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Consumer Memory and Price Fluctuations in Commodity Markets: An Integrodifferential Model

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A model for the dynamics of price adjustment in a single commodity market is developed. Nonlinearities in both supply and demand functions are considered explicitly, as are delays due to production lags and storage policies, to yield a nonlinear integrodifferential equation. Conditions for the local stability of the equilibrium price are derived in terms of the elasticities of supply and demand, the supply and demand relaxation times, and the equilibrium production-storage delay. The destabilizing effect of consumer memory on the equilibrium price is analyzed, and the ensuing Hopf bifurcations are described.

KEY WORDS: Commodity markets; time delays; stability; Hopf bifurcation.
AMS (MOS) SUBJECT CLASSIFICATIONS: 34K15, 45J05, 90A16.

1. INTRODUCTION

Trade cycles, business cycles, and fluctuations in the price and supply of various commodities have attracted the attention of economists for well over 100 years and possible more than thousands of years (Weidenbaum and Vogt, 1988). Early authors often attributed these fluctuations to random factors, e.g., the weather for agricultural commodities (Kalecki, 1952; Slutzky, 1937).

Other workers speculated that economic cycling or fluctuations might be an inherent endogenous dynamical behavior characteristic of unstable economic systems (Ezekiel, 1938). A number of business cycle models postulating the existence of nonlinearities to account for limit cycle

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behavior have played a fundamental role in sharpening the debate between the proponents of the exogenous versus endogenous (or stochastic versus deterministic) schools [cf. Zarnowitz (1985) and the references therein].

The recent explosive developments of modern dynamical system theory (Glass and Mackey, 1988; Guckenheimer and Holmes, 1983; Lasota and Mackey, 1985) have shed new light on this debate. The possibility that economic fluctuations may reflect underlying periodic or chaotic dynamics in nonlinear economic systems has been explored in various contexts [Gabisch and Lorenz (1987), Goodwin *et al.* (1984), Grandmont and Malgrange (1986) and references therein].

Interestingly enough, this recent rekindled interest in the techniques from dynamical systems theory has almost exclusively ignored the potential role of production delays in generating fluctuations in economic indicators [Howroyd and Russel (1984) is an exception]. This omission is surprising for two reasons. The first is historical, because Ricci (1930), Schultz (1930), and Tinbergen (1930) almost simultaneously utilized the known lag between the initiation of production decisions and the delivery of goods to discuss commodity cycles in a discrete time mathematical framework that became known as cobweb theory (Ezekiel, 1938; Kaldor, 1933; Leontief, 1934; Waugh, 1964). Further, Kalecki (1935, 1937, 1943, 1952, 1972), Haldane (1933), Goodwin (1951), and Larson (1964) all developed continuous time theories of cyclic economic behavior formulated as delay differential equations. Secondly, the broad spectrum of dynamic behaviors to be found in nonlinear delay differential equations is now well documented (Glass and Mackey, 1979, 1988; an der Heiden, 1979, 1985; an der Heiden and Walther, 1983; an der Heiden and Mackey, 1982, 1987; an der Heiden *et al.*, 1981; Kaczmarek and Babloyantz, 1977; Lasota, 1977; Mackey and Glass, 1977; Mackey and an der Heiden, 1984; Mates and Horowitz, 1976; Nisbet and Gurney, 1976; Peters, 1980; Saupe, 1982; Walther, 1981, 1985; Wazewska-Czyzewska and Lasota, 1976).

In this paper, we analyze an integrodifferential equation to model the price dynamics of a single commodity market. Supply and demand schedules are explicitly accounted for, as well as production plus storage delays and consumer memory. This model is derived in Section 2. In Section 3, we consider the local stability of the equilibrium price, relating this stability to the relevant economic parameters. The limiting case of no consumer memory is considered first, and then the bifurcations introduced by finite average consumer memory are treated. Stability of the equilibrium price is lost at a supercritical Hopf bifurcation, giving rise to an oscillatory variation in the commodity price. When a finite average consumer memory is operative, there is a possibility for subcritical Hopf bifurcations, and an arbitrary number of sub- and supercritical Hopf bifurcations. In Section 4, the well-known discrete time cobweb formulation and the continuous time models of Haldane and Larson are formally derived as limiting cases of our model.

The main results of this paper are contained in Section 3, where we investigate the destabilizing role of consumer memory on the equilibrium price. In particular, we find that stability switches (Cooke, 1985) can be introduced by the existence of a finite average consumer memory.

2. FORMULATION OF THE MODEL

In considering the dynamics of price, production, and consumption of a particular commodity, we assume that relative variations in *market price* $P(t)$ are governed by the equation

$$\frac{1}{P}\frac{dP}{dt}=f(D(P_D), S(P_S)), \tag{2.1}$$

where $D(\cdot)$ and $S(\cdot)$, respectively, denote the *demand* and *supply functions* for the commodity in question. The arguments of the demand and supply schedules are given by P_D (*demand price*) and P_S (*supply price*), respectively, rather than simply the current market price P, for reasons detailed below. It is further assumed that the minimum demand is always exceeded by the maximum supply,

$$\min_{P_D} D(P_D) \leqslant \max_{P_S} S(P_S). \tag{2.2}$$

The *price change function* $f(D, S)$ relates the relative change in market price $[(dP/dt)/P]$ to the imbalance between demand and supply, and satisfies the conditions:

(i) $f(D, S)=0$ when $D=S$; and

(ii) $\dfrac{\partial f}{\partial D}=f_D \geqslant 0,$

$\dfrac{\partial f}{\partial S}=f_S \leqslant 0.$

In a simple case f might be given by

$$f(D, S)=D-S.$$

2.1. Demand Price

In specifying how consumer behavior affects commodity demand, we assumed that this behavior is governed by an integration of information regarding past prices. Thus, demand for a commodity is a weighted function (P_D) of past prices. In calculating this weighted function, we assume that, at time t, the consumer attaches a weight $K_D(t-u)$ to a past market price $P(u)$, where $-\infty \leqslant u \leqslant t$, and that the weighted average of all of these past prices is just the demand price:

$$P_D(t) = \int_{-\infty}^{t} K_D(t-u)\, P(u)\, du. \tag{2.3}$$

The weighting function $K_D(q)$, the *demand price kernel*, is assumed to be normalized:

$$\int_0^{\infty} K_D(q)\, dq = 1.$$

2.2. Supply Price

To complete the formulation of the model, the relation between current market price P and the supply price P_S must be specified. We assume that producers, like consumers, take past market prices into account when making decisions to initiate alterations in production using an analogous normalized weighting function, the *supply price kernel* $K_S(q)$. For most commodities, there is a finite minimum time $T_{\min} \geqslant 0$ that must elapse before a decision to alter production is translated into an actual change in supply. In agricultural commodity markets, this delay is related to biological constraints (e.g., the gestation plus growth period). Then the supply price P_S is given by

$$P_S(t) = \int_{-\infty}^{t-T_{\min}} K_S(t-T_{\min}-u)\, P(u)\, du, \tag{2.4}$$

in complete analogy with the demand price.

