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Noise-Induced Asymptotic Periodicity in a Piecewise Linear Map

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We examine asymptotically periodic density evolution in one-dimensional maps perturbed by noise, associating the macroscopic state of these dynamical systems with a phase space density. For asymptotically periodic systems density evolution becomes periodic in time, as do some macroscopic properties calculated from them. The general formalism of asymptotic periodicity is examined and used to calculate time correlations along trajectories of these maps as well as their limiting conditional entropy. The time correlation is shown to naturally decouple into periodic and stochastic components. Finally, asymptotic periodicity is studied in a noise-perturbed piecewise linear map, focusing on how the variation of noise amplitude can cause a transition from asymptotic periodicity to asymptotic stability in the density evolution of this system.

KEY WORDS: Asymptotic periodicity; asymptotic stability; density evolution; noise; Keener map; Boltzmann conditional entropy.

1. INTRODUCTION

The study of highly irregular behavior in one-dimensional maps often deals with deterministic transformations that exhibit chaotic motion in a phase space. However, complex behavior in the iterates of maps may also be observed when one-dimensional maps are stochastically perturbed by noise. In this case the evolution of iterates is truly random, even when the underlying noise-free map has periodic solutions.

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Often, the unpredictable behavior of trajectory evolution in nonlinear dynamical systems can be simplified if one examines their behavior in terms of density evolution.[1] This alternative viewpoint has particular appeal when applying the concepts of nonlinear dynamics to many problems in statistical physics,[2] and offers an immediate connection with the mathematical discipline of ergodic theory which developed from the early work of Boltzmann and Gibbs.

This paper examines the property of asymptotic periodicity[3] in the density evolution of one-dimensional noise-perturbed maps. The remainder of this section contrasts the trajectory versus density evolution in noise-perturbed maps and introduces the noise-induced Markov operator, a linear integral operator that governs the flow of densities in these systems. Section 2 introduces the dynamical concept of asymptotic periodicity, and gives general techniques for the calculation of the autocorrelation function and the conditional entropy of an asymptotically periodic system. Finally, in Section 3 we illustrate noise-induced asymptotic periodicity in the map

$$x_{t+1} = S(x_t) + \xi_t \mod 1 \tag{1}$$

where S is the map[4] given by

$$S(x) = \alpha x + \beta \mod 1 \tag{2}$$

The ξ_t in (1) are uniformly distributed on $[0, \theta]$, while the parameters α and β are adjusted to give period-$(m+1)$ solutions in the map S. It is of particular interest to examine how varying the noise amplitude leads to a transition from asymptotic periodicity to asymptotic stability in density evolution.

1.1. Trajectory Versus Density Evolution

Chaotic attractors, for which the motion of a time series through them is ergodic with respect to some invariant measure, can be found in several deterministic systems, such as the logistic[5] and hat maps[6] for certain ranges of parameter values. Another class of systems in which this type of ergodic attractor arises is in maps possessing simple periodic solutions, which are stochastically perturbed by noise. While it is not entirely surprising that such a situation should arise, the stochastic perturbation has transformed a completely predictable problem to one where we can at most know the boundaries of the phase space attractor, but not the motion of a trajectory through it. This is reminiscent of the situation encountered when dealing with the N-body problem. To make the study of such noise-perturbed maps more tractable, we may argue, as Gibbs did when dealing with the N-body problem, that a macroscopic state of a system described by such maps is not in general given by a single point in phase space, but rather a collection, or *ensemble*, of points distributed according to some density. The evolution of a system, in this formalism, is therefore given by the evolution (or flow) of densities. In this approach, exact values are replaced by ensemble averages or expectation values weighted by the phase space density.

For the systems of statistical mechanics in the thermodynamic limit (number of particles $N \to \infty$, volume $V \to \infty$), it is assumed that the evolution of densities attains the density of the canonical ensemble Z. For low-dimensional noise-perturbed maps, however, the flow of densities may display several types of behavior, never attaining an equilibrium density.

1.2. Noise-Induced Markov Operators and the Evolution of Densities

The evolution of densities under the action of a dynamical system S is described by a Markov operator which we denote by P. Formally, any linear operator $P^t: L^1 \to L^1$ that satisfies

$$P^t f \geqslant 0 \qquad \text{and} \qquad \int_X P^t f(x)\,dx = \int_X f(x)\,dx$$

for $f \geqslant 0$ is called a *Markov operator*,[1] where X denotes the phase space on which S operates. Throughout this paper we deal with the subset of L^1 functions which are normalized to one. This set of densities is denoted by D. It is clear that when a Markov operator acts on a density it yields another density. Beginning with an ensemble of phase space points representing some macroscopic state of a system, and distributed according to an initial density f_0, one unit of time (iteration) later the new density state of the system f_1 is given by $f_1 = Pf_0$. For a deterministic one-dimensional map S, defined on $X = [0, 1]$ and additively perturbed by noise ξ distributed with density $h(x)$, so $x_{n+1} = S(x_n) + \xi$, P is given[3] by

$$Pf(x) = \int_0^1 h(x - S(y))\,f(y)\,dy \tag{3}$$

Markov operators may possess a *stationary density* f^*. This density satisfies $Pf^* = f^*$ and may be associated with a state of thermodynamic equilibrium of a dynamical system.

The evolution of densities under (3) characterizes P as well as the dynamical map[1] S. Three general behaviors may be displayed by the sequence $\{P^t f_0\}$. These are *ergodicity*, *mixing*, and *exactness*. In all three cases the system possesses an invariant density f^*. However, the three behaviors differ in the way the sequence $\{P^t f_0\}$ converges to f^*.

Of the three cases above, exactness implies the strongest form of convergence of $\{P^t f_0\}$. Mathematically, a system is said to be exact[1] if and only if

$$\lim_{t \to \infty} |P^t f_0 - f^*| = 0$$

for all initial densities f_0. Exactness may be considered as the analogue of an approach to equilibrium from all initial preparations of a system.

Mixing implies a weak form of convergence of $\{P^t f_0\}$. In particular, for any L^∞ function $\mathscr{F}$, a system is mixing[1] if and only if

$$\lim_{t \to \infty} \langle P^t f_0, \mathscr{F} \rangle = \langle f^*, \mathscr{F} \rangle$$

for all initial densities f_0. Mixing systems spread densities throughout the accessible phase space, as determined by the support of f^*.

Ergodicity implies the weakest form of convergence of $\{P^t f_0\}$. For ergodic[1] systems

$$\lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{t-1} \langle P^n f_0, \mathscr{G} \rangle = \langle f^*, \mathscr{G} \rangle$$

for all $f_0 \in D$ and any L^∞ function $\mathscr{G}$.

Exactness implies mixing, which in turn implies ergodicity. However, ergodicity alone does not constrain the sequence $\{P^t f_0\}$ to become asymptotically equal to f^*.

2. ASYMPTOTIC PERIODICITY

Asymptotic periodicity is a type of density evolution that may be displayed by one-dimensional maps by themselves[7] or under the influence of noise. Without loss of generality, the phase space of these maps will be taken as $[0, 1]$. For asymptotically periodic systems the sequence $\{P^t f_0\}$ satisfies the Komornik spectral decomposition theorem (ref. 8, Theorem 1). This theorem states that for any initial density f_0,

$$Pf_0(x) = \sum_{i=1}^{r} \lambda_i(f_0)\, g_i(x) + Qf_0(x) \tag{4}$$

where the functions g_i form a sequence of r densities satisfying $g_i g_j = 0$ if $i \neq j$, $i, j = 1,..., r$, so the supports of the g_i densities, denoted $\text{supp}\{g_i\}$, are disjoint. Also the g_i satisfy $Pg_i = g_{\alpha(i)}$, where $\alpha(i)$ is a permutation on the numbers $\{1, 2,..., r\}$. For ergodic systems $\alpha(i)$ must be a cyclic permutation.[1] The scaling coefficients $\lambda_i(f_0)$ are linear functionals of the initial density f_0 given by

$$\lambda_i(f_0) = \int_0^1 Y_i(x)\, f_0(x)\, dx$$

where $\{Y_i\}$ is a sequence of L^∞ functions. The symbol Q is called the transient operator, and satisfies $\|Q^t f_0\| \to 0$ as $t \to \infty$. From (4) the tth iterate, $P^t f_0$, may be written as

$$P^t f_0(x) = \sum_{i=1}^{r} \lambda_i(f_0)\, g_{\alpha^t(i)}(x) + Q^t f_0(x) \tag{5}$$

Allowing the transient operator to decay and noting that the permutation $\alpha(i)$ is invertible, we may write

$$P^t f_0(x) = \sum_{i=1}^{r} \lambda_{\alpha^{-t}(i)}(f_0)\, g_i(x) \tag{6}$$

From (6), it is easy to verify that the (necessarily unique) invariant density of an ergodic asymptotically periodic system is given by

$$f^*(x) = \frac{1}{r} \sum_{i=1}^{r} g_i(x) \tag{7}$$

Equation (6) describes a density evolution that is periodic in time. At a given time $P^t f_0$ may be visualized as a linear combination of the basis states g_i, each scaled by a weighting factor $\lambda_{\alpha^{-t}(i)}(f_0)$. Since we are dealing with densities, the $\lambda_i(f_0)$ must sum to 1. Each coefficient $\lambda_i(f_0)$ gives a probabilistic measure of an asymptotically periodic system being in a basis state g_i. Scaling of more than one basis state implies that the system has a probability of being in more than one basis state. When only one term is present in the sum of (6), the system will be found in one g_i state at any given time.

