a density and since f_R is unique, $g = f_R$. Since $\{f_{n_k}\}$ was arbitrary, (b) follows. □

7.2 Approximation in the Space of Continuous Piecewise Linear Functions

In the previous section, we constructed a sequence of finite-rank operators on the space of piecewise constant functions Δ_n which converges to P_R in L^1 and whose fixed points converge to the fixed point of P_R, i.e., f_R. In order to improve the rate of convergence, it is natural to generalize the piecewise constant approximation scheme to higher order polynomials.

In [28], an approximating sequence of finite-rank operators on the space of continuous piecewise linear functions was constructed for general piecewise C^2 and expanding transformations on $\mathbf{R}^N$ using the Giusti definition of bounded variation [35]. In this section, we obtain an extension of that result to random maps composed of Jabłoński transformations. Since the Giusti definition reduces to Tonelli definition in two dimensions [41], we restrict ourselves to the unit square $X = I^2$ so that we can apply our results of previous sections. We now proceed to construct an appropriate space of continuous piecewise linear functions and sequences of finite-dimensional operators analogous to P_n and Q_n of Section 7.1.

Divide the interval $0 \leq x \leq 1$ into M equal parts with $h = \frac{1}{M}$, and divide the interval $0 \leq y \leq 1$ into N equal parts with $k = \frac{1}{N}$. Let $I_i = [x_{i-1}, x_i]$, $x_i = ih$ for $i = 1, \ldots, M$, and $J_j = [y_{j-1}, y_j]$, $y_j = jk$ for $j = 1, \ldots, N$. Now divide each rectangle $X_{ij} = I_i \times J_j$ into two simplices $\text{conv}\{(x_{i-1}, y_{j-1}),(x_{i-1}, y_j),(x_i, y_j)\}$ and $\text{conv}\{(x_{i-1}, y_{j-1}),(x_i, y_{j-1}),(x_i, y_j)\}$, where $\text{conv}A$ denotes the convex hull of A. Thus, we obtain a triangulation of I^2 into a family of $2MN$ simplices; each with area $\frac{hk}{2}$. In the following, let $v_{ij} = (x_i, y_j)$, $i = 1, \ldots, M$, $j = 1, \ldots, N$ denote the vertices of these simplices. For any vertex $v_{ij} = (x_i, y_j)$ of the triangulation, let

$$V_{ij} = \bigcup\{\sigma : v_{ij} \text{ is vertex of } \sigma, \text{ where } \sigma \text{ is a triangulation of } I^2\}.$$

Let $\#(i,j)$ denote the number of simplices that are contained in V_{ij} and let Δ_c be the space of continuous piecewise linear functions corresponding to the above triangulation. Then Δ_c is a linear subspace of $L^1(I^2)$ of dimension $c = (M+1)(N+1)$. Let $\phi_{ij} \in \Delta_c$ be such that $\phi_{ij}(v_{rs}) = \delta_{ir}\delta_{js}$, i.e., ϕ_{ij} is 1 at the vertex v_{ij} and is 0 at all other vertices of the triangulation. Then $\{\phi_{ij}\}$'s are linearly independent and $\Delta_c = \text{span}\{\phi_{ij}\}$, $i = 1, \ldots, M$ and $j = 1, \ldots, N$, and therefore [28] form a basis for Δ_c with the following properties:

(a) $\| \phi_{ij} \|_1 = \frac{\#(i,j)}{6} hk$.

(b) $\phi_{ij} \geq 0$ and $\sum_{i=0}^{M} \sum_{j=0}^{N} \phi_{ij} = 1$.

(c) If $f \in \Delta_c$ then $f = \sum_{i=0}^{M} \sum_{j=0}^{N} f_{ij}\phi_{ij} = 1$ if and only if $f(v_{ij}) = f_{ij}$ for $i = 1, \ldots, M$ and $j = 1, \ldots, N$.

Define $Q_c : L^1(I^2) \to \Delta_c$ by

$$Q_c = \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{2}{\#(i,j)hk} \left(\int_{V_{ij}} f dm \right) \phi_{ij}.$$

Then Q_c is a Markov operator on $L^1(I^2)$ [28, Remark 3.1]. Furthermore, we have:

Lemma 7.3 *For $f \in L^1$, the sequence $Q_n f \to f$ in L^1 as $c \to \infty$.*

Proof. As $c \to \infty$, $\Delta_c \to 0$. Therefore, the result follows immediately from Lemma 3.2 of [28]. □

We have the following lemma analogous to Lemmas 3.4 of [28] using the fact that the Giusti definition of bounded variation used there reduces to that of Tonnelli definition used here.

Lemma 7.4 *Let $f \in BV(I^2)$. Then there exists a constant $\rho > 0$ such that*

$$\bigvee^{I^2} Q_c f \leq \rho \bigvee^{I^2} f.$$

For $f \in \Delta_c$, define $P'_c = Q_c \circ P_R$. Since both P_R and Q_c are Markov operators, $P'_c : \Delta_c \to \Delta_c$ is a Markov operator of finite rank.

