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Commodity Price Fluctuations: Price Dependent Delays and Nonlinearities as Explanatory Factors

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This paper develops a price adjustment model for a single commodity market with state dependent production and storage delays. Conditions for the equilibrium price to be stable are derived in terms of a variety of economic parameters. When stability of the equilibrium price is lost a Hopf bifurcation occurs, giving rise to an oscillatory commodity price with a period between two and four times the equilibrium production–storage delay. *Journal of Economic Literature* Classification Numbers: 022, 131, 213, 214. © 1989 Academic Press, Inc.

I. INTRODUCTION

Trade cycles, business cycles, and fluctuations in the price and supply of various commodities have attracted the attention of economists for well over 100 years. Early authors [35, 43, 50, 55] often attributed these fluctuations to random factors, e.g., the weather for agricultural commodities. On the other hand, others have speculated that economic cycling might be an inherent behaviour characteristic of unstable economic systems [4, 10, 15, 27, 29, 30, 36, 51, 54, 58, 59]. Their work and that of others [3, 8, 13, 16, 17, 37] has played a fundamental role in sharpening the debate between the proponents of the exogenous versus endogenous schools (cf. [63] and the references therein). Recent developments in nonlinear dynamics in applied mathematics [18, 40], physics and chemistry [42, 53], and biology and medicine [14, 47] have played a role in this discussion.

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Interestingly enough, this newer economic modelling has almost entirely ignored the potential role of production delays in generating economic fluctuations ([28] is an exception). This omission is surprising for several reasons. Historically, Ricci [48], Schultz [52], and Tinbergen [57] almost simultaneously utilized the known lag between the initiation of production decisions and the delivery of goods to discuss commodity cycles in a discrete time mathematical framework that became known as cobweb theory [10, 41, 62]. Further, Kalecki [31–35], Haldane [19], Goodwin [15], and Larson [38] all developed delay differential equation models of cyclic economic behaviour. Finally, it is now known that a broad spectrum of dynamic behaviors can be found in nonlinear delay differential equations [22–26, 39, 44–46, 60].

This paper develops a continuous time model for the price dynamics of a single commodity market, formulated as a delay–differential equation. Nonlinearities in supply and demand schedules are explicitly accounted for, as well as production plus storage delays that may be functions of the market price.

II. Formulation of the Model

In considering the dynamics of price, production, and consumption of a particular commodity, assume that relative variations in market price $P(t)$ are governed by a simple balance between demand and supply:

$$\frac{1}{P}\frac{dP}{dt} = D(P) - S(P_S). \tag{1}$$

In Eq. 1, $D(\cdot)$ and $S(\cdot)$ respectively denote the demand and supply functions for the commodity in question, and it is assumed that the minimum demand is always exceeded by the maximum supply, $\min_{P_D} D(P_D) \leqslant \max_{P_S} S(P_S)$.

The argument of the demand schedule is taken as $P(t)$ in keeping with the simplest assumption that consumers base all buying decisions on the current market price.

However, the argument of the supply schedule, P_S, is more complicated because of two factors incorporated in this model. First, for most commodities there is a finite minimum time $T_{\min} \geqslant 0$ that must elapse before a decision to alter production is translated into an actual change in supply. For example, in agricultural commodity markets $T_{\min}$ is related to biological constraints, e.g., gestation plus growth period. Second, certain commodities, once produced, may be stored for a variable period of time (denoted by Δ) until market prices are deemed advantageous for selling by the producer. Typically, it would be expected that as market prices increase, the storage period is likely to fall with the maximum storage period ($\Delta_{\max}$) occurring when the market price is in the neighborhood of the production price. Furthermore, if the market price falls very much below the production price then the storage period may again fall as producers attempt to recoup as much of their investment as possible. Thus, the total production delay T (the total elapsed time between the initiation of changes in production and the final alteration of supply) may be either a monotone decreasing or a humped function of current market price, $T(P) = T_{\min} + \Delta(P)$, where $T_{\min} \leqslant T(P) \leqslant T_{\min} + \Delta_{\max}$.

The ultimate consequence of these two facets of supply dynamics is that it is only the market price at a time $t - T(P)$ in the past which can have any effect on the current supply price $P_S(t)$. Thus, the supply price $P_S(t)$ is just the delayed market price $P(t - T(P))$,

$$P_S(t) = P(t - T(P)). \tag{2}$$

Equations (1) and (2), in conjunction with a specification of the functions $D(P)$, $S(P_S)$, and $T(P)$, complete the formulation of the model when an initial function $P(t_0)$, $-(T_{\min} + \Delta_{\max}) \leqslant t_0 \leqslant 0$, is given. The model offers a flexible framework within which one may consider the dynamics of a variety of commodity markets. The uniqueness of the model lies in the dynamics of market price being governed by a nonlinear delay–differential equation with a state dependent delay.