A special case of the spectral decomposition theorem occurs when $r = 1$ in (4). In this situation the flow of densities approaches the invariant density of the system g_1, and $\{P^t f_0\}$ is exact. Exactness in asymptotically periodic systems is also known as *asymptotic stability*. The transition from asymptotic periodicity to asymptotic stability in a noise-perturbed system will be the focus of the last section.

Expectation values of a measurable quantity O at a time t are given by weighting O over the density $P^t f_0$. It is clear from (6) that for asymptotically periodic systems $\langle O \rangle$ will generally be periodic in time. The time dependence of the oscillation is found from (6) to be

$$\langle O \rangle(t) = \sum_{i=1}^{r} \lambda_{\alpha^{-t}(i)}(f_0) \langle O(x) \rangle_i \tag{8}$$

where

$$\langle O(x) \rangle_i = \int_{\text{supp}\{g_i\}} O(x)\, g_i(x)\, dx$$

2.1. The Autocorrelation Function in Asymptotically Periodic Systems

From the densities $P^t f_0$ it is possible to calculate all nonequilibrium properties of the system represented by the map S. As an illustration, the time correlation function is calculated for asymptotically periodic systems using the properties of the decomposition (5) and (6). We assume that we are dealing with an ergodic asymptotically periodic system, so the permutation $\alpha(i)$ is cyclic,[1] i.e., $\alpha(i) = (i+1) \bmod r$, where r is the number of elements in the sum of (5) or (6).

Defining the autocorrelation function as $R_{xx}(\tau) \equiv \langle x_t x_{t+\tau} \rangle$, the time correlation function is $C(\tau) = R_{xx}(\tau) - \langle x \rangle^2$. The autocorrelation function can be written using the properties of the Markov operator as

$$R_{xx}(\tau) = \int_0^1 x P^\tau(x f^*(x))\, dx \tag{9}$$

From the invariant density (7) we have

$$P^\tau\{x f^*(x)\} = \frac{1}{r} \sum_{i=1}^{r} P^\tau\{x g_i(x)\} \tag{10}$$

Also, by (4) we may write $P\{x g_i(x)\} = \lambda_i(x g_i)\, g_{\alpha(i)}(x) + Q\{x g_i(x)\}$, where only one term appears since

$$\lambda_i(x g_i) = \int_{\text{supp}\{g_i(x)\}} Z_i(x)(x g_i(x))\, dx \tag{11}$$

Hence,

$$P^\tau\{x g_i(x)\} = \lambda_i(x g_i)\, g_{\alpha^\tau(i)}(x) + Q^\tau\{x g_i(x)\} \tag{12}$$

Since Markov operators satisfy $\|P^\tau f_0\| = \|f_0\|$, and $\|Q^t(f_0)\| \to 0$ as $t \to \infty$, we have

$$\lambda_i(x g_i) = \langle x \rangle_i \tag{13}$$

Substituting (13) into (12), using the definition (9) of the autocorrelation, and noting that

$$\langle x\rangle = \frac{1}{r}\sum_{i=1}^{r} \langle x\rangle_i \qquad \text{and} \qquad \langle x\rangle^2 = \frac{1}{r^2}\sum_{i,j=1}^{r} \langle x\rangle_i \langle x\rangle_j \tag{14}$$

the time correlation function $C(\tau)$ takes the form

$$C(\tau) = \frac{1}{r}\sum_{i=1}^{r}\left[\langle x\rangle_{\alpha^\tau(i)} - \frac{1}{r}\sum_{j=1}^{r}\langle x\rangle_j\right]\langle x\rangle_i + \sum_{i=1}^{r}\xi_i(\tau) \tag{15}$$

where the $\xi_i(\tau)$ are defined by

$$\xi_i(\tau) = \frac{1}{r}\int_0^1 xQ^\tau(xg_i(x))\,dx \tag{16}$$

By the properties of the transient operator Q the terms $\xi_i(\tau) \to 0$ as $\tau \to \infty$.

The first term of (15) is periodic due to the cyclicity of the permutation $\alpha(i)$. To see this, recall that $\alpha^\tau(i) = (i+\tau) \bmod r$. Extract the $j = (i+\tau) \bmod r$ term of the sum in the curly brackets of (15) and add it to single term $\langle x\rangle_{\alpha^\tau(i)}$. The first term of (15) can then be rewritten as

$$\frac{1}{r}\sum_{i=1}^{r}\left[\frac{r-1}{r}\langle x\rangle_{\alpha^\tau(i)} - \frac{1}{r}\sum_{j,\, j\neq i+\tau}^{r}\langle x\rangle_j\right]\langle x\rangle_i \tag{17}$$

With the aid of the identity

$$\sum_{m=1}^{n-1} e^{2\pi i(mz)/n} = -1 \tag{18}$$

where n and z are integers, with z nonzero, (17) can be written as

$$\frac{1}{r^2}\sum_{i=1}^{r}\left\{\sum_{j=1}^{r}\langle x\rangle_j\left[\sum_{m=1}^{r-1}\exp\left(2\pi i\frac{m(i+\tau-j)}{r}\right)\right]\right\}\langle x\rangle_i \tag{19}$$

The sum $(i+\tau)$ in the exponent of (19) is understood to be modulo r. We make the substitution $k = m+1$ in (19) and use the fact that $e^{2\pi i(I \bmod r)/r} = e^{2\pi i(I/r)}$, hence dropping the "modulo" notation. Also, we define the *discrete frequencies* $\omega_j \equiv 2\pi(j-1)/r$. With these substitutions in (15) the time correlation function becomes

$$C(\tau) = \sum_{m=2}^{r} |\psi(\omega_m)|^2\, e^{i\omega_m\tau} + \sum_{i=1}^{r}\xi(\tau) \tag{20}$$

where

$$\psi(\omega_m) \equiv \frac{1}{r}\sum_{k=1}^{r}\langle x\rangle_k\, e^{i\omega_m(k-1)} \tag{21}$$

Note that with the substitution $k=m+1$ in (19) the periodic part of (20) begins at $m=2$.

An interesting property of asymptotically periodic systems is evident from (20). Namely, the correlation $C(\tau)$ function naturally decomposes into sustained periodic and decaying stochastic components. This decoupling of the time correlation function into two independent components can be understood as follows. Asymptotically periodic systems have r disjoint attracting regions of their phase space X whose union is given by

$$\bigcup_{i=1}^{r} \operatorname{supp}\{g_i\}$$

Each of the regions $\operatorname{supp}\{g_i\}$ maps onto each other cyclically according to $\alpha(i)$. All ensembles of initial conditions will asymptotically map into these regions (i.e., all densities will decompose). Thus a time series will also visit these supports periodically, and we expect a periodic component in the time correlation function. However, iterates of the time series which return into any one of the $\operatorname{supp}\{g_i\}$ are described by a density g_i, and so there must exist a stochastic component of the correlation function [the second term of (20)]. This component is seen to decay to zero by (16).

2.2. The Conditional Entropy for Asymptotically Periodic Systems

Assuming the existence of a density f describing the thermodynamic state of a system at a time t, Gibbs introduced the concept of the *index of probability*, given by $-\log f(x)$. Weighting the index of probability by the density f, he introduced what is now known as the *Boltzmann–Gibbs entropy*, given by

$$H(f) = -\int_X f(x) \log f(x)\, dx$$

It can be shown[9,10] that the Boltzmann–Gibbs entropy is the only (up to a multiplicative constant) entropy definition satisfying the property of being an extensive quantity, which a mathematical analog of the thermodynamic entropy should have.

The Boltzmann–Gibbs entropy can be generalized by introducing the *conditional entropy*. If f and g are two densities such that $\operatorname{supp} f \subset \operatorname{supp} g$, then the conditional entropy of the density f with respect to the density g is defined as

$$H_c(f \mid g) = -\int_X f(x) \log\left[\frac{f(x)}{g(x)}\right] dx \tag{22}$$

Since $H_c(f \mid g) = 0$ when $f = g$, this implies that the conditional entropy is a measure of how close the states characterized by f and g are to each other. Moreover, using the Gibbs inequality, it can be shown[1] that the conditional entropy satisfies

$$H_c(f \mid g) \leqslant 0 \tag{23}$$

When dealing with asymptotically periodic systems the limiting conditional entropy takes on a particularly transparent form, clearly expressing that $\lim_{t \to \infty} H_c(P^t f_0 \mid f^*)$ is dependent on the initial preparation of the system through f_0. To see this, use the invariant density (7) along with the asymptotic decomposition (6) and the orthogonality of the g_i to obtain

$$\begin{aligned} &\lim_{t \to \infty} H_c(P^t f_0 \mid f^*) \\ &\quad = \sum_{i=1}^{r} \int_X \lambda_{\alpha^{-t}(i)}(f_0)\, g_i(x) \log[r \lambda_{\alpha^{-t}(i)}(f_0)]\, dx \\ &\quad = \sum_{i=1}^{r} \int_X [\lambda_{\alpha^{-t}(i)}(f_0)\, g_i(x)]\{\log[\lambda_{\alpha^{-t}(i)}(f_0)] + \log(r)\}\, dx \end{aligned} \tag{24}$$