Now we order the basis $\{\phi_{ij}\}$ in the following way:

$$\phi_{00}, \ldots, \phi_{M0}, \phi_{M1}, \ldots, \phi_{0N}, \ldots, \phi_{MN},$$

and $\{v_{ij}\}$, $\{V_{ij}\}$, and $\#(i,j)$ are ordered accordingly. In this order, we may write the basis functions as $\phi_1, \ldots, \phi_c$ with $c = (M+1)(N+1)$. Let $\bar{P}'_c = (u_{rs})$ be the corresponding $c \times c$ representation matrix of P'_c under this basis. Then $P'_c f_c = f_c$ for $f_c = \sum_{i=1}^{c} w_i \phi_i$ if and only if $w\bar{P}'_c = w$, where $w = (w_1, \ldots, w_c)$ is the row vector. We then have:

Lemma 7.5 *There exists a density $f_c \in \Delta_c$ such that $P'_c f_c = f_c$.*

Proof. The proof is similar to Lemma 3.5 of [28]. By definition,

$$P'_c \phi_r = Q_c \circ P_R \phi_r = \sum_{s=1}^{c} \frac{2}{\#(s)hk} \left(\int_{V_s} P_R \phi_r dm \right) \phi_s.$$

Therefore, the representation $\bar{P}'_c = (u_{rs})$ of P'_c under the natural basis $\{\phi_1, \ldots, \phi_c\}$ is given by

$$u_{rs} = \frac{2}{\#(s)hk} \int_{V_s} P_R \phi_r dm,$$

for $r, s = 1, \ldots, c$. The matrix $\bar{P}_c$ is non-negative. Let $\xi = (\xi_1, \ldots, \xi_c)^T$ be such that $\xi_r = \frac{\#(r)}{6}$ for $r = 1, \ldots, c$. Then using the properties of ϕ_i's, we have for each $r = 1, \ldots, c$,

$$\bar{P}'_c \xi_r = \sum_{s=1}^{c} u_{rs} \xi_s = \sum_{s=1}^{c} \frac{2}{\#(s)hk} \left(\int_{V_s} P_R \phi_r dm \right) \frac{\#(s)}{6} = \xi_r.$$

Therefore, $\bar{P}'_c \xi = \xi$ and there exists a non-negative row vector $w \neq 0$ such that $w\bar{P}'_c = w$. This completes the proof. □

Lemma 7.6 *If $\rho\alpha < 1$, then for all stationary densities $f_c \in \Delta_c$ of P_c, the sequence $\{\bigvee^{I^2} f_c\}$ is bounded.*

Proof. From Proposition 3.1 and Lemma 7.4, we have

$$\begin{aligned} \bigvee^{I^2} P_c f &= \bigvee^{I^2} Q_c \circ P_R f \leq \rho \bigvee^{I^2} P_R f \\ &\leq \rho\alpha \bigvee^{I^2} f + K' \parallel f \parallel_1, \end{aligned}$$

where $K' = \rho K > 0$. Since $\rho\alpha < 1$, this implies $\bigvee^{I^2} f_c \leq \frac{\rho K'}{1-\rho K'}$, and so $\{\bigvee^{I^2} f_c\}$ is uniformly bounded. □

As in Section 7.1, P'_c can be extended to whole of L^1 by defining $P_c f = P'_c Q_c f$. Then, using Lemma 7.3, we obtain the following result which is analogous to Lemma 7.2:

Lemma 7.7 *For $f \in \Delta_c$, the sequence $P_c f \to P_R f$ in L^1 as $c \to \infty$.*

We then have the following convergence result.

Theorem 7.2 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \dots, l\}$ be a random map, where each $\tau_k : I^2 \to I^2$ be a piecewise C^2 (not necessarily expanding) Jabłoński transformation, satisfying condition (2). If $\rho\alpha < 1$, where α and ρ are defined in Proposition 3.1 and Lemma 7.4 respectively, then*

(a) *there exists a sequence of finite rank operators P_c with stationary densities f_c on the space of continuous piecewise linear functions such that $P_c \to P_R$ in $L^1(I^2)$.*

(b) *If R has a unique ACIM μ_R with density f_R, then $f_c \to f_R$ in $L^1(I^2)$.*

Proof. (a) follows directly from Lemmas 7.5 and 7.7. (b) is similar to that of Theorem 7.1(b) and follows from Lemmas 7.6 and 2.3. □

Remark 7.1 We presented two methods of approximating the invariant f_R density of a random map R by approximating the Frobenius-Perron operator P_R. An alternative approach, could be approximating the map R itself instead of P_R. To be precise, let $R = \{\tau_k, p_k, k = 1, 2\}$, $\tau_k : I^N \to I^N$ be a given random map with a unique invariant density f_R and let $R_n = \{\tau_{kn}, p_k, k = 1, 2\}$, $n = 1, 2, \dots$ be a sequence of random maps with a unique invariant density f_{Rn} such that $R_n \to R$, as $n \to \infty$ uniformly. We then have the question: *does f_{Rn} converge to f_R?* This requires establishing a *compactness theorem*, for the invariant densities associated with the family of approximating transformations, which has been shown to hold for (deterministic) one dimensional piecewise expanding [39] and non-expanding [72], and higher dimensional [17], [2] maps. We leave this problem for future research.

