Analog simulation of delayed dynamics: bifurcations and chaos.

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Abstract

An electronic analog computer has been constructed to model the behavior of a delayed feedback system described by an inhomogeneous nonlinear first order delay differential equation (D.D.E). We give a description of circuit design and performance. The circuit allows the rapid reproduction of analytic results for fixed delays and constant initial functions (I.F's), thus demonstrating the feasibility of this approach for the investigation of D.D.E dynamics. Periodic solutions, bifurcations and aperiodic solutions have been observed. Using the analog computer we discovered multistability in a paradigm equation for mixed delayed feedback systems with non-constant I.F's.

Introduction

Delays are ubiquitous. The impossibility of instantaneously transmitting information between two systems poses a fundamental constraint on any theory describing physical interactions. The accurate modeling of physical processes remains possible if the dynamics of the system being considered do not have a dramatic dependence on delays. If the time scale of the process under consideration is comparable to the time scale of the delays then it becomes essential to construct a model that depends on these delays. Typical systems for which such problems arise are delayed feedback systems. These are most often encountered in biological sciences since they regulate the processes that maintain living organisms. The electronic analog computer we constructed integrates a class of such systems with mixed piecewise constant delayed feedback. We focused our attention on the following equation:

$$\frac{dx}{dt} = -\alpha x(t) + F[x(t-\tau)] \qquad (1)$$

where $F(y) = c$ if $y \in [\theta_1, \theta_2]$ and $F = 0$ otherwise; $c>0$.

Equation (1) was first discussed by Mackey and an der Heiden as a paradigm equation for mixed delayed feedback systems [1]. It was used by A. Longtin and J. Milton to describe complex oscillations in the human pupil light reflex [2].

The analog computer

Analog simulations might appear to be an archaic way to investigate D.D.E's. In fact several methods to solve these equations have been developped both analytically and numerically. So far, only the simplest equations in simple situations (absence of noise, constant initial functions, piecewise constant feedback functions) have been solved. The use of a task specific analog computer allows us to study *in real time* the behavior of a physical system built to be accurately described by a D.D.E. It also facilitates rapid scans of parameter space and the study of the bifurcation diagram as a parameter is varied continuously.

Our circuit is a closed loop oscillator. The circuit simulating the integral equation equivalent to (1) amplifies the present signal, delays it to transform it according to the desired feedback function simultaneously, sums the resulting signals, integrates the sum and then equates the result of the integration with the initial signal. These operations are schematically represented in Fig 1.

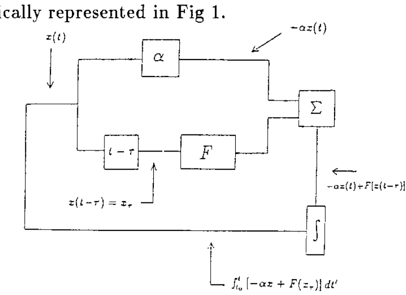

Figure 1. *Schematic diagram of the analog computer. The voltage x(t) satisfies the integral delay equation:* $x(t) - x(t_0) = \int_{t_0}^{t} -\alpha x(t') + F[x(t'-\tau)]dt'$, *equivalent to the original D.D.E. We simulate the integral equation to avoid a differentiation stage in the circuit.*

The amplification, summing, and integration are performed by standard operational amplifiers. The delay is provided by an analog delay line (a bucket brigade device or B.B.D.). Comparators and logic switches are used to create the desired feedback functions. The turn-on state of the B.B.D is the initial function for the system. By interfacing the analog computer with a digital computer we can program any initial function (I.F) into the delay line. Control of the I.F's is of crucial importance when we study multisability.

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Fig. 2 shows some of the signals obtained with the analog computer using constant initial functions. They display quantitative agreement with the analytic solutions when the limit cycles are simple.