Also, since the permutation $\alpha(i)$ is invertible, we have

$$\sum_{i=1}^{r} \lambda_{\alpha^{-t}(i)}(f_0) = \sum_{i=1}^{r} \lambda_i(f_0) \tag{25}$$

Thus, defining

$$H_c^{\infty}(P^t f_0 \mid f^*) \equiv \lim_{t \to \infty} H_c(P^t f_0 \mid f^*) \tag{26}$$

we may reexpress the limiting conditional entropy as

$$H_c^{\infty}(P^t f_0 \mid f^*) = -\log(r) - \sum_{i=1}^{r} \lambda_i(f_0) \log \lambda_i(f_0) \tag{27}$$

Noting that the $0 \leqslant \lambda_i(f_0) \leqslant 1$ for all i, we obtain

$$-\log(r) \leqslant H_c^{\infty}(P^t f_0 \mid f^*) \leqslant 0 \tag{28}$$

When an initial density f_0 is localized over one of $\operatorname{supp}\{g_i\}$, then $\{P^t f_0\}$ will asymptotically cycle through the sequence $\{g_i\}$. In this case there is only one component to the spectral decomposition (6) at any time t. According to (27), this situation is one of lowest conditional entropy and $H_c^{\infty}(P^t f_0 \mid f_0) = -\log(r)$. Physically, this implies that the initial ensemble, described by f_0, will evolve through $\bigcup_i \operatorname{supp}\{g_i\}$ in the most localized manner possible. At any time t, only a "pure state" g_i is needed to describe the statistical properties of the system. In general any f_0 whose support runs over the boundary of some g_i will cause $P^t f_0$ to decompose into a linear combination of several densities g_i. At a time t the members of an ensemble are now less localized, and more information is required to determine their distribution through the phase space. As a result, the conditional entropy of a linear combination of several densities g_i has a higher conditional entropy than a single pure state g_i.

Asymptotically periodic systems can also be shown to be irreversible. It has been shown[11] that if P is a Markov operator, then the conditional entropy must satisfy

$$H_c(P^t f \mid f^*) \geqslant H_c(f \mid f^*) \tag{29}$$

By the decomposition (5), the density sequence $\{P^t f_0\}$ settles onto a periodic cycle after a sufficiently long transient. The rate at which the transient decays is controlled by the transient term $Q^t f_0$ in the expansion (5). Since $\|Q^t f_0\| \to 0$ as $t \to \infty$, the conditional entropy of asymptotically periodic systems increases uniformly to the limiting value given by (27), satisfying

$$\Delta H_c(P^t f_0 \mid f^*) \geqslant 0 \tag{30}$$

where Δ denotes the temporal change in H_c. In the special case when f_0 is a linear combination of the states g_i there is no transient term in the expansion (5). In that case the equality in (30) holds.

Equation (27) shows that the unique, limiting conditional entropy of an asymptotically periodic system settles down to a value uniquely determined by the density of the initial preparation of the system, while the iterates $P^t f_0$ remain asymptotically periodic. This implies that all density states within the cycle to which $\{P^t f_0\}$ converges are of the same entropy with respect to the stationary density (7).

3. NOISE-INDUCED ASYMPTOTIC PERIODICITY IN THE KEENER MAP

We now turn to an explicit example of a stochastically perturbed dynamical system that displays asymptotic periodicity. This is the system (1), (2) and we restrict α to $0 < \alpha < 1$. Density evolution in this system is described by the Markov operator

$$Pf(x) = \int_0^1 f(y)\{h(x - S_1(y)) + h(x - S_2(y))\}\, dx \tag{31}$$

where S_1 and S_2 describe the two branches of the map in (2). The support of the kernel $K(x, y)$ in (31) is illustrated in Fig. 1. The width in the x direction of the support branches is θ, where θ is such that $0 \leqslant \xi \leqslant \theta$ in (1). As the Markov operator (31) can been shown[1] to satisfy the conditions of asymptotic periodicity, the density $P^t f_0$ of (31) can be spectrally decomposed as a linear combination of a sequence of densities $\{g_i\}$, i.e.,

$$P^t f_0(x) = \sum_{i=1}^{r} \lambda_{\alpha^t(i)}(f_0)\, g_i(x) + Q^{t-1} f_0(x) \tag{32}$$

In (32) the dependence of $\lambda_{\alpha^t(i)}(f_0)$ and g_i on θ is implied. Hence, $\{P^t f_0\}$ can be either asymptotically periodic ($r > 1$) or stable ($r = 1$). It will be shown that a transition between the two behaviors can be induced by changes in the noise amplitude. Moreover, there will always exist one stationary density f^* for the operator (32), given by

$$f^*(x) = \frac{1}{r} \sum_{i=1}^{r} g_i(x)$$

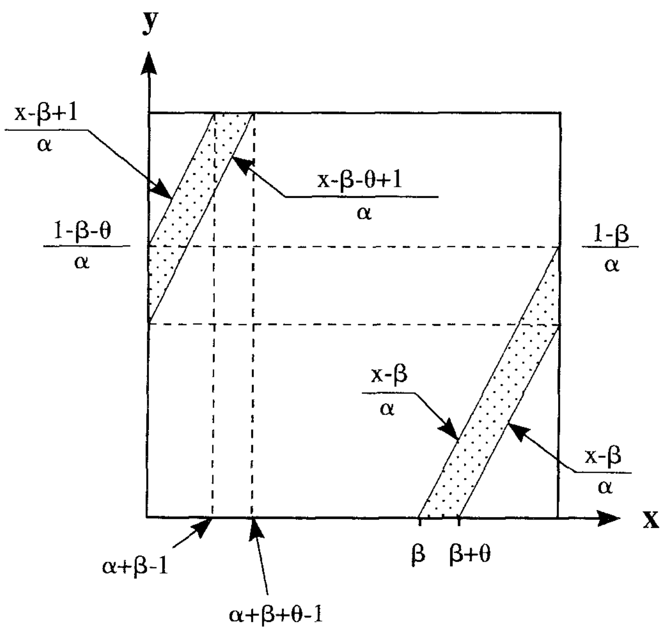

Fig. 1. The support of the kernel $K(x, y)$ of the integral equation (31). Note that its form is just the inverse of the map (2) with bands bounded by straight-line branches. The width of the bands in the x direction is θ.

which is the average of all the elements of the $\{g_i\}$ sequence. Unless otherwise stated, a stationary density refers to this f^*.

3.1. Support Structure of the Sequence $\{P^t f_0\}$

In this subsection we establish conditions by which to determine the regions on [0, 1] where $P^t f_0$ in (31) [or (32)] is nonzero for a given noise amplitude θ. This region is called the support of $P^t f_0$ and is denoted by $\text{supp}\{P^t f_0\}$. The support structure is diagrammatically represented in what is called a support bifurcation diagram. This is a figure illustrating where a density is nonzero, on the vertical axis, as a function of noise amplitude θ on the horizontal axis. In the limit $t \to \infty$, and for a particular choice of f_0 defined below, the support bifurcation diagram of $\{P^t f_0\}$ converges to that of $\text{supp}\{f^*\}$ (or equivalently the union of the supports of the g_i).

To assure that after a sufficiently long transient the supports of the $\{P^t f_0\}$ will coincide with those of f^* in (7), it is clear from (32) that all $\lambda_i(f_0)$ must be nonzero. Thus we must have

$$\lambda_i(f_0) = \int_0^1 Y_i(x)\, f_0(x)\, dx > 0$$

Since the Y_i are not identically equal to zero, an initial density entirely supported on the unit interval guarantees that $\lambda_i(f_0) > 0$ for all i. Thus, the densities $\{P^t f_0\}$ will go to a limiting sequence that satisfies

$$\lim_{t \to \infty} \text{supp}\{P^t f\} = \text{supp}\{f^*\} \tag{33}$$

where f^* denotes the stationary density (7). This is because the limit of the sequence $\{P^t f_0\}$ and f^* are both nonzero on the supports of the g_i. Thus, in order to map the attracting region of phase space on which the densities g_i are supported, or equivalently the supports of the stationary density (7), one need only begin with an initial density f_0 satisfying

$$\text{supp}\{f_0\} = [0, 1] \tag{34}$$

There is yet another result that is useful in establishing the rate at which the support of $P^t f_0$ converges to that of f^*. For the operator (31) it is not difficult to show that if the supports of two successive densities, say $P^{t_0-1} f_0$ and $P^{t_0} f_0$, are identical, then all subsequent densities will have the same supports. Mathematically, if $\text{supp}\, f_0 = [0, 1]$ and

$$\text{supp}\{P^{t_0-1} f_0\} = \text{supp}\{P^{t_0} f_0\} \tag{35}$$

for some t_0 and θ, then

$$\lim_{t \to \infty} \operatorname{supp}\{P^t f_0\} = \operatorname{supp}\{P f^{t_0} f\} = \operatorname{supp}\{f^*\} \tag{36}$$

Equation (36) may be used as follows. If the support bifurcation diagrams of $P^{t_0-1}f_0$ and $P^{t_0}f_0$ overlap for certain values of θ, then these regions remain "frozen" for all later times. For these values of θ we say that the attracting region of phase space (the union of the g_i) has been established by the t_0th iteration.