Equations (2.1), (2.3), and (2.4), in conjunction with a specification of the functions $f(D, S)$, $D(P_D)$, and $S(P_S)$, complete the formulation of the model when an initial function $P(t_0)$, $-T_{\min} \leqslant t_0 \leqslant 0$, is given. The uniqueness of the model lies in the dynamics of market price being governed by a nonlinear integrodifferential equation.

3. EQUILIBRIUM AND STABILITY

In this section, we determine the steady state or equilibrium price for the model derived in the previous section, and investigate the local stability of the equilibrium price as a function of the various economic factors.

Denote the demand and supply prices P_D and P_S for which demand and supply are equal by P^*. Since it has been specified that $f(D, S) = 0$ when $D = S$, it is clear that

$$\frac{1}{P^*}\frac{dP^*}{dt} = 0.$$

Thus, P^* is the *equilibrium*, or *steady state*, *price* of the commodity, and is implicitly determined by the relation

$$D(P^*) = S(P^*).$$

Because of the specified properties of D and S in Eq. (2.2), at least one equilibrium price must exist, though the existence of more than one equilibrium price is not excluded.

In examining the stability of the equilibrium price P^*, we would like to determine, as generally as possible, the conditions under which P^* is *globally asymptotically stable.* Because of the inherent nonlinearity of the model as formulated, this global stability cannot, in general, be determined. One must be content with an examination of the *local asymptotic stability* of the equilibrium price P^*, i.e., stability under small perturbations. The local stability of the equilibrium price does not, by itself, offer any insight into the global stability of P^* without additional study. However, the local instability of the equilibrium price in turn guarantees the global instability of P^*, and may indicate the existence of limit cycle or other behavior in the full model.

We expand all nonlinearities in the model derived in the previous section in a Taylor's series about the equilibrium price P^*, and discard all nonlinear terms. We then define a new variable $z(t) = P(t) - P^*$ and ultimately find that $z(t)$ satisfies the linear variational equation

$$\frac{dz}{dt} = P^*[f_D^* D_{P_D}^* z_D + f_S^* S_{P_S}^* z_S]. \tag{3.1}$$

In Eq. (3.1), the symbols have the following meaning:

$$f_D^* = f_D(D^*) \qquad \text{where} \quad D^* = D(P^*)$$

$$f_S^* = f_S(S^*) \qquad \text{where} \quad S^* = S(P^*)$$

$$D_{P_D}^* = \left.\frac{\partial D}{\partial P_D}\right|_{P_D = P^*}, \qquad S_{P_S}^* = \left.\frac{\partial S}{\partial P_S}\right|_{P_S = P^*}$$

The value of $D_{P_D}^*$ gives the slope of the demand function D with respect to the consumer weighted demand price P_D evaluated at the equilibrium price

P^*, with a corresponding meaning for $S^*_{P_S}$. Finally, the variables z_D and z_S in Eq. (3.1) are given by the integrals

$$z_D(t) = \int_{-\infty}^{t} K_D(t-u)\, z(u)\, du, \tag{3.2}$$

and

$$z_S(t) = \int_{-\infty}^{t-T_{\min}} K_S(t - T_{\min} - u)\, z(u)\, du. \tag{3.3}$$

The coefficients in the linear variational Eq. (3.1) for $z(t)$ can be rewritten in terms of the elasticities of demand and supply. By definition, the *elasticity of demand*, e_D, is given by

$$e_D = -\frac{D^*_{P_D}}{D^*/P^*},$$

while the *elasticity of supply* is, correspondingly,

$$e_S = \frac{S^*_{P_D}}{S^*/P^*}.$$

Thus, (3.1) may be rewritten as

$$\frac{dz}{dt} = -A_D z_D - A_S z_S, \tag{3.4}$$

where $A_D = e_D/T_D$, $A_S = e_S/T_S$, and $T_D = (D^* f^*_D)^{-1}$ and $T_S = -(S^* f^*_S)^{-1}$ are, respectively, the *local demand* and *supply relaxation times.*

In order to determine when the equilibrium market price P^* is locally stable, it must be determined when the linear Eq. (3.4) has solutions $z(t)$ that approach zero. To do this, make the (usual) *Ansatz* $z(t) = \exp(vt)$, where v is a (generally complex) eigenvalue to be determined. Clearly, v will depend on some or all of the economic parameters e_D, T_D, e_S, T_S, and $T_{\min}$, as well as the demand and supply price kernels K_D and K_S. The values or ranges of these parameters such that $\mathrm{Re}(v) < 0$ must be determined.

Substituting $z(t) = \exp(vt)$ into Eqs. (3.2), (3.3), and (3.4) results in the integral eigenvalue equation

$$v + A_D \int_0^{\infty} K_D(q) e^{-vq}\, dq + A_S \int_{-T_{\min}}^{\infty} K_S(q + T_{\min}) e^{-vq}\, dq = 0 \tag{3.5}$$

to be solved for v. Although it is possible to obtain some very restrictive sufficient conditions for $\operatorname{Re}(\nu) < 0$ from (3.5), thus implying the local asymptotic stability of the equilibrium price P^*, their application is analytically difficult. Thus, it is more instructive to consider the local stability problem after a specification of potentially interesting forms for the demand and supply price kernels, K_D and K_S, to obtain some insight into how various economic parameters in this model affect the local asymptotic stability of the equilibrium price P^*. A simple case that may be dealt with analytically is considered here. It involves making specific assumptions about producer and consumer behaviors that are reflected in properties of the supply and demand price kernels.

From the supply side, we assume that producers, in making decisions to alter production, pay little attention to past price and base production alteration decisions primarily on current prices. Thus, the supply price kernel is a Dirac delta function $K_S(q) = \delta(q)$. Under this assumption from Eq. (2.4), the supply price P_S is simply $P_S(t) = P(t - T_{\min})$.

In terms of consumer behavior, we consider a class of consumers who base buying decisions on a weighted average of past prices. In addition, we suppose that the consumers have a fading memory, more accurate for more recent prices. This assumption is equivalent to assuming that the demand price kernel is a monotone decreasing function, which we take to be an exponential function

$$K_D(q) = [\exp(-q/T_c)]/T_c,$$

where the constant T_c is to be interpreted as the average length of the consumer memory. With this assumption, the demand price then takes the form

$$P_D(t) = T_c^{-1} \int_{-\infty}^{t} [\exp -(t-u)/T_c]\, P(u)\, du$$

and the commodity price dynamics are completely specified by

$$\frac{1}{P}\frac{dP}{dt} = f(D(P_D(t)), S(P(t - T_{\min}))). \tag{3.6}$$

With these assumptions, the eigenvalue Eq. (3.5) takes the explicit transcendental form

$$\lambda + \frac{Q}{1 + R\lambda} + e^{-\lambda T} = 0. \tag{3.7}$$

where $\lambda = \nu/A_S$, $Q = A_D/A_S$, $R = A_S T_c$, and $T = A_S T_{\min}$. We now determine, for a fixed value of the parameter R, the region in the space of parameters Q and T for which all roots of λ Eq. (3.7) satisfy $\mathrm{Re}(\lambda)<0$. This is a variant of the method of D subdivision (Kolmanovskii and Nosov, 1986), since one of the delays, T, is taken as a parameter. (Usually, the coefficients are considered variable, for fixed values of the delays.)