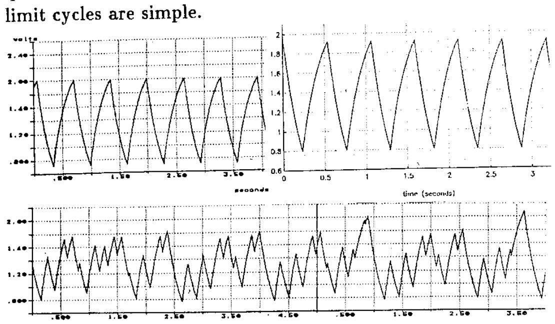

Figure 2. *The most simple limit cycle obtained electronically is plotted here (top left) and compared to an analytic solution of equation (1) (top right) for the same parameter values and I.F. As θ_1 is increased a series of bifurcations lead to chaotic behavior. The bottom trace shows a portion of a signal produced by the analog computer for which no period was found.*

When the limit cycles are more complicated the strict quantitative agreement with the theoretical predictions is lost. However it is kept up to a shift in parameter space. This is due to the fact that dynamically relevant parameters cannot be measured accurately on the circuit when the limit cycles are too complicated.

This is not a serious problem. The oscillator is not designed to reproduce the most erratic solutions of equation (1) because any analog circuit will have a finite cutoff frequency. Rather we want to use this novel tool to further our understanding of delayed dynamics. We focus our attention on the influence of non-constant I.F.'s on solution behavior.

Multistability

Suppose that a physical process can be described with a D.D.E. Then the initial preparation of the setup is in fact the function defined on the initial interval [$-\tau$,0]. The nature of the influence on the evolution of a system by its initial preparation is not well understood in the case of delayed dynamics. The main difficulty arising when the problem is approached analytically being that D.D.E's are ***infinite dimensional*** equations. The existence of a solution requires the specification of a function defined everywhere on the initial interval.

Here the problem is approached using the analog computer. The turn-on state of the B.B.D is the initial function. The oscillator can be started with constant initial conditions. The bifurcation diagram obtained when θ_1 is increased slowly towards θ_2 (keeping all other parameters in the equation constant) can then be studied. After increasing θ_1 we lower it and find *hysteresis* in the bifurcation diagram. That is, different solutions of equation (1) are found at the same parameter values depending on whether θ_1 is being raised or lowered. This indicates the possible existence of multistability in the system. In fact multistability in D.D.E's has been observed [3], but only by varying the value of a constant I.F. In our case the I.F's are non-constant since they are solutions of (1).

The next step in the study of multistability is the control of non-constant I.F's.

The B.B.D is a sampling device. It is interfaced with a digital computer which sends the desired I.F to the delay line through a D/A converter. A switching circuit closes the loop when the delay line is filled and the circuit starts oscillating. For the sake of simplicity we choose to illustrate the possible existence of multistability with I.F's which cross both θ_1 and θ_2 only once. The crossing times of θ_1 and θ_2 on the initial interval are labeled t_1 and t_2. Fig 3. shows two solutions corresponding to the same parameter values for different initial conditions. This result is not a *proof* of multistability but a strong indication of its presence in the system.

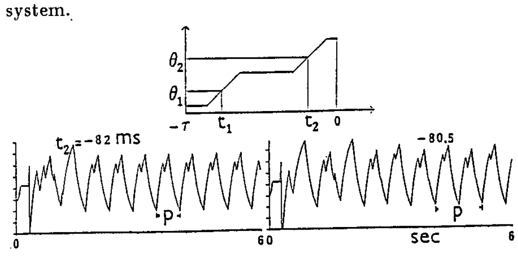

Figure 3. *A change in the parameter t_2 of the initial function (top) leads to a doubling of the period (P) of the electronically generated solutions (left vs right) with all other parameters held constant.*

This result is interesting because period doubling bifurcations are usually produced by changes in the parameters, not by changes in the initial conditions. The analog computer is not accurate enough and further analytical work is required to understand the nature of these bifurcations. Nevertheless it has given us a flavor of the complexity displayed by delayed dynamics, and it has put forward the problem of initial conditions, too often set aside when dealing with delay dependent models.

Conclusion

This is to our knowledge the first attempt ever made to construct an analog computer designed to investigate the dynamics of a delay differential equation. The high degree of stability of our system and the possibility of real time analysis make it a privileged mean of investigating the complex behavior of D.D.E's. This is not a tool for rigourous analysis, but it has highlighted the sensitivity of these systems to initial conditions.

References

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