3.2. An Algorithm for Obtaining $\operatorname{supp}\{P^t f_0\}$

Figure 1 schematically depicts the domain of the kernel $K(x, y)$ of (31) for uniform noise distributed on $[0, \theta]$. It is clear that $\operatorname{supp}\{P^t f\}$ is obtained from $\operatorname{supp}\{P^{t-1} f\}$ by allowing the former to play the role of $f(y)$ in (31), intersecting the domain of the y variable with the domain of $K(x, y)$, and retaining the x projection of this resulting set. Hence, from the diagram any initial density supported on a subset $[a, b]$ of $[0, 1]$ will iterate, under (31), into a density whose supports are in

$$(L \cup U) \cap [0, 1] \tag{37}$$

where the two sets L and U are defined by

$$L = [\alpha a + \beta - 1, \alpha b + \beta + \theta - 1]$$

and

$$U = [\alpha a + \beta, \alpha b + \beta + \theta]$$

respectively, and the splitting is induced by the modulo operator. The two sets L and U originate due to the intersection of $[a, b]$ with the unbounded domain between the two pairs of lines defining the boundaries of $K(x, y)$ in (31). The intersection of $L \cup U$ with $[0, 1]$ ensures that the resulting supports lie inside the unit interval. Note that the two component sets of (37) are shifted from each other by -1. Thus, we define L as the *lower* set and the second set U as the *upper* set.

As an example, consider an initial density f_0 entirely supported on $[0, 1]$. After one iteration a new density f_1 emerges whose supports for a given θ are given by

$$\operatorname{supp}\{f_1\} = [0, \min(\beta + \theta + \alpha - 1, 1)] \cup [\max(\beta, 1), 1] \tag{38}$$

The support bifurcation diagram of (38) can be viewed as the support of f_0 with a removed wedge defined by

$$\Delta_{11} = [\beta + \theta + \alpha - 1, 1] \tag{39}$$

Thus,

$$\text{supp}\{f_1\} = \text{supp}\{Pf_0\} = \text{supp}\{f_0\} \cap \Delta_{11}^c \tag{40}$$

where c denotes the set complement. Allowing (40) to become the domain of $f(y)$ in (31), a second density f_2 emerges whose supports are given as the supports of f_1 intersected with the complement of the union of the wedge-shaped sets,

$$\Delta_{21}[(\beta + \theta)(\alpha + 1) - \alpha - 1, (\alpha + 1)\beta - 1] \tag{41}$$

and

$$\Delta_{22}[(\beta + \theta)(\alpha + 1) + \alpha(\alpha - 1), (\alpha + 1)\beta] \tag{42}$$

Thus,

$$\text{supp}\{P^2 f_0\} = \text{supp}\{Pf_1\} \cap (\Delta_{21} \cup \Delta_{22})^c \tag{43}$$

Continuing in this fashion produces, via (37), an algorithm by which the support structure of the tth density in the sequence $\{P^t f_0\}$ is obtained. To find a general expression of these supports as a function of time is a formidable task, as it is extremely sensitive to the choice of parameters β and α. As an example of this sensitivity, Figs. 2 and 3 illustrate the support

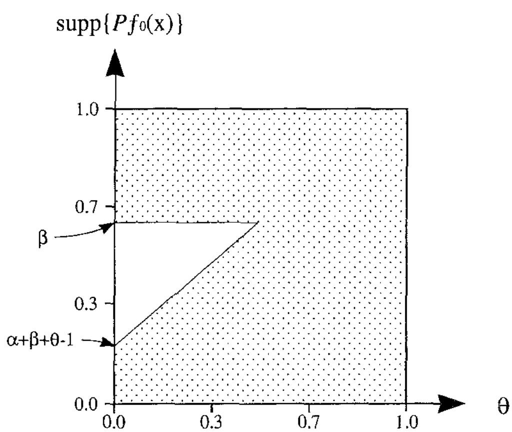

Fig. 2. The support bifurcation diagram of $\text{supp}\{Pf_0\}$ when $\text{supp}\{f_0\} = [0, 1]$. The dotted area is the region of support.

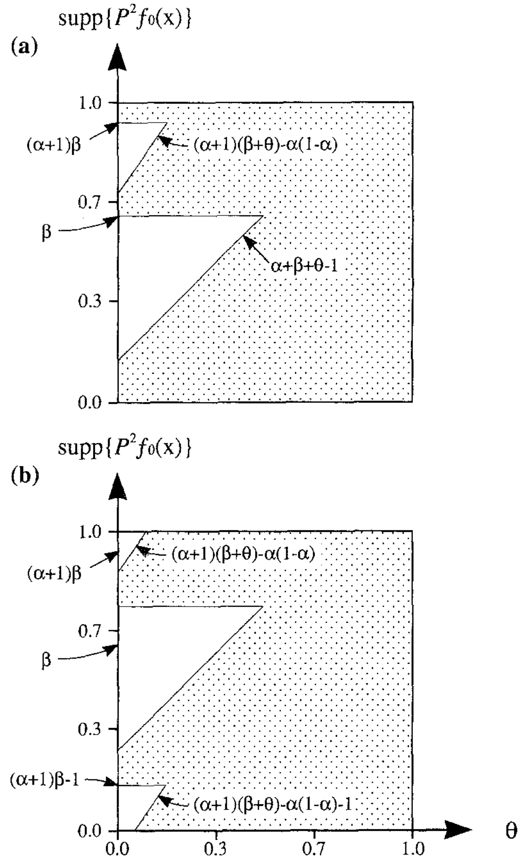

Fig. 3. (a) The support bifurcation diagram of supp$\{Pf_0\}$ for $\beta < 1/(\alpha+1)$. (b) supp$\{Pf_0\}$ for $\beta > 1/(\alpha+1)$. Note the sensitivity of bifurcations of supp$\{P^t f_0\}$ to the choice of β. The dotted area is the region of support.

bifurcation sets of Pf_0 and $P^2 f_0$, respectively. In Fig. 2, $\beta = 1/(\alpha+1)$. In Fig. 3a, β is slightly less than $1/(\alpha+1)$. In Fig. 3b, β is slightly greater than $1/(\alpha+1)$.

3.3. The Supports of f^* for $\beta = 1/(\alpha^m + \cdots 1)$

Using the procedure developed in the last subsection, it is possible to derive an analytic expression defining the supports of the union of the g_i densities when

$$\beta = \frac{1}{\alpha^m + \cdots + 1}, \qquad m = 1, 2, \ldots \tag{44}$$

This choice of β corresponds to period-$(m+1)$ solutions of the noise-free map (2). The addition of noise to (2) with β as in (44) will transform the attracting region of phase space from a sequence of points to a sequence of bands, destroying any predictability related to trajectories of the system (1). However, we will show that the flow of densities does become asymptotically periodic, restoring exact predictability in the statistical sense.

Starting with an initial density satisfying $\operatorname{supp}\{f_0\} = [0, 1]$ and applying (37) repeatedly, the supports of f^* with β as in (44) can be shown[12] to be

$$\operatorname{supp}\{f^*\} = \begin{cases} \left[0, \dfrac{\beta+\theta}{1-\alpha} - \dfrac{1}{1-\alpha^{m+1}}\right] \cup \Pi^*(\alpha, \beta), & 0 \leqslant \theta \leqslant \theta_{\text{crit}}(m+1) \\ [0, \beta+\theta+\alpha-1] \cup \Pi(\alpha, \beta, m+1), & \theta_{\text{crit}}(m+1) < \theta \leqslant 1 \end{cases} \tag{45}$$

where

$$\Pi^*(\alpha, \beta) = \bigcup_{i=0}^{m-1} \left[\beta \sum_{k=0}^{i} \alpha^k, \frac{\beta+\theta}{1-\alpha} - \frac{\alpha^{i+1}}{1-\alpha^{m+1}}\right] \tag{46}$$

and

$$\Pi(\alpha, \beta, m+1) = \bigcup_{i=0}^{m-1} \left[\left(\sum_{k=0}^{i} \alpha^k\right)\beta, \left(\sum_{k=0}^{i+1} \alpha^k\right)(\beta+\theta) + \alpha^{i+2} - \alpha^{i+1}\right] \tag{47}$$

while

$$\theta_{\text{crit}}(m+1) \equiv \frac{\alpha^m(1-\alpha)}{\alpha^m + \cdots + 1} \tag{48}$$

It is straightforward to show that (45) remains invariant under one iteration using (37).

Figures 4a–4c show the evolution of $\operatorname{supp}\{P^t f_0\}$ to the set (45) when $m = 1$. The solid lines in Fig. 4c depict $\operatorname{supp}\{f^*\}$. The line $\theta = \theta_{\text{crit}}(2)$ divides θ space into two parts. For $0 \leqslant \theta \leqslant \theta_{\text{crit}}(2)$ it can be shown that density evolution becomes asymptotically periodic with period two, with the two dotted regions corresponding to the supports of the *two* g_i densities in (32). To the right of $\theta_{\text{crit}}(2)$ density evolution becomes asymptotically stable. In this region the dotted areas, even when disjoint for some values of θ, correspond to the support of *one* invariant density f^*.