In the case $R=0$, Eq. (3.7) becomes a quasi-polynomial, the roots of which are well known. Hayes (1950) has completely characterized the conditions under which the eigenvalues obtained as solutions of (3.7) will satisfy $\mathrm{Re}(\lambda)=0$.

Using our notation and restricting ourselves to the case at hand, we have the following:

Theorem 3.1. *(Hayes). Let $R=0$ and $Q>0$. Then all of the roots λ of Eq. (3.7) satisfy* $\mathrm{Re}(\lambda)<0$ *if and only if*

(1) $Q>1$ *(3.8)*

or (2a) $Q\leqslant 1$ *(3.9a)*

and (2b) $T<T_{\mathrm{crit}}$ *(3.9b)*

where $$T_{\mathrm{crit}}=\frac{\arccos(-Q)}{[(A_S)^2-(A_D)^2]^{1/2}}. \tag{3.9c}$$

and the inverse cosine takes its value in the interval $[\frac{1}{2}\pi, \pi]$.

The Hayes criteria are presented graphically in Fig. 1. Noting that Eq. (3.9c) may be rewritten in the form

$$A_S T_{\mathrm{crit}}=\frac{\cos^{-1}(-A_D/A_S)}{[1-(A_D/A_S^2]^{1/2}}, \tag{3.10}$$

the Hayes stability criteria may be examined by plotting them in the $A_S T_{\mathrm{min}}$ vs. (A_D/A_S) plane. Since both A_D and A_S are positive, attention need only be confined to the first quadrant of this plane that is naturally divided into two separate regions by the conditions (3.8) and (3.9a). The division between these two areas is indicated by the dashed vertical line at $(A_D/A_S)=1$ in Fig. 1. From condition (3.8), for all combinations of the four parameters (e_D, T_D, e_S, T_S) falling into region I (cf. Fig. 1), the equilibrium price P^* is locally stable irrespective of the value of the total production delay T_{min}. However, region II is naturally divided into two subregions, IIa and IIb: the boundary between them is indicated by the solid curved line that is the graph of Eq. (3.10). In the limit as $(A_D/A_S)\to 0$, $A_S T_{\mathrm{crit}}\to \frac{1}{2}\pi$ as indicated on the graph of Eq. (3.10). Thus, from inequality (3.9b), for all values of the parameters such that a point $(A_D/A_S, A_S T_{\mathrm{min}})$ lies in region IIa, the equilibrium price will be stable.

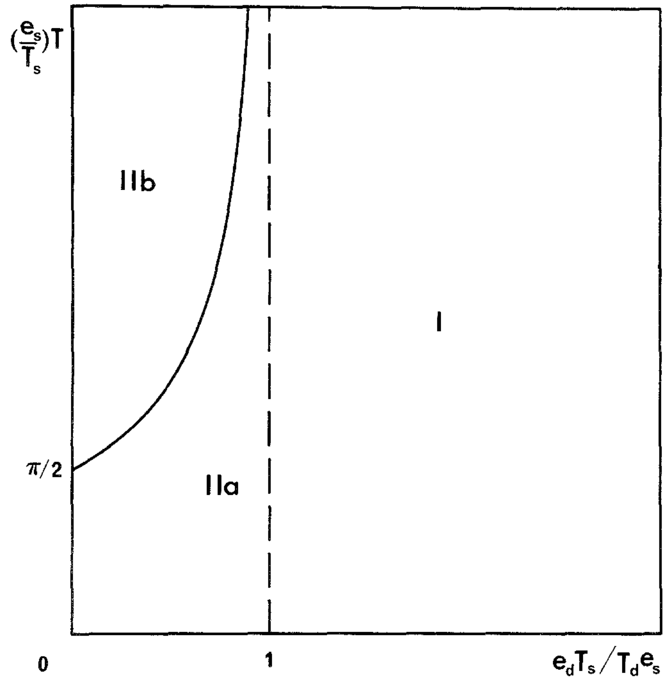

Fig. 1. A graphical representation of the Hayes criteria of Theorem 3.1. Any combination of the parameters (e_D, T_D, e_S, T_S, T) falling into region I or IIa corresponds to locally stable equilibrium prices P^*, while parameter sets in region IIb corresponds to an unstable equilibrium price. The solid concave up curve is the graph of Eq. (3.9c).

Once this point passes into region IIb, the equilibrium price becomes unstable.

The effects of an alteration of model parameters on an initially stable equilibrium price as one parameter is varied at a time, holding the other four constant, are summarized in Table I.

These results can be interpreted as follows. Because of moderately precise knowledge of current market conditions on the part of commodity traders, the supply relaxation time T_S is expected to be less than the demand relaxation time T_D, $T_S < T_D$. Thus, the ratio $A_D/A_S = (e_D/e_S)(T_S/T_D)$ should be less than the ratio of demand to supply elasticities. This, coupled with the fact that $0 \leqslant A_D/A_S < 1$ is a necessary condition for instability, would suggest that highly responsive and well-informed commodity marketing schemes with elasticities of supply exceeding elasticities of demand are primary contributors to commodity price fluctuations.

To this point, only variations in the various economic parameters leading to a loss of stability of the equilibrium price P^* have been con-

Table I. Summary of Stability Results

Parameter varied	Regions visited	Stability of equilibrium
Increased production delay	I → I IIA → IIB	Always stable Eventually unstable
Decreased demand elasticity	I → IIA if $e_S T_{min}/T_S < \frac{1}{2}\pi$	Always stable
or Increased demand relaxation	I → IIA → IIB if $e_S T_{min}/T_S > \frac{1}{2}\pi$	Eventually unstable
Increased supply elasticity or Decreased supply relaxation	I → IIA → IIB	Eventually unstable

sidered. When stability is lost, the relation $T = T_{crit}$ between the production lag T and the critical combination of the elasticities of supply and demand and the supply and demand relaxation times holds. Using Eq. (3.9c) gives the explicit relation

$$T = \frac{\arccos(-A_D/A_S)}{[(A_S)^2 - (A_D)^2]^{1/2}} \tag{3.11}$$

defining when $\mathrm{Re}(\lambda) = 0$. The graph of Eq. (3.11) is the convex curve in Fig. 1 and the graph defines a locus of points in parameter space for which $\mathrm{Re}(\lambda) = 0$. Whenever the five parameters e_D, T_D, e_S, T_S, and T_{min} satisfy Eq. (3.11), then the eigenvalue Eq. (3.7) has a purely imaginary solution $\lambda = \pm i\omega_H$. Below and to the right of this curve, $\mathrm{Re}(\lambda) < 0$ (local stability), while above the curve $\mathrm{Re}(\lambda) > 0$ (local instability), as can be checked explicitly since

$$\left.\frac{d(\mathrm{Re}\,\lambda)}{dT}\right|_{\lambda = i\omega_H} = \frac{\omega_H^2}{(\omega_H T)^2 + (1 + QT)^2} > 0.$$