8 Applications to Entropy and Correlation Functions

8.1 Asymptotic Periodicity, Asymptotic Stability and Conditional Entropy

Let $P : L^1 \to L^1$ be a Markov operator. In this section, we draw the connections between the properties of asymptotic periodicity, asymptotic stability and entropy of the sequence of densities $\{P^n f\}$. Assuming the existence of a density f on the phase space X ($\subset I^N$ in our case) describing a thermodynamic state of a system at a particular time, Gibbs introduced the concept of the *index of probability*, given by $\ln f$. Weighting the index of probability by the density f, the *Boltzmann-Gibbs entropy* is given by [65],

$$H_{BG}(f) = -\int_X f \ln f dm,$$

and is considered to be standard mathematical analog of the thermodynamic entropy [80]. H_{BG} adequately describes the behavior of the entropy of a density under the action of a Markov operator with a uniform stationary density but leads to limitations for Markov operators with a nonconstant one [65, Example 3.7]. This has led to a generalization of Boltzmann-Gibbs entropy, the *conditional entropy*.

Definition 8.1 Let $f, g \in \mathcal{D}$ such that supp$f \subset$ supp$g$. Then the *conditional entropy* of the density f with respect to g is,

$$H_c(f \mid g) = -\int_X f \ln\left(\frac{f}{g}\right) dm.$$

Since g is a density, $H_c(f \mid g)$ is always defined. If g is a constant density, $g = 1$ and $H_c(f \mid 1) = H_{BG}(f)$. It can be shown [65] that

$$H_c(f \mid g) \leq 0. \tag{21}$$

(The equality holds if $f = g$.) Let $P : L^1 \to L^1$ be a Markov operator. By [86],

$$H_c(Pf \mid Pg) \geq H_c(f \mid g), \tag{22}$$

for $f, g \in \mathcal{D}$, supp$f \subset$ supp$g$. Therefore, if $g = f^*$ is a stationary density of P then by (21) $H_c(Pf \mid Pf^*) \geq H_c(f \mid f^*)$, and so the conditional entropy with respect to the stationary density is *always non-decreasing* and *bounded above by zero*. Therefore, it follows from Remark 9.2.4 of [58] that $H_c(Pf \mid f^*)$ always converges, but not necessarily to zero as $n \to 0$. Denote $\lim_{n\to\infty} H_c(P^n f \mid f^*) \equiv H_c^\infty(P^n f \mid f^*)$ and call H_c^∞ the *asymptotic conditional entropy*. Since supp$f \subset$ suppf^* implies supp$Pf \subset$ supp$Pf^* =$ suppf_R, H_c^∞ is well defined. We can define the asymptotic Boltzmann-Gibbs entropy analogously.

We can now examine the asymptotic behavior of the entropy of a sequence of densities $\{P_R^n f\}$ in our case of a random map R.

A consequence of asymptotic periodicity is that thermodynamic equilibrium of the system may consist of a sequence of metastable states which are visited periodically. Its implication for the asymptotic Boltzmann-Gibbs entropy is reflected in the following result:

Proposition 8.1 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 (not necessarily expanding) Jabłoński transformation satisfying condition (2). Let R have a unique constant invariant density. Then the asymptotic Boltzmann-Gibbs entropy of $\{P_R^n f\}$ oscillates periodically in a metastable equilibrium in the following sense:*

$$H_{BG}^\infty(P_R^n f) = H_{BG}^\infty(P_R^{n+m} f) \text{ if } \alpha^m(i) = i,$$

where α is a permutation of $i \in \{1, \ldots, r\}$ and $\alpha^p(i) \neq i$, $p < m$.

Proof. From Theorem 4.2, $\{P_R^n f\}$ is asymptotically periodic. Therefore, by (13), $\alpha^m(i) = i$ implies that $\lim_{n\to\infty} P_R^n f = \lim_{n\to\infty} P_R^{n+m} f$ as $n \to \infty$. Now, using the definition of H_{BG}^∞, the proof follows. □