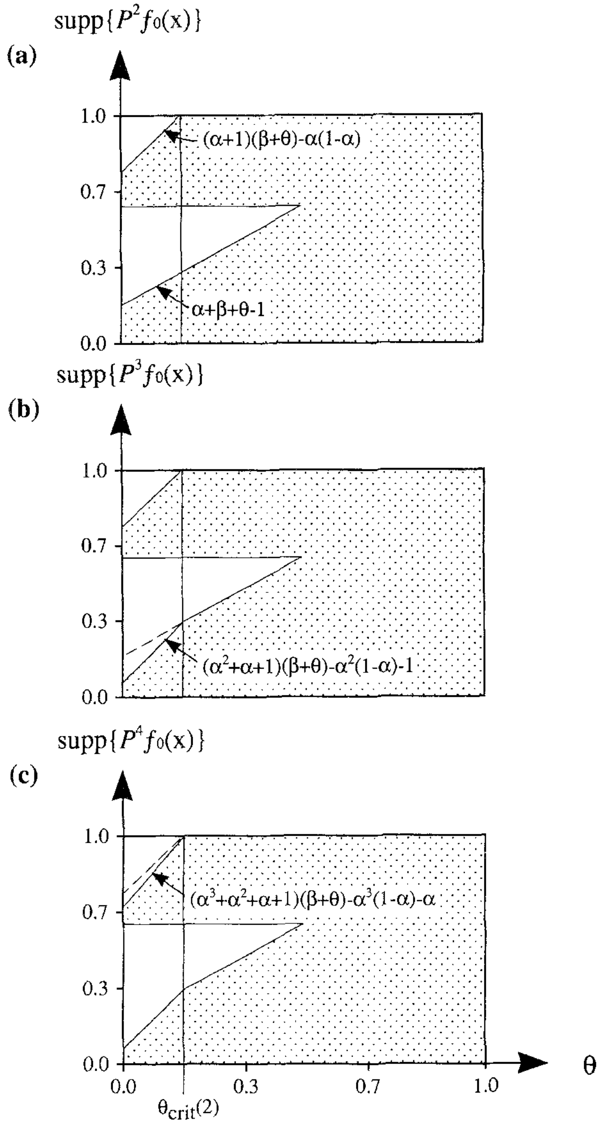

Fig. 4. The evolution of supp$\{P^t f_0\}$ for $\theta \leqslant \theta_{\text{crit}}(2)$, when $\beta = 1/(\alpha+1)$, $\alpha = 1/2$. (b) The first support of part (a) is modified. (c) The second support of part (a) is modified. This process continues cyclically. The dotted area is the asymptotic region of support.

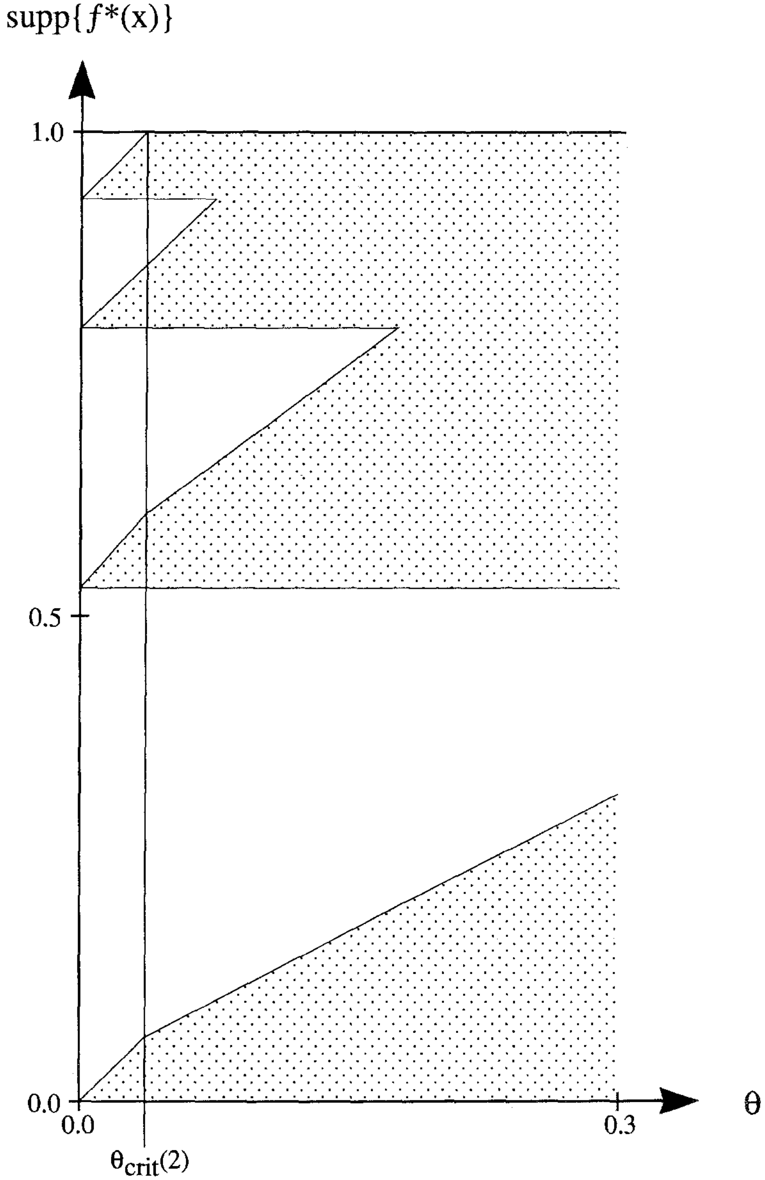

Fig. 5. The support structure of $\lim_{t\to\infty} \text{supp}\{P^t f_0\}$. For $\theta \leqslant \theta_{\text{crit}}(4)$ the limiting support structure is given by (45), where $\text{supp}\{f_0\} = [0, 1]$ is assumed. The dotted area is the region of support.

Figure 5 illustrates the set (45) outlining the invariant density (or the support of the union of the g_i) when $m = 3$. Analogous statements to those made above can be said concerning density evolution to the left and right of $\theta_{\rm crit}(4)$.

3.4. Asymptotic Periodicity for $\theta \leqslant \theta_{\rm crit}(m+1)$

When showing that the set (45) remains invariant under one application of (37), we also discover that for $\theta \leqslant \theta_{\rm crit}(m+1)$ the $(m+1)$ com-

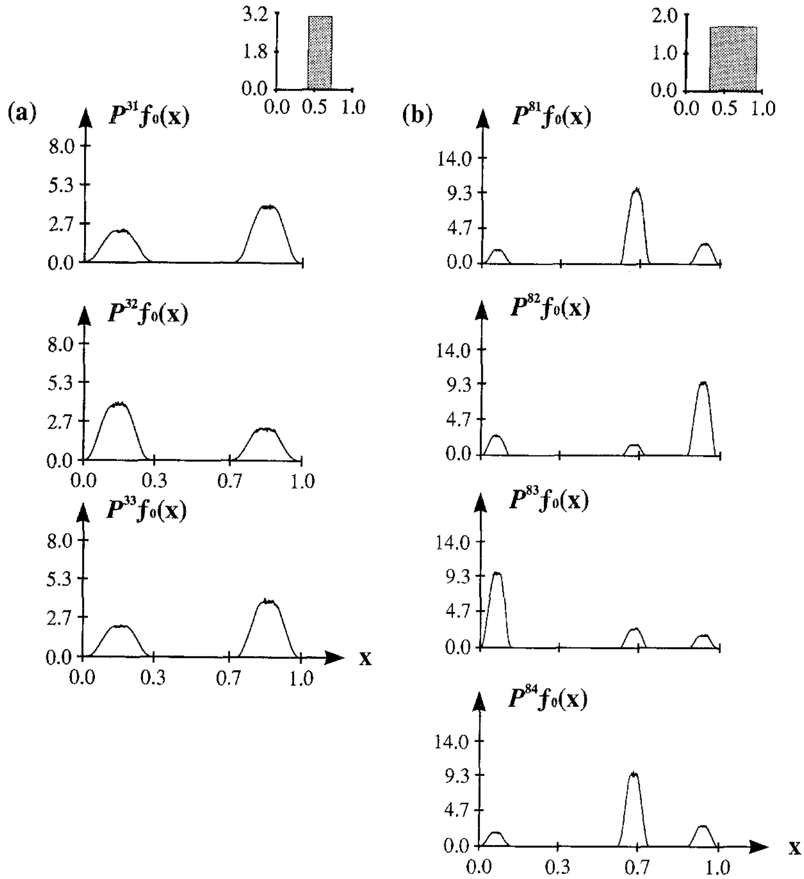

Fig. 6. The emergence of period-two and period-three noise-induced asymptotic periodicity at the onset points (a) $\theta_{\rm crit}(2)$, $\beta = 1/(\alpha+1)$, $\alpha = 1/2$ and (b) $\theta_{\rm crit}(3)$, $\beta = 1/(\alpha^2+\alpha+1)$, $\alpha = 1/2$. (a) A transient of 30 densities has been discarded, and $P^{31}f_0$, $P^{32}f_0$, $P^{33}f_0$ are shown. Since $P^{31}f_0 = P^{33}f_0$, the sequence $\{P^t f_0\}$ repeats with period two. The initial density, shown in the inset, was uniform over [0.4, 0.7]. (b) A transient of 80 densities has been discarded and $P^{81}f_0$ through $P^{84}f_0$ are shown. Since $P^{81}f_0 = P^{84}f_0$, the sequence $\{P^t f_0\}$ repeats with period three. The initial density, shown in the inset, is uniform on [0.3, 0.9].

ponents of (45) map onto each other in a cyclic fashion. In other words, the first set maps exactly onto the second, the second exactly onto the third,..., and the $(m+1)$th exactly onto the first. Since (31) possesses a spectral decomposition, we therefore conclude that each of these $(m+1)$ components of (45), in the θ range specified, is the domain of one of the $(m+1)$ g_i densities in (31). Thus, the sequence $\{P^t f_0\}$ will become asymptotically periodic with period $(m+1)$ for $\theta \leqslant \theta_{\text{crit}}(m+1)$.