For the combination of parameters defined by Eq. (3.11), the linear variational equation has an oscillatory solution and its period may be calculated exactly. Indeed we have the following:

Theorem 3.2. *Let Eq. (3.11) be satisfied for some values of the parameters T, A_D, and A_S. The Eq. (3.4) has a periodic solution of period*

$$T_H = \frac{2\pi T}{[(A_S)^2 - (A_D)^2]^{1/2}}. \tag{3.12}$$

Furthermore,

$$2T \leqslant T_H \leqslant 4T. \tag{3.13}$$

Proof. Let $\lambda = i\omega$ in Eq. (3.7). Separating the real and imaginary parts leads to

$$A_D = -A_S \cos(\omega T_{\min}) \tag{3.14a}$$

and

$$\omega = A_S \sin(\omega T_{\min}). \tag{3.14b}$$

Squaring both equations and adding them yields

$$\omega = [(A_S)^2 - (A_D)^2]^{1/2}$$

for the frequency of the solutions $\sin(\omega t)$ and $\cos(\omega t)$ of Eq. (3.4), or the period given by Eq. (3.12).

From Eq. (3.14a), $\omega_H T = \cos^{-1}(-A_D/A_S)$, and thus Eq. (3.12) can be written as

$$T_H = \frac{2\pi T_{\min}}{\cos^{-1}(-A_D/A_S)}.$$

Since, as stated above, $\cos^{-1}(-A_D/A_S) \in (\frac{1}{2}\pi, \pi)$, Eq. (3.13) follows. ∎

Remarks. (i) Eq. (3.13) makes explicit the role of the elasticities of supply and the price relaxation times in determining the period of the periodic solution. (ii) The model predicts, by Eq. (3.13), that when the equilibrium price becomes locally unstable there will be an oscillation in market price with a period that is between two and four times the production lag T.

When $T < T_H$, it is of interest to determine what happens to this periodic orbit. A supercritical Hopf bifurcation will take place (Stech, 1985) provided that certain nondegeneracy conditions are fulfilled by the nonlinear terms in Eq. (3.6). This Hopf bifurcation is marked by the passing of a pair of complex conjugate eigenvalues from the left-hand to the right-hand side of the complex plane. Just as they cross the imaginary axis [when Eq. (3.11) is satisfied], the loss of stability of the equilibrium price is accompanied by the birth of a cyclic oscillation of period T_H in the market price $P(t)$ near P^*. An easy but quite involved computation using the method of Stech (1985) yields the following:

Theorem 3.3. *Consider Eq. (3.6) with* $P_D(t) = P(t)$, $R = 0$, *and* $(A_D/A_S) < 1$. *Then there is an* $\varepsilon > 0$ *such that, for a value of the parameters* A_D, A_S, *and* $T_{\min}$ *close to and above the curve defined by Eq. (3.11), there exists an orbitally asymptotically stable periodic solution* $y(t)$ *of Eq. (3.6), satisfying* $\| y - P^* \| < \varepsilon$, *provided that*

$$\mathrm{Re}\{[3h_3(\phi, \phi, \bar{\phi}) + 2h_2(\bar{\phi}, a_2 e^{2\omega i}) + 2h_2(\phi, a_0)]/[1 - Te^{-T\omega i}]\} < 0, \qquad (3.15)$$

where

$$\phi(s) = e^{2\omega i}, \qquad a_2 = h_2(\phi, \phi)/(2\omega i + Q + e^{-2\omega i T}),$$

$$a_0 = 2h_2(\phi, \bar{\phi})/(Q+1),$$

$$h_2(\phi, \zeta) = C\phi(0)\,\zeta(0) + D\phi(-T)\,\zeta(-T) + \frac{E}{2}\{\phi(0)\,\zeta(-T) + \phi(-T)\,\zeta(0)\},$$

$$\begin{aligned} h_3(\phi, \zeta, \eta) = {} & F\phi(0)\,\zeta(0)\,\eta(0) + G\phi(-T)\,\zeta(-T)\,\eta(-T) \\ & + \frac{H}{3}[\phi(0)\,\zeta(-T)\,\eta(-T) + \phi(-T)\,\zeta(0)\,\eta(-T) \\ & + \phi(-T)\,\zeta(-T)\,\eta(0)] \end{aligned}$$

and the Taylor expansion about P^* *in Eq. (2.1) is given by*

$$\begin{aligned} Pf(D(P), S(P_s)) = {} & A(P - P^*) + B(P_s - P^*) + C(P - P^*)^2 \\ & + D(P_s - P^*)^2 + E(P - P^*)(P_s - P^*) + F(P - P^*)^3 \\ & + G(P_s - P^*)^3 + H(P - P^*)(P_s - P^*)^2 \end{aligned}$$

This periodic solution is unique up to a phase shift.

If we now incorporate consumer memory into Eq. (3.6), then this corresponds to $R > 0$ in Eq. (3.7). In this case, the preceding calculations essentially carry over, but the algebraic complications increase somewhat.

As a first step, we determine the values of the parameters Q, R, and T for which Eq. (3.7) possesses pure imaginary roots λ. We proceed by fixing $R > 0$, and considering Q and T as parameters. We have

Theorem 3.4. *Consider Eq. (3.7) for a fixed value of* $R > 0$. *Then the values of* T *giving pure imaginary roots* $\lambda = i\omega$ *of Eq. (3.7) are given by*

(i) if $R < \sqrt{2} - 1$, *then* $Q \in [0, 1)$ *and*

$$T_+ = \frac{2^{1/2} R \arccos(-2Q/[1 + 2RQ + R^2 + [(1+R^2)^2 + 4RQ(R^2-1)]^{1/2})}{\{R^2 + 2QR - 1 + [(1+R^2)^2 + 4RQ(R^2-1)]^{1/2}\}} \qquad (3.16a)$$

(ii) if $\sqrt{2}-1<R<1$, *let* $Q_c=(1+R^2)^2/\{4R(1-R^2)\}$; *then for* $Q\in[0, Q_c)$, *T is given by Eq. (3.16a). Further, for* $Q\in[1, Q_c)$, *there is another family of roots given by*

$$T_-=\frac{2^{1/2}R\arccos(-2Q/[1+2RQ+R^2-[(1+R^2)^2+4RQ(R^2-1)]^{1/2})}{\{R^2+2QR-1-[(1+R^2)^2+4RQ(R^2-1)]^{1/2}\}} \tag{3.16b}$$

(iii) if $R>1$, *for* $Q\in[0,\infty)$, $T=T_+$ *as given by Eq. (3.16a) and, for* $Q\in[1,\infty)$, $T=T_-$ *as given by Eq. (3.16b).*

Remark. The region of definition of each family of roots in the parameter plane (R, Q) is shown in Fig. 2.