Now, let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 (not necessarily expanding) Jabłoński transformation satisfying condition (2). Then, by Theorem 4.2, P_R is constrictive and so by (13) and (14) in the Spectral Decomposition Theorem 4.1, we can write

$$P_R^n f \approx \sum_{i=1}^{r} \lambda_{\alpha^{-n}(i)}(f) g_i. \tag{23}$$

Furthermore, by Proposition 4.1

$$f^* = \frac{1}{r} \sum_{i=1}^{r} g_i \tag{24}$$

is a stationary density of P_R. Let R have a unique ACIM μ_R with density $f_R > 0$. By Remark 2.1, f_R is stationary density of P_R and since f_R is unique, $f^* = f_R$. Then by (23) and (24) and the orthogonality of the g_i's from Theorem 4.1(b), we have,

$$\begin{aligned} H_c^\infty(P_R^n f \mid f_R) &= -\lim_{n\to\infty} \int_{I^N} P_R^n f \ln\left(\frac{P_R^n f}{f_R}\right) dm \\ &= -\sum_{i=1}^{r} \int_{I^N} \lambda_{\alpha^{-n}(i)}(f) g_i \ln(r\lambda_{\alpha^{-n}(i)}(f)) dm. \end{aligned}$$

Since the permutation $\alpha(i)$ is invertible and the scaling coefficients λ_i's add up to 1, we obtain

$$H_c^\infty(P_R^n f \mid f_R) = -\ln r - \sum_{i=1}^{r} \lambda_i(f) \ln[\lambda_i(f)].$$

Furthermore, since $0 \le \lambda_i \le 1$, for all i, it follows using (21) that $-\ln r \le H_c^\infty(P_R^n f \mid f_R) \le 0$. Now let each τ_k, $k = 1, \ldots, l$, be expanding and P_{τ_k} be its corresponding Frobenius-Perron operator. Then by Remark 4.1, P_{τ_k} is constrictive and therefore has, say, r_k densities in its spectral representation. Therefore, using Theorem 4.3 we finally have

$$-\min_{1\le k\le l} \ln r_k \le -\ln r \le H_c^\infty(P_R^n f \mid f_R) \le 0.$$

Hence, for asymptotically periodic sequence of densities $\{P_R^n f\}$ corresponding to the random map R, we obtain a bound for the asymptotic conditional entropy. We can also obtain a partial converse of this result.

Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is nonsingular. Let R have an ACIM μ_R with density $f_R > 0$. If there is a constant $c > 0$ such that for every bounded $f \in \mathcal{D}$, $H_c(P_R^n f \mid f_R) \ge -c$ for sufficiently large n, then by Theorem 6.6 of [65] the sequence of densities $\{P_R^n f\}$ is asymptotically periodic. We state the above as:

Theorem 8.1 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is nonsingular. Let $f_R > 0$ be a stationary density of P_R. Then*

(a) *if each τ_k is a piecewise C^2 (not necessarily expanding) Jabłoński transformation satisfying condition (2) and if f_R is unique, we have*

$$-\ln r \le H_c^\infty(P_R^n f \mid f_R) \le 0, \tag{25}$$

where r is the number of densities in the spectral representation of P_R. Furthermore, if each τ_k is piecewise expanding then P_{τ_k}, $k = 1, \ldots, l$, is constrictive (having say r_k densities its spectral representation), we have

$$-\min_{1\le k\le l} \ln r_k \le -\ln r \le H_c^{\infty}(P_R^n f \mid f_R) \le 0.$$

(b) *if there is a constant $c > 0$ such that for every bounded $f \in \mathcal{D}$,*

$$H_c(P_R^n f \mid f_R) \ge -c$$

for sufficiently large n, then the sequence of densities $\{P_R^n f\}$ is asymptotically periodic.

From Theorems 4.5 and 8.1(a), we may conclude

Proposition 8.2 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \dots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and expanding Jabłoński transformation on the partition $\mathcal{P}_k = \{D_{k_1}, \dots, D_{k_m}\}$ related to which the number $M_N^{(k)}$ is defined. Let $\mathcal{V}_k$ be the set of vertices of elements of $\mathcal{P}_k$ which lie in the interior of I^N. Let P_R have r densities in its spectral representation and let $f_R > 0$ be its unique stationary density. For each $k = 1, \dots, l$ let P_{τ_k} be the Frobenius-Perron operator corresponding to τ_k, having r_k densities in its spectral representation, such that the permutation $\{\alpha_k(1), \dots, \alpha_k(r_k)\}$ of $\{1, \dots, r_k\}$ is cyclical. If $\min_{1\le k\le l} \frac{\Lambda_k}{M_N^{(k)}} > 1$, then*

$$-\min_{1\le k\le l} \ln \mathcal{V}_k = -\min_{1\le k\le l} \ln r_k \le -\ln r \le H_c^{\infty}(P_R^n f \mid f_R) \le 0.$$

For random maps composed of piecewise linear Markov transformations, we can make the estimate of Theorem 8.1(a) more precise.