Figure 6a numerically illustrates period-two noise-induced asymptotic periodicity of the sequence $\{P^t f_0\}$ with f_0 uniform on $[0.4, 0.7]$ with $\theta = \theta_{\text{crit}}(2)$. Note that both g_1 and g_2 are present in the decomposition (32), but weighted differently. Figure 6b illustrates period-three noise-induced asymptotic periodicity with $\theta = \theta_{\text{crit}}(3)$ and with the initial density chosen uniform on $[0.3, 0.9]$. Note that at any given time the largest contribution to the probability density $P^t f_0$ comes from one of the g_i densities, with some small contribution from the other two.

The fraction of an initial ensemble, distributed according to f_0, that settles onto the supports of the densities g_i represents $\lambda_i(f_0)$ of the spectral decomposition (32). Also, it can be shown[12] that iterates of the sequence $\{P^t f_0\}$ decompose into the $(m+1)$ supports defined by (45) by the $(m+1)$th iteration. When

$$\beta = \frac{1}{\alpha^m + \alpha^{m-1} + \cdots + \alpha + 1} \qquad \text{and} \qquad \theta \leqslant \theta_{\text{crit}}(m+1) \tag{49}$$

we thus have

$$\lambda_i(f_0) = \int_{\text{supp}_i} P^{m+1} f_0(x)\, dx, \qquad i = 1, 2,..., m+1 \tag{50}$$

where supp_i is the boundary defined by (45).

Equation (50) is easily obtained for the period-two case, when $m = 1$. Using the explicit form of the operator (31), we have

$$Pf_0(x) = \begin{cases} \displaystyle\int_{(x-\beta-\theta+1)/\alpha}^{\min[1, (x-\beta+1)/\alpha]} f_0(y)\, dy, & 0 \leqslant x \leqslant \alpha + \beta + \theta - 1 \\ \displaystyle\int_{\min[0, (x-\beta-\theta)/\alpha]}^{(x-\beta)/\alpha} f_0(y)\, dy, & \beta \leqslant x \leqslant 1 \end{cases} \tag{51}$$

When $\theta \leqslant \theta_{\text{crit}}(2)$, the functions in (51), after an infinite number of (even) oscillations, will become $\lambda_1(f_0)\, g_1$ and $\lambda_2(f_0)\, g_2$, respectively. The domains of (51) are the supports of the two g_i densities. Thus, integrating (51) over their respective domains yields the scaling coefficients $\lambda_i(f_0)$:

$$\begin{aligned}\lambda_1(f_0) &= \int_0^{\alpha+\beta+\theta-1} \int_{(x-\beta-\theta+1)/\alpha}^{\min[1,(x-\beta+1)/\alpha]} f_0(y)\,dy\,dx \\ \lambda_2(f_0) &= \int_\beta^1 \int_{\min[0,(x-\beta-\theta)/\alpha]}^{(x-\beta)/\alpha} f_0(y)\,dy\,dx\end{aligned} \tag{52}$$

The dependence of the asymptotic evolution of $\{P^t f_0\}$ on f_0, with $\beta = 1/(\alpha+1)$, $\alpha = 1/2$, and $\theta = 0.14 < \theta_{\text{crit}}(2)$, is shown numerically for three choices of initial density f_0 in Fig. 7. In Fig. 7a the asymptotic sequence $\{P^t f_0\}$ "bounces" between g_1 and g_2, while in Figs. 7b and 7c the period-two asymptotic decomposition involves both g_1 and g_2. Note that although the initial densities in Figs. 7b and 7c are supported on [0.2, 0.6], the associated scaling coefficients $\lambda_1(f_0)$ and $\lambda_2(f_0)$ are different in each case.

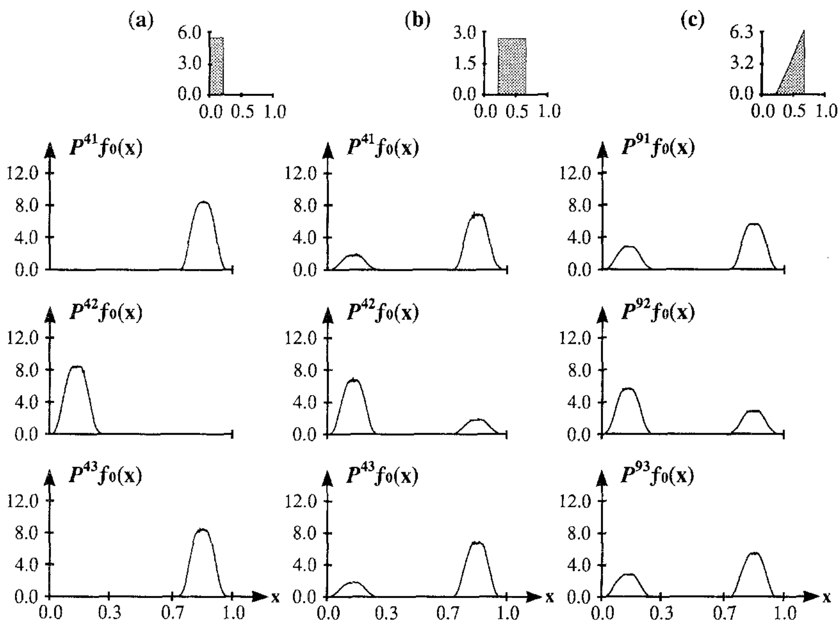

Fig. 7. Numerical simulations of period-two noise-induced asymptotic periodicity of $P^t f_0$ for three initial densities f_0. In all parts, $\beta = 1/(\alpha+1)$, $\alpha = 1/2$, and $\theta = 0.14 < \theta_{\text{crit}}(2)$. (a, b) Forty transient densities have been discarded and the iterates $P^{41}f_0$, $P^{42}f_0$, $P^{43}f_0$ are shown. Since $P^{41}f_0 = P^{43}f_0$, the sequence $\{P^t f_0\}$ repeats with period two. (c) Ninety transients have been discarded and $P^{91}f_0 = P^{93}f_0$. In parts (a) and (b) the initial densities f_0, shown in the insets, are uniform over [0, 0.2] and [0.2, 0.6], respectively, while in part (c), $f_0(x) = 12.5(x - 0.2)$ for $x \in [0.2, 0.6]$.

Figure 8 illustrates period-three asymptotic periodicity in the evolution of the sequence $\{P^t f_0\}$. Now $\beta = 1/(\alpha^2 + \alpha + 1)$, $\alpha = 1/2$, and $\theta = 0.068 < \theta_{crit}(3)$. Each figure corresponds to a different initial density f_0. Note how a slight change in f_0 changes the number of scaling coefficients $\lambda_i(f_0)$, $i = 1, 2, 3$, in the period-three asymptotic decomposition of $\{P^t f_0\}$.

3.5. Transition to Asymptotic Stability for $\theta > \theta_{crit}(m+1)$

To now, asymptotic periodicity has only been considered for values of

$$\theta \leqslant \theta_{crit}(m+1) \qquad \text{and} \qquad \beta = \frac{1}{\alpha^m + \cdots + 1}$$