Proof. Separating the real and imaginary parts of Eq. (3.7) after letting $\lambda=i\omega$, we obtain

$$\cos(\omega T)=-Q/(1+R^2\omega^2) \tag{3.17a}$$

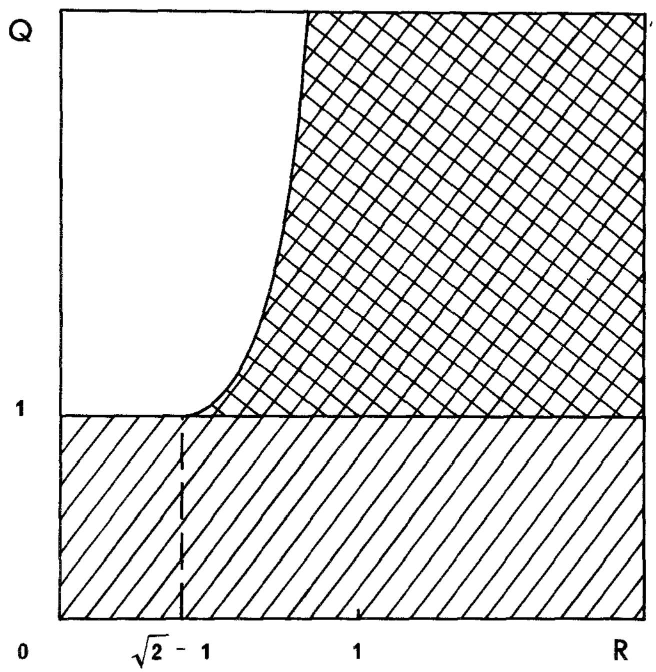

Fig. 2. An illustration of the parameter values giving pure imaginary roots of Eq. (3.7). The boundary curve for $\sqrt{2}-1\leqslant R<1$ is defined by $Q=(1+R^2)^2/[4R(1-R^2)]$. The roots ω_+ (▨) and ω_- (▧) are given by Eq. (3.19).

and

$$\sin(\omega T) = \omega[1 - RQ/(1 + R^2\omega^2)]. \tag{3.17b}$$

Since roots of Eq. (3.7) come in conjugate pairs, we assume without loss of generality that $\omega > 0$ in Eqs. (3.17). Adding the squares of the right-hand sides of Eqs. (3.17) yields a quadratic polynomial in the variable ω^2, with coefficients that are polynomials in the parameters Q and R:

$$R^2\omega^4 + (1 - 2QR - R^2)\omega^2 + Q^2 - 1 = 0. \tag{3.18}$$

Solving this last equation for the roots ω^2 yields

$$\omega_{\pm}^2 = \frac{R^2 + 2QR - 1 \pm [(R^2 + 2QR - 1)^2 - 1)^2 - 4R^2(Q^2 - 1)]^{1/2}}{2R^2} \tag{3.19}$$

Since ω is real, the roots defined by Eqs. (3.19) only exist when the discriminant in Eq. (3.18) is positive. This occurs for all values of Q when

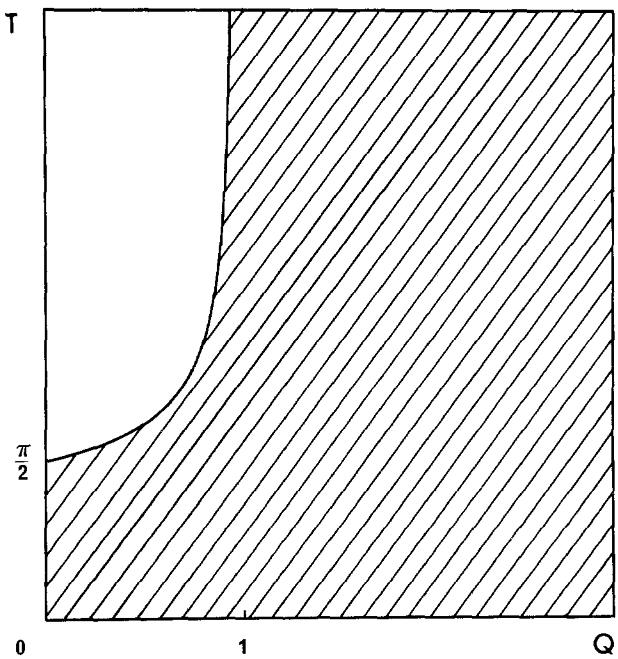

Fig. 3. A graphical representation of the values of the parameters Q and T leading to stability of the null solution in Eq. (3.4): local stability occurs for parameter values in the hatched region. Here, $R = 0.35$, but the diagram is qualitatively unchanged for values of R between 0 and $\sqrt{2} - 1$ (cf. Theorems 3.4 and 3.5).

$R > 1$, but only for $Q < (1 + R^2)^2/[4R(1 - R^2)]$ if $R < 1$. Furthermore, the values of the roots $\omega_{\pm}^2$ must be positive. For ω_-, this is equivalent to $Q > 1$; for ω_+, this introduces the restriction that $Q < 1$ whenever $R < 2^{1/2} - 1$. In each of these regions, the square roots can be taken in Eq. (3.19), the positive root being kept by the remark above. Substituting these values back into Eq. (3.17a), we obtain Eqs. (3.16a) and (3.16b). ▌

Once the roots given by Theorem 3.4 are known, we have to determine how their real parts vary as a function of the parameters. In particular, among the infinite branches given by Eqs. (3.16), we must find which ones are parts of the boundary of the domain of stability of Eq. (3.7). This problem is resolved in the following:

Theorem 3.5. *Consider Eq. (3.7), and denote T_+^j (resp. T_-^j) the root given by Eq. (3.16a) [resp. (3.16b]) for which the arccosine takes its value in the interval $(\pi/2 + 2\pi j,\ \pi + 2\pi j)$ [resp. $(\pi/2 + 2\pi j,\ \pi + 2\pi j)$ if $R < 1$ and $(\pi + 2\pi j,\ 3\pi/2 + 2\pi j)$ if $R > 1$], where j is any positive integer. Then the*

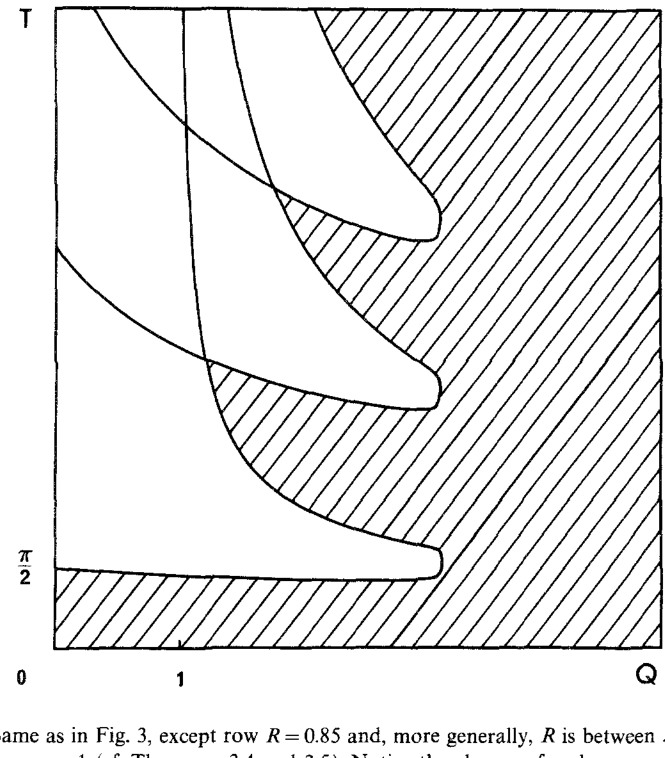

Fig. 4. Same as in Fig. 3, except row $R = 0.85$ and, more generally, R is between $\sqrt{2} - 1$ and 1 (cf. Theorems 3.4 and 3.5). Notice the change of scale.