Let $R(x) = \{\tau_k(x), p_k, k = 1, \dots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise linear and expanding Jabłoński transformation having the Markov property with respect to a common partition $\mathcal{P} = \{D_1, \dots, D_m\}$. By Theorem 3.1, R has an ACIM, say μ_R. Suppose μ_R is unique with density $f_R > 0$. From Theorem 3.4(a), $f_R \in \mathcal{S}$, i.e., $f_R = \sum_{j=1}^M C_j \chi_{D_j}$ for some (positive) constants $C_1, \dots, C_M$. We therefore have, for any $f \in \mathcal{D}$,

$$\begin{aligned}
H_c(P_R^n f \mid f_R) &= -\int_{I^N} P_R^n f \ln\left(\frac{P_R^n f}{\sum_{j=1}^M C_j \chi_{D_j}}\right) dm \\
&= -\sum_{j=1}^M \int_{D_j} P_R^n f \ln\left(\frac{P_R^n f}{C_j}\right) dm \\
&= -\sum_{j=1}^M \int_{D_j} P_R^n f \ln(P_R^n f) dm + \sum_{j=1}^M \int_{D_j} P_R^n f \ln(C_j) dm \\
&\le -\int_{I^N} P_R^n f \ln(P_R^n f) dm + \left(\sum_{j=1}^M \ln(C_j)\right) \int_{I^N} P_R^n f dm.
\end{aligned}$$

Since $P_R f \in \mathcal{D}$, we finally have $H_c(P_R^n f \mid f_R) \le H_{BG}(P_R^n f) + C$, where $C = \sum_{i=1}^M \ln(C_j)$. Denoting $\lim_{n\to\infty} H_{BG}(P_R^n f) \equiv H_{BG}^{\infty}(P_R^n f)$, and using Theorems 4.6 and 8.1(a), our next result can be stated as:

Theorem 8.2 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \dots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is a piecewise linear and expanding Jabłoński transformation with the Markov property* *with respect to a common partition $\mathcal{P}$. Let P_{τ_k} and P_R be the Frobenius-Perron operators corresponding to τ_k and R, having r_k and r densities in their spectral representations respectively. Then*

$$-\min_{1\le k\le l} \ln r_k \le -\ln r \le H_c^{\infty}(P_R^n f \mid f_R) \le H_{BG}^{\infty}(P_R^n f) + C \le C.$$

We now prove that, as a consequence of asymptotic stability, the asymptotic conditional entropy *does* attain its maximum and vice versa, i.e., whenever the asymptotic conditional entropy attains its maximum, the system is asymptotically stable.

Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is piecewise C^2 and expanding Jabłoński transformation. Then if any one of the maps τ_k is exact, it can be shown (as in the proof of Theorem 5.1) that P_R is exact. Let f_R be the stationary density of P_R with respect to which it is exact. If f_R is unique, then exactness of P_R is equivalent to asymptotic stability if $\{P_R^n\}$. Thus, from Theorem 5.1 and Theorem 7.7 of [65], we have the following result:

Theorem 8.3 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map, where each $\tau_k : I^N \to I^N$ is nonsingular. Then*

(a) *if each τ_k is a piecewise C^2 and expanding Jabłoński transformation, and if any one of the maps τ_k, $k = 1, \ldots, l$, is exact, we have*

$$H_c^{\infty}(P_R^n f \mid f_R) = 0.$$

(b) *if $H_c^{\infty}(P_R^n f \mid f_R) = 0$, the sequence of densities $\{P_R^n f\}$ is asymptotically stable.*

8.2 Asymptotic Periodicity, Asymptotic Stability and Correlation Functions

In this section, we consider the consequence of asymptotic periodicity and asymptotic stability of $\{P_R^n f\}$ on both auto-correlation and time-correlation functions.

Let $\tau : X \to X$, X a compact metric space, be a nonsingular map with a unique ACIM with density f^*.

Definition 8.2 For any two bounded integrable functions $\sigma, \eta : X \to \mathbf{R}$, the *auto-correlation function* of σ with η is defined as

$$C_{\sigma,\eta}^{AC}(T) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sigma(\tau^{t+T}(x))\eta(\tau^t(x))$$

and the *time-correlation function* $C_{\sigma,\eta}^{TC}$ is defined as

$$C_{\sigma,\eta}^{TC}(T) = C_{\sigma,\eta}^{AC}(T) - <\sigma><\eta>$$

where

$$<\sigma> = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sigma\tau^t(x).$$

In [65, Chapter 5, §C], it has be shown that

$$C^{AC}_{\sigma,\eta}(T) = \int_{I^N} \sigma(x) P^T_\tau[\eta(x) f^*(x)] dm(x),$$

which we can use, in an analogy with the deterministic case, to define correlation functions in case of random maps. Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map with a unique ACIM μ_R with density f_R, where each $\tau_k : I^N \to I^N$ is nonsingular.

Definition 8.3 Let σ and η be as in Definition 8.2. Then, the *auto-correlation function* in case of R is defined by

$$C^{AC}_{\sigma,\eta}(T) = \int_{I^N} \sigma(x) P^T_R[\eta(x) f_R(x)] dm(x). \tag{26}$$

The *time-correlation function* in case of R is defined using the relation in Definition 8.2 and (26).