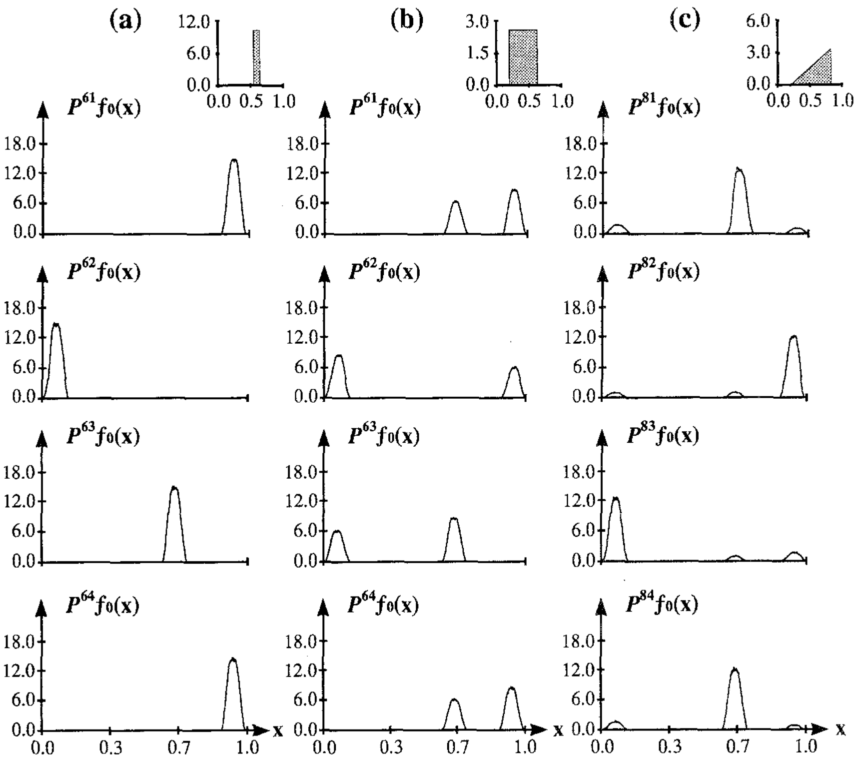

Fig. 8. Numerical simulation of period-three noise-induced asymptotic periodicity of $P^t f_0$ for three initial densities f_0. In all three parts, $\beta = 1/(\alpha^2 + \alpha + 1)$, $\alpha = 1/2$, and $\theta = 0.068 < \theta_{crit}(3)$. (a, b) Sixty transients have been thrown out, showing $P^{61}f_0$, $P^{62}f_0$, $P^{63}f_0$, and $P^{64}f_0$. Since $P^{61}f_0 = P^{64}f_0$, the sequence $\{P^t f_0\}$ repeats with period three. (c) A transient of 80 densities has been discarded and $P^{81}f_0 = P^{84}f_0$. The initial densities f_0 for parts (a) and (b) are uniform on [0.6, 0.7] and [0.2, 0.6], respectively, and in part (c), $f_0(x) = (50/9)(x - 0.2)$ for $x \in [0.2, 0.8]$.

What about the evolution of $P^t f_0$ above $\theta_{\text{crit}}(m+1)$? For $\theta > \theta_{\text{crit}}(m+1)$, *mixing* of an ensemble of initial conditions will take place. For $\theta > \theta_{\text{crit}}(m+1)$, all initial density states will evolve to one unique stationary density. Thus, from the considerations of Section 1 the operator (31), in this θ range, is exact or asymptotically stable. The transition from asymptotic periodicity to asymptotic stability when $\theta > \theta_{\text{crit}}(m+1)$ is proved first for the case when $m=1$.

Theorem 1. Consider the system (1), (2) with $\beta = 1/(\alpha+1)$ and $\theta > \alpha(1-\alpha)/(\alpha+1)$. If f_0 is an initial density, then the sequence of iterates $\{P^t f_0\}$ of the Markov operator (31) is asymptotically stable or exact.

Proof. By (45), the attracting region of phase space for $\theta > \theta_{\text{crit}}(2)$ is given by

$$[0, \beta+\theta+\alpha-1] \cup [\beta, 1] \tag{53}$$

Suppose f_0 is supported on (53). If not, then iterate once so Pf_0 will be supported on (53). Next, partition the subset of [0, 1] defined by (53) as

$$\left(\bigcup_{i=1}^{m_1} S_i\right) \cup \left(\bigcup_{i=1}^{m_2} S_i'\right)$$

where

$$S_i \subset (0, \beta+\theta+\alpha-1)$$

and

$$S_i' \subset (\beta, 1)$$

Here $S_i \cap S_j = \varnothing$ if $i \neq j$ and $S_i' \cap S_j' = \varnothing$ if $i \neq j$. Since Markov operators are linear, let us consider first just one component of f_0 (or Pf_0) supported on S_k or S_k'. Say (without loss of generality) that it is $S_k = (a_0, b_0)$. We wish first to show that the restriction of f_0 to S_k, written as $f_0' = f_0 1_{S_k}(x)$, will eventually spread out to fill the entire attracting region of phase space given by (53).

Iterate f_0' twice, via (37), keeping only that part of $P^2 f_0'$ which returns to the set $(0, \beta+\theta+\alpha-1)$. Then iterate this remainder twice more, again retaining only that part in the set $(0, \beta+\theta+\alpha-1)$, etc. By continuing this procedure and considering the times $t=2n$, $n=1, 2, 3,\ldots$, some algebra yields

$$\text{supp}\{P^{2n} f_0'\} = [\max(0, q_1), \min(\alpha+\beta+\theta-1, q_2)] \tag{54}$$

where

$$q_1 = \alpha^{2n} a_0 + \beta \sum_{k=0}^{2n-1} \alpha^k - \sum_{k=0}^{n-1} \alpha^{2k}$$
$$q_2 = \alpha^{2n} b_0 + (\beta + \theta) \sum_{k=0}^{2n-1} \alpha^k - \sum_{k=0}^{n-1} \alpha^{2k} \tag{55}$$

Taking the limit $n \to \infty$, (54) becomes

$$\lim_{n \to \infty} \operatorname{supp}\{P^{2n} f'_0\} = \left[0, \min\left(\beta + \theta + \alpha - 1, \frac{\beta + \theta}{1 - \alpha} - \frac{1}{1 - \alpha^2}\right)\right] \tag{56}$$

Similarly, when $t = 2n + 1$, $n = 1, 2, 3,...$,

$$\operatorname{supp}\{P^{2n+1} f'_0\} = [\max(\beta, q_3), \min(1, q_4)] \tag{57}$$

where now

$$q_3 = \alpha^{2n+1} a_0 + \beta \sum_{k=0}^{2n} \alpha^k - \alpha \sum_{k=0}^{n-1} \alpha^{2k}$$
$$q_4 = \alpha^{2n+1} b_0 + (\beta + \theta) \sum_{k=0}^{2n} \alpha^k - \alpha \sum_{k=0}^{n-1} \alpha^{2k} \tag{58}$$

Thus,

$$\lim_{n \to \infty} \operatorname{supp}\{P^{2n+1} f'_0\} = \left[\beta, \min\left(1, \frac{\beta + \theta}{1 - \alpha} - \frac{\alpha}{1 - \alpha^2}\right)\right] \tag{59}$$

Now, for $\theta > \theta_{\text{crit}}(2)$, (56) ad (59), respectively, satisfy

$$\frac{\beta + \theta}{1 - \alpha} - \frac{1}{1 - \alpha^2} > \beta + \theta + \alpha - 1$$

and

$$\frac{\beta + \theta}{1 - \alpha} - \frac{\alpha}{1 - \alpha^2} > 1$$

Thus there must exist an even integer M_0 such that

$$\operatorname{supp}\{P^{M_0} f'_0\} = [0, \beta + \theta + \alpha - 1] \tag{60}$$

Iterating (60) once more via (37) and retaining both the *upper* (U) and *lower* (L) sets that emerge gives

$$P^{M_0 + 1} f'_0 = [0, \varepsilon] \cup [\beta, 1]$$

where $\varepsilon < \beta + \theta + \alpha + 1$. Allowing $[0, \varepsilon]$ to play the role of $[a_0, b_0]$ above, eventually $[0, \varepsilon]$ (after an infinite number of even iterations) will spread over the entire interval $[0, \beta + \theta + \alpha - 1]$. Thus,

$$\lim_{n \to \infty} \operatorname{supp}\{P^{2n}P^{M_0}f'_0\} = (0, \beta + \theta + \alpha - 1) \cup (\beta, 1)$$

and so the restriction of f_0 to S_k spreads out fill the attracting domain of f^* given by (53). Since S_k was an arbitrary component of a density on $[0, 1]$, any component of an initial density will spread to cover the whole attracting part of the phase space when $\theta > \theta_{\text{crit}}(2)$, i.e.,

$$\lim_{t \to \infty} \operatorname{supp}\{P^t f_0\} = (0, \beta + \theta + \alpha - 1) \cup (\beta, 1) \tag{61}$$

for $n > N_0(f_0)$, where $N_0(f_0)$ is some integer and $f_0 \in D$. Lasota and Mackey (ref. 3, Theorem 5.6.1) have shown that if condition (61) holds for operators like (31), then the iterates $\{P^t f_0\}$ will be *asymptotically stable* for all $f_0 \in D$. This completes the proof. ∎

A generalization of the proof of Theorem 1 can be extended to the case when

$$\beta = \frac{1}{\alpha^m + \cdots + 1}$$

and

$$\theta > \theta_{\text{crit}}(m+1)$$

The idea is to begin with an initial density supported on the supports of the attracting region of phase space, given by (45), with $t = m + 1$. If this is not the case, then we just iterate $(m+1)$ times and then $P^{m+1}f_0$ will be supported on the attracting region. Then, by algebra similar to that used in the proof of the case $m = 1$, it is demonstrated that any subinterval of an initial density will fill up all of the attracting region of phase space given by (45). This condition again suffices to cause $P^t f_0$ to converge strongly to f^* for $\theta > \theta_{\text{crit}}(m+1)$.