positive values of Q and T for which all roots λ of Eq. (3.7) satisfy $\mathrm{Re}(\lambda) < 0$ *are given, according to the values of R, by*

A. $R < \sqrt{2} - 1$: $\quad Q > 1 \quad$ *or* $\quad Q < 1 \quad$ *and* $\quad T^0_+$

B. $\sqrt{2} - 1 < R < 1$: $\quad Q > \dfrac{(1+R^2)^2}{4R(1-R^2)}$

or $\quad Q < \dfrac{(1+R^2)^2}{4R(1-R^2)} \quad$ *and* $\quad T < T^0_+ \quad$ *or* $\quad T^j_- < T < T^{j+1}_+$

C. $R > 1$: $\quad 0 < T < T_+ \quad$ *or* $\quad T^j_- < T < T^{j+1}_+$.

Remark. Each of these three cases is illustrated in Figs. 3–5.

Proof. From Eq. (3.17) it is clear that $\cos(\omega_\pm T) \leqslant 0$. It is easy to see, from Eqs. (3.17b) and (3.19), that $\sin(\omega_+ T)$ is always positive, and that

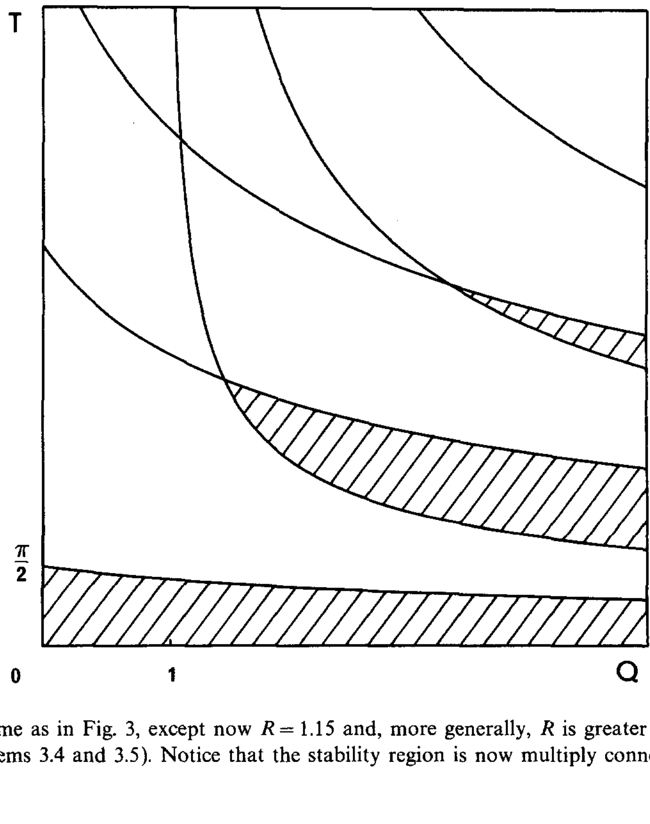

Fig. 5. Same as in Fig. 3, except now $R = 1.15$ and, more generally, R is greater than 1 (cf. Theorems 3.4 and 3.5). Notice that the stability region is now multiply connected.

$\sin(\omega_- T)$ is positive if and only if $R<1$. For a fixed value of R, the appropriate branches of the inverse cosine functions in Eqs. (3.16) thus have the ranges specified in the definitions of T_+^j and T_-^j. Three distinct cases must be analyzed. Consider first case A, defined by the requirement that $R<\sqrt{2}-1$. Then only branches T_+^j give rise to imaginary roots of Eq. (3.7), as follows from Theorem 3.4. Given the expression of Eq. (3.16a) for all of these branches, it is easy to see that $\lim_{Q\downarrow 0} T_+^j(Q)=\pi/2+2\pi j$, $\lim_{Q\uparrow 1} T_+^j(Q)=\infty$, and no two branches T_+^j and T_+^k can intersect when $j\neq k$. For a fixed value of Q less than 1, an increase in values of T thus leads to a successive crossing of all branches T_+^j in increasing order of the integer j. An implicit differentiation of Eq. (3.7) leads to

$$\left.\frac{d}{dT}\right|_{\lambda=i\omega}(\operatorname{Re}\lambda) = \frac{[(1+R^2\omega^2)^2-RQ(RQ+2)]\,\omega^2}{\{\omega^2(R^2+2RT)-(1+TQ-RQ\}^2+\omega^2\{2R+T+TRQ+\omega^2R^2T\}^2} \tag{3.20}$$

and use of Eqs. (3.16) then gives

$$\left.\frac{d}{dT}\right|_{\lambda=i\omega_+}(\operatorname{Re}\lambda)>0. \tag{3.21}$$

On any branch $T=T_+^j(Q)$, eigenvalues of Eq. (3.7) cross from the left half to the right half of the complex parameter plane λ, and Eq. (3.7) thus has all of its roots with negative real part when $T<T_+^0$ if $Q<1$. [When $Q>1$, there are from Theorem 3.4 no values of T giving imaginary roots of Eq. (3.7).]

At the value $R=\sqrt{2}-1$, a significant change occurs in the region of stability, as shown in Fig. 4, and remains so for values of R between $\sqrt{2}-1$ and 1. In this case, values of Q greater than $Q_c=(1+R^2)^2/4R(1-R^2)$ give a stable steady state for all values of T (see Theorem 3.4). When $Q<Q_c$, there is a finite value of T at which a solution of Eqs. (3.17) exists, and is given by Eq. (3.6). In Eq. (3.16b), the curves T_-^j are only defined for $Q\in(1,Q_c)$ whereas the curves T_+^j are defined, by Eq. (3.16a), for $Q\in(0,Q_c)$. Unlike the previous case, however, the variation of the real part of an eigenvalue λ of Eq. (3.7) is not uniform on all curves defined by Eqs. (3.16). Indeed, Eq. (3.21) still holds, but we have

$$\left.\frac{d}{dT}\right|_{\lambda=i\omega_-}(\operatorname{Re}\lambda)<0. \tag{3.22}$$

From Eqs. (3.16), one computes $\lim_{Q \downarrow 0} T^j_+(Q) = 2\pi j + \pi/2$,

$$\lim_{Q \uparrow Q_c} T^j_+(Q) = \lim_{Q \uparrow Q_c} T^j_-(Q) = 2R \frac{1-R^2}{-R^4+6R^2-1} \times \arccos\left[\frac{-(1+R^2)^2}{R(3+R^2-R^4)}\right],$$

and

$$\lim_{Q \downarrow 1} T^j_-(Q) = \infty.$$

For increasing values of T at a fixed value of $Q \in (1, Q_c)$, there is thus a loss of stability at the values $T = T_+(Q)$ defined by Eq. (3.16a), and (re)gain of stability at the values $T = T_-(Q)$ defined by Eq. (3.16b). Furthermore, for a fixed value of Q less than 1, roots of Eq. (3.7) can only become pure imaginary by crossing from the left-hand side to the right side of the complex plane and, thus, once stability of the stationary solution has been lost, it cannot be gained by increasing T (see Fig. 4).