We then have the following result relating asymptotic stability of $\{P^n_R f\}$ and *asymptotic* auto-correlation and *asymptotic* time-correlation functions.

Theorem 8.4 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map with a unique ACIM μ_R with density f_R, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 and expanding Jabłoński transformation. If any one of the maps τ_k is exact, then*

$$\lim_{T\to\infty} C^{AC}_{\sigma,\eta}(T) = <\sigma><\eta>$$

and

$$\lim_{T\to\infty} C^{TC}_{\sigma,\eta}(T) = 0.$$

Proof. Since one of the maps τ_k is exact, by Theorem 5.1 R is exact and hence mixing. Therefore, by Corollary 4.4.1(b) of [58] we have

$$\lim_{n\to\infty} < P^n_R(\eta f_R), \sigma > = < \eta f_R, 1 >< f_R, \sigma > . \tag{27}$$

The result then follows from (26), (27) and Definition 8.3. □

Now, let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map with a unique ACIM μ_R with density f_R, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 (not necessarily expanding) Jabłoński transformation satisfying condition (2). Then by Theorem 3.2, $\{P^n_R f\}$ is asymptotically periodic and so by Theorem 3.1, we have the spectral representation (8), i.e.,

$$P^T_R f = \sum_{i=1}^{r} \lambda_i(f) g_{\alpha^T(i)} + Q_T f.$$

Also, since P_R is constrictive, by (23) we have

$$f_R = \frac{1}{r}\sum_{i=1}^{r} g_i.$$

Combining these with (26), we obtain

$$C^{AC}_{\sigma,\eta}(T) = \sum_{i=1}^{r} \lambda_i(\eta(x)) \int_{I^N} g_{\alpha^T(i)}(x)\sigma(x) + Q_T\eta(x) dm(x) \tag{28}$$

Thus, as in [65], we have arrived at the following result:

Theorem 8.5 *Let $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, l\}$ be a random map with a unique ACIM, where each $\tau_k : I^N \to I^N$ is a piecewise C^2 (not necessarily expanding) Jabłoński transformation satisfying condition (2). Let σ and η be as in Definition 8.2. Then the auto-correlation function $C^{AC}_{\sigma,\eta}$ (and hence the time-correlation function $C^{TC}_{\sigma,\eta}$) seperates into sustained periodic and decaying stochastic components represented by expression (28).*

8.3 Examples

In this section, we consider examples which illustrate some of the previous results.

Example 8.1 Consider a random map $R(x) = \{\tau_k(x), p_k, k = 1, \ldots, 4\}$, where $\tau_1, \ldots, \tau_4$ are defined as in Example 5.1 and $\tau_5 : I^2 \to I^2$ be the map τ_2 of Example 3.1. Let $p_k = \frac{1}{20}$ for $k = 1, \ldots, 4$ and $p_5 = \frac{1}{5}$. Then R satisfies condition (2) with

$$\sum_{k=1}^{5} \sup_j \frac{p_k}{\mid \varphi'_{k,ij}(x_i) \mid} = \frac{2}{3} < 1.$$

Therefore, by Theorem 4.2, the sequence $\{P^n_R f\}$ is asymptotically periodic. By Theorem 5 of [14] τ_5 has a unique ACIM and so by Proposition 3.3, R has a unique ACIM. Hence, applying Theorem 8.1(a) we have an upper and lower bound on the asymptotic conditional entropy of $\{P^n_R f\}$ as in (25).

Remark 8.1 By an argument similar to that of Example 5.1, it follows that $\{P^n_{\tau_k} f\}$ is not asymptotically periodic for any of the maps $\tau_k, = 1, \ldots, 4$, in Example 8.1. Thus, we conclude that $\{P^n_R f\}$ can have asymptotic periodicity even if none of the $\{P^n_{\tau_k} f\}$'s are asymptotically periodic.

Example 8.2 Consider a random map $R(x) = \{\tau_1(x), \tau_2(x), p_1, p_2\}$, where $\tau_k : I^2 \to I^2$ are defined on a common partition $\mathcal{P} = \{D_1, \ldots, D_9\}$ shown in Fig. 1. Let τ_1 be the map τ_2 of Example 3.1 and τ_2 be defined by

$$\tau_2(x_1, x_2) = (\varphi_{2,1j}(x_1), \varphi_{2,2j}(x_2)),$$

where

$$\varphi_{2,ij}(x) = \begin{cases} 1 - 3x & \text{if } 0 \le x < \frac{1}{3} \\ 3x - 1 & \text{if } \frac{1}{3} \le x < \frac{2}{3} \\ -3(x-1) & \text{if } \frac{2}{3} \le x \le 1 \end{cases},$$

$i = 1, 2$ and $j = 1, \ldots, 9$. $\varphi_{2,ij}$ is shown in Fig. 4. Let $p_k = \frac{1}{2}$ for $k = 1, 2$. Then both τ_1 and τ_2 are piecewise linear and expanding Jabłoński transformations with Markov property on the partition $\mathcal{P}$. As in Example 3.1, τ_1 has a unique ACIM. It is easy to see that τ_1 satisfies condition (17), since it satisfies the much stronger communication property. Therefore, τ_1 is exact and so by Proposition 5.1, the sequence of densities $\{P^n_R f\}$ has asymptotic stability. By Proposition 3.3, R has a unique ACIM, which by Theorem 5.3 is an SRB measure. By Theorem 8.3(a) the asymptotic conditional entropy of $\{P^n_R f\}$ attains its maximum value, i.e, 0.