The origin of this transition to asymptotic stability above the critical onset value $\theta_{\text{crit}}(m+1)$ is the nonlinearity induced by the modulo operator of the map (2). The β values (44) give rise to period-$(m+1)$ solutions when $\theta = 0$. However, the addition of noise destroys this periodic condition, giving rise to a noisy periodicity if the noise amplitude is low enough. The kernel of (31) is nothing more than the inverse of the map (2) with bands rather than lines. Hence, when $0 \leqslant \theta \leqslant \theta_{\text{crit}}(m+1)$, a point in this noisy orbit will jump from one band of (53) to another with probability one, although the motion within any one of these bands is completely stochastic. We saw that an ensemble (density) of initial conditions evolving in such a manner gave rise to asymptotic periodicity of the sequence $\{P^t f_0\}$. The stochasticity within each of these bands is entirely determined in terms of the statistical properties of the densities g_i. When $\theta > \theta_{crit}(m+1)$, however, a noisy orbit in the mth band has a finite probability of jumping to either the $(m+1)$th band or back into the first band. This is because of the modulo operation in (2), coupled with the fact that the maximum noise amplitude $\theta > \theta_{crit}(m+1)$ can cause a trajectory to

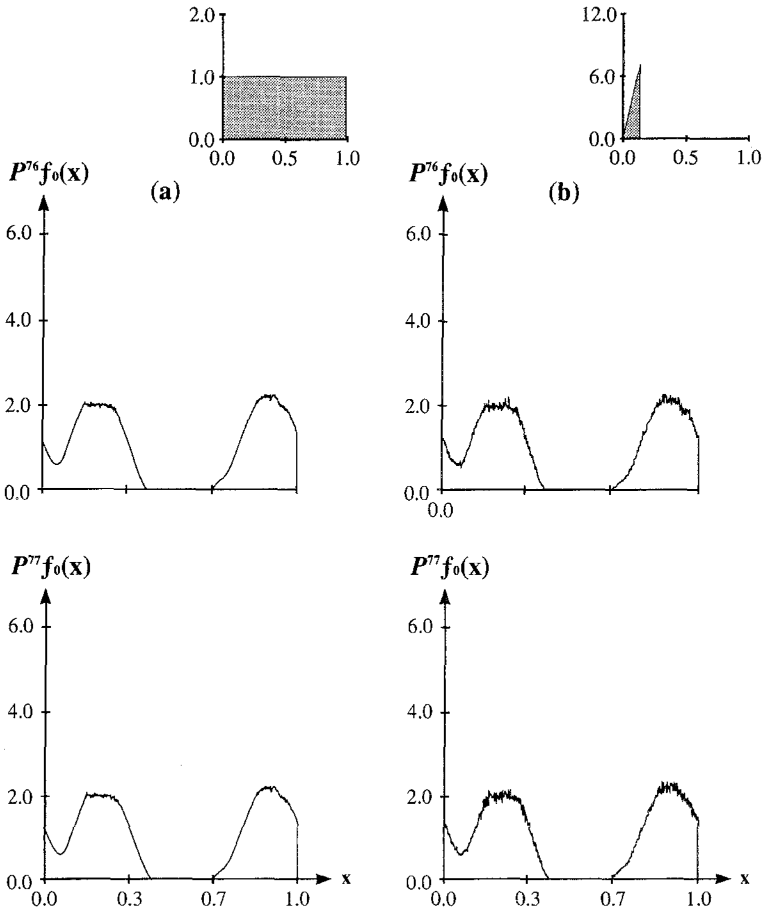

Fig. 9. Numerical illustration of the strong convergence of $P^t f_0$ to a stationary density f^* for two choices of f_0. In both parts $\beta = 1/(\alpha + 1)$, $\alpha = 1/2$, and $\theta = 0.24 > \theta_{crit}(2)$, and a transient of 75 iterations has been discarded. Note in both parts that $P^{76} f_0 = P^{77} f_0$. The initial densities f_0 were uniform on $[0, 1]$ for part (a) and $f_0(x) = 50x$ for $x \in [0, 0.2]$ for part (b).

overshoot $x = 1$. This effect will eventually *mix* an ensemble of initial conditions throughout the attracting region of phase space given by (45). As a result, $\{P^t f_0\}$ settles toward an asymptotically stable invariant density. It is for this reason that we say that each of the components of (45), with $t = m + 1$, is in fact part of one density g_i, $\theta > \theta_{\text{crit}}(m+1)$.

Figure 9 numerically illustrates asymptotic stability of $P^t f_0$ to f^* when $\theta = 0.24 > \theta_{\text{crit}}(2)$, $\beta = 1/(\alpha+1)$, and $\alpha = 1/2$ for two choices of initial density f_0. In Figs. 10a and 10b asymptotic stability of $\{P^t f_0\}$ to f^* is illustrated for $\theta = 0.14 > \theta_{\text{crit}}(3)$, $\beta = 1/(\alpha^2 + \alpha + 1)$, and $\alpha = 1/2$. Once more two different initial densities f_0 were chosen.

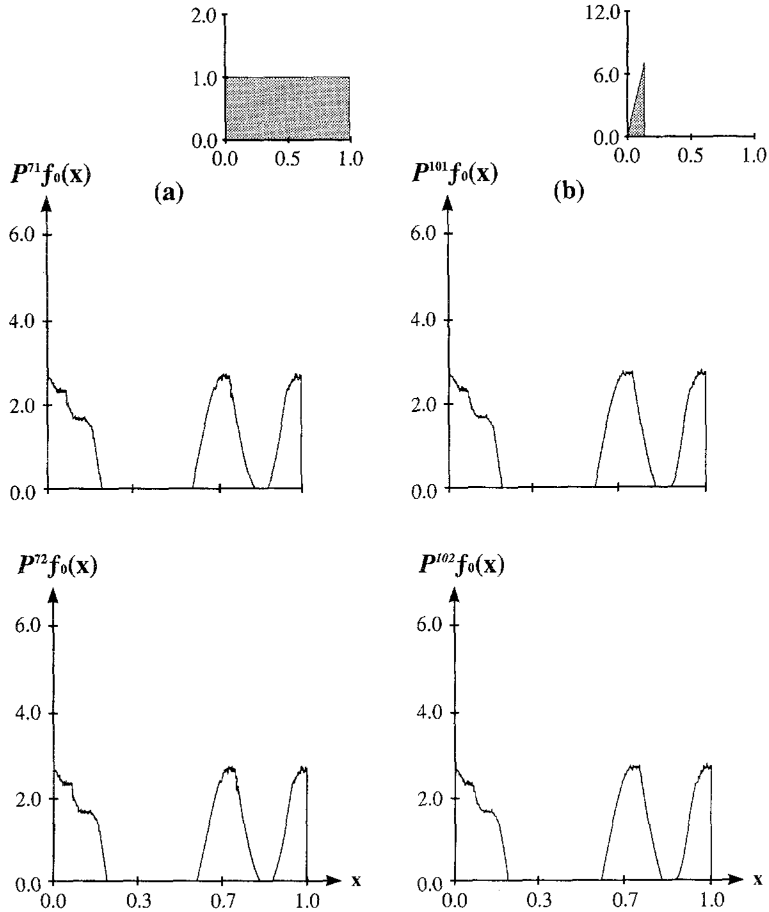

Fig. 10. Strong convergence of $P^t f_0$ to f^* for two different f_0. In both figures $\beta = 1/(\alpha^2 + \alpha + 1)$, $\alpha = 1/2$, $\theta = 0.14 > \theta_{\text{crit}}(3)$. (a) A transient of length 70 iterations has been discarded showing $P^{71} f_0 = P^{72} f_0$, where f_0 is uniform on $[0, 1]$. (b) After 100 transients have been discarded, $P^{101} f_0 = P^{102} f_0$. The initial density was $f_0(x) = 50x$ on $[0, 0.2]$.

4. SUMMARY

We have considered the flow of densities $P^t f_0$ under the action of the map (2) stochastically perturbed by noise. The evolution of the supports of the densities $P^t f_0$ were studied, giving a general algorithm for obtaining $\text{supp}\{P^t f_0\}$. Analytic expressions for $\text{supp}\{P^t f_0\}$ and $\text{supp}\{f^*\}$ can be obtained when β is such that the noise-free map (2) has period-$(m+1)$ solutions. For these values of β, the iterates $P^t f_0$ of the Markov operator (31) were shown to be asymptotically periodic over a restricted region of θ space and asymptotically stable over another. The states of a periodic cycle show a clear dependence on the initial density f_0 with which the system is prepared. In the stable regime, however, $P^t f_0$ strongly converges toward f^* independent of f_0. Also, when $P^t f_0$ is asymptotically periodic, the periodicity of these phase space averages is of the same time scale over which the states $P^t f_0$ vary. For θ in the asymptotically stable range, all periodic variations die out as $P^t f_0$ approaches the equilibrium density f^*.

The noise-induced transition between asymptotic periodicity to asymptotic stability, as θ is varied, is also discussed. The transition can be associated with a switch from several multiply coexisting equilibrium states in $P^t f_0$ to one globally stable state f^*. Since the states $P^t f_0$ were assumed to represent thermodynamic states of a dynamical system, the transition would therefore mirror a change from periodic to stable properties of macroscopic observables for the system described by (1).

An interesting feature to be noted is that the periodic and stable dynamics of $P^t f_0$ are brought about by the addition of noise to the system (2). A noisy orbit or trajectory cannot yield precise information about the future evolution of a system. However, the density sequence $\{P^t f_0\}$ can yield this information in the form of phase space averages.

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