When $R > 1$, a "perturbation" of the stability diagram obtained for $R \in (\sqrt{2}-1, 1)$ (Fig. 4) leads to Fig. 5. It reflects the fact that both ω_+ and ω_- now exist for all values of Q greater than 1, and that ω_+, of course, exists also for $Q \in (0, 1)$. At a fixed value of Q greater than 1, the calculations from the paragraph above indicate that, as T increases, stability is lost at $T = T_+$ and gained at $T = T_-$. The main difference from the previous case is that there are no values of Q for which P^* is stable at all values of T. In other words, when $R > 1$, for a fixed value of Q, there is always a value of T at which the steady state P^*, which is stable for T small enough, loses its stability: the region of absolute stability has disappeared. ∎

As in the more simple case of $R = 0$, nondegeneracy conditions must be fulfilled for a Hopf bifurcation to occur at the values of T and Q for which Eqs. (3.17) are satisfied. Namely, only one pair of eigenvalues of Eq. (3.7) must be on the imaginary axis, and all others must lie in the left-hand half of the complex plane: this is clear from the continuous dependence of λ on Q and T in Eq. (3.7) and the calculation above. There is also a condition involving the nonlinear terms of Eq. (3.6). When this condition is fulfilled, a supercritical Hopf bifurcation occurs, and a stable periodic solution appears when the steady state is unstable. The period of this solution can be estimated, and will depend upon the value R.

Let us write T_H for the period of the oscillating solution at the bifurcation value. Then $T_H = 2\pi/\omega_H$, where ω_H is the value of the root of

Eqs. (3.17). Using the explicit expressions of Eqs. (3.16) and Eq. (3.19), we obtain

$$T_H = \frac{2\pi}{\arccos(\omega_+)} T_+ \qquad \text{or} \qquad \frac{2\pi}{\arccos(\omega_-)} T_-$$

according to whether $\omega_H = \omega_+$ or $\omega_H = \omega_-$. As mentioned above, the inverse cosine function takes its value either in an interval $(\pi/2 + 2\pi j, \pi + 2\pi j)$ or $(\pi + 2\pi j, 3\pi/2 + 2\pi j)$, where j is an integer, depending on the value of R if $\omega_H = \omega_-$. Thus,

$$2T \leqslant T_H \leqslant 4T \qquad \text{if} \quad \omega_H = \omega_+ \tag{3.23a}$$

$$\omega_H = \omega_- \qquad \text{and} \qquad R < 1$$

$$\tfrac{4}{3}T \leqslant T_H \leqslant 2T \qquad \text{if} \quad \omega_H = \omega_- \qquad \text{and} \qquad R > 1, \tag{2.36b}$$

when $j = 0$, which gives the bifurcation line present for all values of R.

The model thus predicts that, when the equilibrium price becomes unstable, the induced oscillation in market price will have a period between two and four times the production lag T, when consumer memory is sufficiently poor. When an improvement in the latter occurs, however, this induced oscillation may have a period of either between two and four times the production delay T *or* between four-thirds and two times the production lag T.

4. RELATION TO OTHER MODELS

In this section, we compare the model presented here with well-studied models from the economic literature.

The discrete time cobweb models that have been so widely exploited in economic modeling are limiting cases of the model developed here. To see this, examine the price adjustment dynamics embodied in Eq. (3.6), viz.

$$\frac{1}{P}\frac{dP}{dt} = f(D(P), S(P(t-T)), \tag{4.1}$$

with $P_D = P$ and $T_{\min} = T$. When the supply and demand relaxation times, T_S and T_D, are very short so price adjustment is quite rapid, then

$$\frac{1}{P}\frac{dP}{dt} = 0$$

so, from Eq. (4.1).

$$f(D(P), S(P(t-T)) = 0.$$

Hence, from the properties of f specified in Eq. (2.3), in this limiting case Eq. (4.1) reduces to an implicit (generally nonlinear) difference equation in $P(t)$:

$$D(P(t)) = S(P(t-T)). \tag{4.2}$$

If the demand schedule D is a locally monotone (and thus invertible) function, then Eq. (4.2) may be written in the explicit form

$$P(t) = F(P(t-T)) \tag{4.3}$$

where $F = D^{-1} \circ S$. When time t is measured in units of the constant lag T, then Eq. (4.2) or the explicit version (4.3) is the limiting cobweb version of our model that should hold under conditions of very rapid price adjustment.

It is well known that the equilibrium price P^*, given by the solution of $D(P^*) = S(P^*)$, is locally asymptotically stable when

$$F'(P^*) < 1 \tag{4.4}$$

and unstable when

$$F'(P^*) > 1. \tag{4.5}$$

Now $F'(P^*) = S'(P^*)/D'(P^*) = e_S/e_D$ so conditions (4.4) and (4.5) are simply the discrete time-limiting cases of conditions (3.8) and (3.9) of the Hayes criteria for the stability of the continuous time model.

Recently a variety of discrete time economic models that may be cast into the form of the map (4.3), or higher dimensional versions, have been studied because of their period doubling bifurcation structure leading to the generation of chaotic time series [Goodwin *et al.* (1984) and references therein]. However, the analogy between discrete maps, such as Eq. (4.3), and delay equations, such as Eq. (4.1), from which they are derived as a singular limit, may have severe limitations as far as the dynamical behavior of each system is concerned (Mallet-Paret and Nussbaum, 1986).

In an apparently little known paper, the British physiologist J. B. S. Haldane developed a linear model for a single commodity market (Haldane, 1933). In this model, $p(t)$ denotes the fractional deviation of commodity price away from its equilibrium value, and Haldane argued that the dynamics of $p(t)$ should be governed by the integrodifferential equation

$$\frac{dp}{dt} = -Ap - B \int_0^\infty g(x)\, p(t-x)\, dx, \tag{4.6}$$

[cf. Haldane (1933), Eq. (2), with notational changes]. In Eq. (4.6), the coefficient A is proportional to the elasticity of demand while B is proportional to the elasticity of supply.

A comparison of Eq. (4.6) with Eq. (3.4) shows that the Haldane model is a special case of that presented here under the assumption that the demand kernel is a delta function and the function g is identified with the supply price kernel, K_S.

Some 30 years later, apparently unaware of Haldane's work, Larson (1964) proposed a "harmonic motion" model for the cyclical behavior of the pork market. A linear relation was assumed between market price P and the quantity of part marketed, that supply being determined by the linear delay differential equation

$$\frac{dS}{dt} = k[P(t-T) - P^*]$$

relating the rate of change of supply to the lagged deviation $P(t-T)$ from the equilibrium price P^*. From these two assumptions, Larson derived the equivalent relations

$$\frac{dz}{dt} = -Rz(t-T) \tag{4.7}$$

and

$$\frac{ds}{dt} = -Rs(t-T), \tag{4.8}$$

where z and s are the deviations from the equilibrium price and supply, respectively. Eqs. (4.7) and (4.8) have oscillatory solutions of period exactly $4T$ for $R=1$ and, to explain the 4-year cycle in pork price and supply, Larson made the ad hoc assumption that $R=1$, along with the reasonable choice of $T=1$ year.