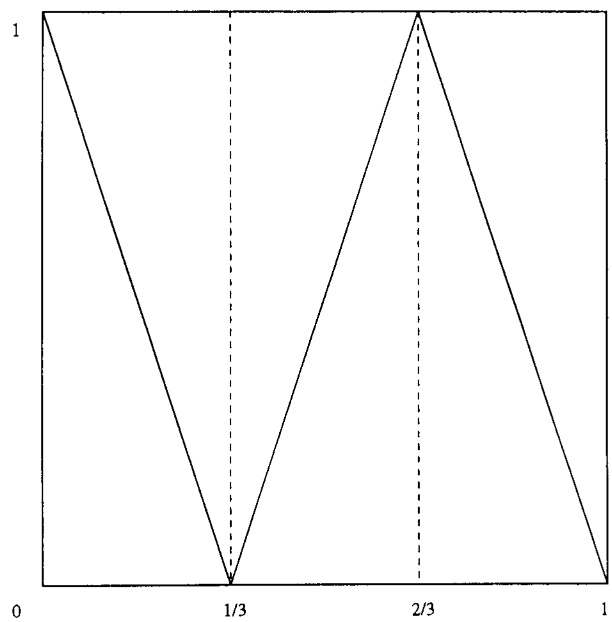

Figure 4: The map $\varphi_{2,ij}$

9 The Inverse Perron-Frobenius Problem

In previous sections, based on the assumption that a random map is given, we studied the asymptotic dynamics of densities of its orbits. Often, all that is available from experimental observations is stochastic data (time series) in the form of an attractor $\mathcal{A}$ and a density f supported on $\mathcal{A}$. We are then faced with the problem of inferring the underlying dynamical system from the data. The inverse problem of constructing the random map (an IFS), given the attractor has been considered in form of the Collage Theorem [4, §9.6, Theorem 3].

The techniques which are employed in direct estimation of a model face many numerical difficulties [56] and are computationally expensive [22]. In [42], a different approach was taken and a method was proposed of determining a one dimensional piecewise linear transformation whose invariant density was given. This problem is called as the *inverse Perron-Frobenius problem.* In this section, we consider the results of [42] in view of random maps. We begin with the following definition, which gives a generalization of a Markov transformation.

Definition 9.1 (Semi-Markov Transformation) A transformation $\tau : I^N \to I^N$ is called a *semi-Markov transformation* with respect to a partition $\mathcal{P} = \{D_1, \ldots, L\}$ if there exist disjoint regions $E_j^{(i)}$ such that for any $i = 1, \ldots, L$, we have

$$D_i = \cup_{j=1}^{k(i)} E_j^{(i)} \text{ and } \tau(E_j^{(i)}) \in \mathcal{P}. \tag{29}$$

The next result generalizes Corollary 3.2 to semi-Markov transformations.

Proposition 9.1 *Let $\tau : I^N \to I^N$ be a piecewise linear and expanding Jabłoński transformation with respect to partition $\mathcal{P}$ satisfying condition (29). Then every invariant density of τ is piecewise constant with respect to $\mathcal{P}$.*

Proof. Using Corollary 3.2, the proof follows along the lines of Theorem 2 of [42]. □

In the case of a semi-Markov transformation τ, the matrix induced by τ is given $\mathcal{M}_\tau = (m_{ij})_{1 \le i,j \le L}$, where

$$m_{ij} = \begin{cases} \mid \det(A_k^{(i)})^{-1} \mid & \text{if } \tau(E_k^{(i)}) = D_j, \\ 0 & \text{otherwise} \end{cases}$$

Definition 9.2 A piecewise linear semi-Markov transformation τ with respect to partition $\mathcal{P}$ is said to be a *3-band transformation* if its induced matrix $\mathcal{M}_\tau = (m_{ij})$ satisfies

$$m_{ij} = 0, \quad \text{if } \mid i - j \mid > 1$$

for any $i = 1, \ldots, Z$.

A property of the invariant density of a 3-band transformation is given by the next lemma.

Lemma 9.1 *Let $\tau : I^N \to I^N$ be a 3-band transformation and let f be any invariant density of τ. Denote $f_i \equiv f \mid_{D_i}$, for each $i = 1, \ldots, L$. Then*

$$m_{i,i-1} \cdot f_i = m_{i-1,i} \cdot f_{i-1},$$

for each $i = 2, \ldots, L$.

Proof. The proof follows along the lines of Theorem 3 of [42]. □

The following lemmas are from [42].

Lemma 9.2 *Let M be a $L \times L$ stochastic matrix and $\mathcal{P}$ be an equal partition of I. Then there exists a semi-Markov transformation τ with respect to $\mathcal{P}$ such that $\mathcal{M}_\tau = M$.*

Lemma 9.3 *Let f be a density on an equal partition of I. Then there exists a (not necessarily unique) piecewise expanding 3-band transformation τ such that f is invariant under τ.*

We now give the main result of the section, whose proof follows from Proposition 9.1 and Lemmas 9.1-9.3, along the lines of Theorem 5 of [42]. It is stated here in one dimension since higher dimensional generalizations of Lemmas 9.2 and 9.3 do not seem feasible at present.

Theorem 9.1 *Let $\mathcal{P}$ be a partition of I and let f_R be an invariant density of a random map R on I, which is piecewise constant with respect to $\mathcal{P}$. Then there exists a (not necessarily* *unique) piecewise linear and expanding semi-Markov transformation τ with respect to $\mathcal{P}$ such that f_R is invariant under τ.*

Remark 9.1 Theorem 3.4(a) guarantees the existence of f_R in Theorem 9.1. Given a density f piecewise constant with respect to a partition $\mathcal{P}$, it is not always possible [42] to find a Markov transformation which leaves it invariant. Theorem 9.1 states that we can solve the problem using semi-Markov transformations. Theorem 9.1 is of interest since it states that even though we may not be able to construct a random map R itself from its given invariant density f, a 3-band matrix can be constructed that can be viewed as a deterministic point transformation τ which has f as its invariant density and so the orbits of τ mimic the dynamics of R in the statistical sense.

10 Conclusion

We have analysed the flow of densities for a large class of random maps from a dynamical point of view. In spite of the diversity of issues investigated, there are many important questions that remain open and may be the object of future research.

We first address some questions directly related to the results of previous sections. In Section 4, we proved the asymptotic stability of $\{P_R^n f\}$ using the fact that P_R is constrictive. It would be of interest to know whether we can have asymptotic stability without this assumption, using other techniques such as the existence of a lower bound function for P_R (in which case the asymptotic stability follows immediately from Theorem 5.6.2. of [58]). Asymptotic stability for random maps composed of nonexpansive continuous maps on locally compact metric spaces has been shown in [60] using such techniques. Now, let a sequence of densities $\{P_R^n f_0\}$ have asymptotic periodicity which is perturbed in such a way that the initial density f_0 is again transformed to a density f_0' (and does not perturb the map R itself). Then, as pointed out in Section 6.1, it would of interest to obtain conditions under which $\{P_R^n f_0'\}$ has asymptotic periodicity. To approximate the invariant densities of more general random maps than considered here, it would be useful to approximate the map R itself and establish a compactness result for invariant densities of a sequence of approximating random maps, as indicated in Remark 7.1. Finally, since a lot of interesting random dynamics takes place in two or more dimensions, it would useful to obtain a higher dimensional analogue of Theorem 9.1.

The results of this paper apply only to random maps which are composed of piecewise-expanding transformations or at most expanding-on-average transformations defined on rectangular partitions. It would be of interest to know whether our results can be extended to random maps composed of non-expanding transformations, in particular, affine transformations which give rise to IFSs such as in [3] or to random maps composed of piecewise-expanding transformations defined on more general domains which are not necessarily rectangular (so the transformations are not of Jabłoński type). For example, it seems possible to extend Corollary 3.2 (and hence Proposition 9.1) to piecewise- expanding transformations of the type considered in [38].

An important question is to what extent do the computer-generated orbits of a dynamical system represent the behavior of the true-orbits. Two tools have been employed in analysing this question [48]: the shadowing property, in a geometrical study, and the existence of an ACIM, in the statistical study of the system under consideration. For certain random maps composed of uniformly expanding hyperbolic transformations a shadowing property was proved in [85], [52], [6]. It is of significance to know whether the shadowing property exists for any class of random maps (composed of non-hyperbolic transformations, such as in our case) that have been studied in this paper. In [37], it was shown for deterministic transformations that if there exist long periodic computer orbits, or long non-periodic orbits which occupy a significant portion of the computer space for all precisions, then the measures derived from computer simulation must approach the ACIM of the theoretical transformation under consideration. Once again, it would of interest to know whether such a result can be proved for random maps considered here.

Applications of the random map framework to an analysis of deterministic systems would be of interest as well. Random maps have been applied [49] to an approximation of statistical quantifiers describing the dynamics of spatially extended systems such as CMLs. Modelling of CMLs by IFSs has been suggested in [18].

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