Eq. (4.7) is simply the linear version of the model presented here (cf. Eq. 3.6) under the assumption that supply price is $P_S(t) = P(t-T)$, and either that the elasticity of demand is identically zero or, equivalently, that the demand relaxation time is infinite. Thus, Larson's harmonic motion commodity market model is another limiting case of the model developed here.

5. DISCUSSION AND CONCLUSIONS

The work presented here furnishes further evidence that the presence of delays in regulatory mechanisms may introduce destabilizing effects.

However, different delayed regulatory actions may lead to different effects, as has been discussed in models for the regulation of populations (Cooke, 1985).

From a mathematical point of view, various avenues have been explored to investigate these effects (Driver, 1963b; Myshkis, 1977; Stech, 1978; Walther, 1976). We will mention only two. The first concerns the consequences of modifying the kernels in integrodifferential equations. Roughly speaking, the more weight that is given to the past, the less stable an equilibrium point is (Hadeler, 1976; Walther, 1976). This is very much in line with our results. The second one deals with the introduction of a second delay, by considering when a delayed regulatory mechanism takes effect. Once again, the longer this second delay is and the less stable the equilibrium point is, in the same sense as in the previous case: the region, in parameter space, leading to an asymptotically stable steady state is reduced when this second delay is increased (Stech, 1978).

In an economic context, our work shows that the production delays in commodity markets are potentially destabilizing factors, as has been pointed out previously (Goodwin, 1951; Haldane, 1933; Kalecki, 1935, 1937, 1943, 1972; Larson, 1964). However, our analysis seems to be the first one displaying an explicit consideration of the roles played by a variety of economic parameters in determining the stability of a single commodity market, and the relation of the period of the oscillation when the market becomes unstable to various economic parameters.

The model that we have presented could be refined in several ways. A very interesting one would be the incorporation of a variable production delay. Indeed, certain commodities, once produced, may be stored for a variable period of time (denoted by Δ) until market prices are deemed advantageous by the producer. Typically, it would be expected that, as market prices increase, the storage period is likely to fall with the maximum storage period ($\Delta_{\max}$) occurring when the market price is in the neighborhood of the production price. Furthermore, if the market price falls very much below the production price, then the storage period may again fall as producers attempt to recoup as much of their investment as possible (the *dumping phenomenon*). For goods that are not perishable, or that do not become obsolete, the maximum storage period would be infinitely long, but this seems unrealistic for most situations and the maximum storage time is usually expected to be finite. In these circumstances, the *total production delay* (the total elapsed time between the initiation of changes in production and the final alteration of supply), T, may either be a monotone decreasing or humped function of current market price, $T(P) = T_{\min} + \Delta(P)$, where $T_{\min} \leqslant T(P) \leqslant T_{\min} + \Delta_{\max}$. The ultimate consequence of this variable storage capability would be that market prices between a time $t - T(P)$ in the past and the present time t cannot have any effect on the current supply price $P_S(t)$. Thus, Eq. (2.4) would be modified to

$$P_S(t) = \int_{-\infty}^{t - T(P)} K_S(t - T(P) - u)\, P(u)\, du.$$

Similar problems with state-dependent delays arise naturally in relativistic electrodynamics in which the motion of two particles is determined by a central interaction, but occurring with a time delay that is an increasing function of the interparticle separation (Driver, 1963a). Also, the control of mammalian platelet production involves a nonlinear feedback operating with a variable delay that depends on the number of circulating platelets (Bélair and Mackey, 1987; Mackey and Bélair, 1988). In both of these examples, however, the state-dependent delay is monotone increasing and therefore quite different than what would be expected in an economic context.

To our knowledge, there has never appeared an explicit consideration of the effect of adding price-dependent storage policies in commodity market models of the type considered here. The results presented here suggest that such policies would be highly destabilizing and would either destabilize a previously stable market situation, or exacerbate an unstable market by an increase in the amplitude and period of oscillations in commodity prices. The mathematical questions raised by problems framed in terms of delay differential equations with state-dependent delays are formidable, and only limited analytical results seem to be available for such systems (Driver, 1963a, b; Myshkis, 1977; Nussbaum, 1974; Sugie, 1988; Winston, 1974). The fact that such problems appear to be of some relevance in economics as well as in biologically derived problems (Bélair and Mackey, 1987; an der Heiden and Mackey, 1987) may prove a spur to mathematicians to study these systems.

Another possibility would be to suppose that, although consumers base their decisions on information concerning past price, there is a non-zero gap in the information available to them; namely, the most recent prices. This gap would correspond to the interval between successive acquisitions on the part of the consumer, and would be present, for example, if only personal information was used. In this case, the demand price kernel K_D would be zero on some interval $[0, G]$. We have every reason to believe that this second delay G would significantly complicate the behavior of Eq. (2.1), for even the linear stability analysis is quite difficult in equations with two delays (Bélair, 1987; Braddock and van den Driessche, 1983; Nussbaum, 1975).

Even if production delays are set identically equal to zero, other choices of demand and supply kernels will give rise to other commodity market model formulations in which oscillations may well arise. As a simple example, let $T_{\min} = 0$ in Eq. (3.6) and, following Fargre (1973), pick the supply and demand kernels to be given by the generic forms

$$K_D = K_D(q; a, m) = a^{m+1} q^m e^{-aq}/m! \qquad a > 0, \qquad m > -1$$

and

$$K_S = K_S(r; b, n) = b^{n+1} r^n e^{-br}/n! \qquad b > 0, \qquad n > -1$$

where (a, b, m, n) are parameters. Then it is a straightforward calculation to show that the system composed of Eqs. (2.1), (2.3), and (2.4) in conjunction with these kernels is equivalent to a system of $(m+n+4)$ ordinary differential equations. Given the fact that nonlinear systems of three ordinary differential equations may have solutions that are aperiodic ["deterministically chaotic" (Guckenheimer and Holmes, 1983)], this procedure of generating a system of ordinary differential equations from the model presented here may lead to solutions displaying irregular behavior. If one retains the production plus storage delay in this procedure, the resulting coupled system will consist of $(m+n+3)$ ordinary differential equations and one nonlinear delay differential equation.

The commodity market model presented here is highly idealized in its consideration of a market isolated from all other economic influences. Many investigators have considered models similar to that presented here embedded in the economic analogue of a thermodynamic "heat bath" [Kalecki (1952) is representative]. In these studies, a loose coupling of a larger extended economy to the primary system offers a source of stochastic shocks to the system being modeled and reproduces the qualitative behavior of real economic situation. Within the context of the model presented here, reduced to a framework of a system of ordinary differential equations as outlined above, the recent work of Lasota and Traple (1986) is likely to be quite important.

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