III. Equilibrium: Stability and Oscillations

This section considers how the stability of the equilibrium price is determined by various economic factors, and what happens when stability is lost.

Denote the equilibrium price at which demand and supply are equal by P^*, so $D(P^*) = S(P^*)$. From Eq. (1) this clearly corresponds to $(dP^*/dt)/P^* = 0$. Because of the specified properties of D and S, at least one equilibrium price must exist, though the existence of more than one equilibrium price is not excluded.

In examining the local stability of the equilibrium price P^* we wish to determine the conditions under which $P(t) \to P^*$ as $t \to \infty$ following some small perturbation of market price away from the equilibrium price P^*, i.e., for perturbations satisfying $|P(t) - P^*|/P^* \ll 1$. It is important to realize that the local instability of the equilibrium price guarantees the global instability of P^*, and may indicate the existence of limit cycle or other behaviour in the full model.

Thus, expand all nonlinearities in the model in a Taylor's series about the equilibrium price P^*, and use the assumed smallness of the deviation of $P(t)$ from P^* to discard all terms in the expansion of order two or higher. Finally, define a new variable $z(t) = P(t) - P^*$ and ultimately find that $z(t)$ satisfies the linear variational equation

$$\frac{dz}{dt} = P^*[D'^* z - S'^* z_S], \tag{3}$$

where $z_S(t) = z(t - T^*)$ and T^* is the total production delay evaluated at the equilibrium price, i.e., $T^* = T(P^*) = T_{\min} + \Delta(P^*)$. In Eq. (3),

$$D'^* = \left.\frac{dD}{dP}\right|_{P = P^*} \quad \text{and} \quad S'^* = \left.\frac{dS}{dP_S}\right|_{P_S = P^*}.$$

Therefore, D'^* gives the slope of the demand function D with respect to the price P but evaluated at the equilibrium price P^*, with a corresponding meaning for S'^*. It is interesting to note that the only effect of the price dependent delay in this model is to lengthen the effective delay via the term $\Delta(P^*)$. It is somewhat surprising that the slope of Δ with respect to P, appearing in terms of the form $\Delta'(P^*)$, does not have any effect on local stability considerations, since this is not the case in other problems [1, 9].

The coefficients in the linear variational Eq. (3) for $z(t)$ can be rewritten in terms of the elasticities of demand and supply given by $e_D = -D'^*/(D^*/P^*)$ and $e_S = S'^*/(S^*/P^*)$, respectively, where $D^* = D(P^*) = S^* = S(P^*)$. Thus (3) becomes

$$\frac{dz}{dt} = -(e_D z + e_S z_S)/T_R, \tag{4}$$

where $T_R = (D^*)^{-1} = (S^*)^{-1}$ is the price relaxation time.

The local asymptotic stability of P^* is equivalent to $z(t) \to 0$ as $t \to \infty$ for small perturbations. Thus, in order to determine when the equilibrium market price P^* is locally stable we must determine when the linear Eq. (4) has solutions $z(t)$ that approach zero. To do this, we make the standard assumption that $z(t) = \exp(\lambda t)$, where λ is a (generally complex) eigenvalue to be determined. Substituting $z(t) = \exp(\lambda t)$ into Eq. (4) results in the transcendental eigenvalue equation

$$T_R + e_D + e_S \exp(-\lambda T^*) = 0. \tag{5}$$

Clearly, λ will be dependent on some or all of the economic parameters e_D, e_S, T_R, T^*. The ranges of these parameters such that $\text{Re}(\lambda) < 0$ must be determined, as this will ensure the local asymptotic stability of P^*.

Hayes [21] has completely characterized the conditions under which the eigenvalues λ obtained as solutions of (5) will satisfy $\mathrm{Re}(\lambda) < 0$. Rewriting the Hayes criteria in a form useful for our purposes, the equilibrium price P^* will be locally stable if and only if *either*

$$\left|\frac{e_D}{e_S}\right| > 1, \tag{6}$$

or

$$0 \leqslant \left|\frac{e_D}{e_S}\right| \leqslant 1 \tag{7a}$$

and

$$T^* < T_{\text{crit}}, \tag{7b}$$

where

$$T_{\text{crit}} = \frac{T_R \cos^{-1}(-e_D/e_S)}{[(e_S)^2 - (e_D)^2]^{1/2}}. \tag{7c}$$

It is easiest to understand the significance of the Hayes criteria for the local stability of the equilibrium price by presenting then graphically as in Fig. 1. Since Eq. (7c) may be rewritten in the form

$$e_S T_{\text{crit}}/T_R = \frac{\cos^{-1}(-e_D/e_S)}{[1 - (e_D/e_S)^2]^{1/2}}, \tag{8}$$

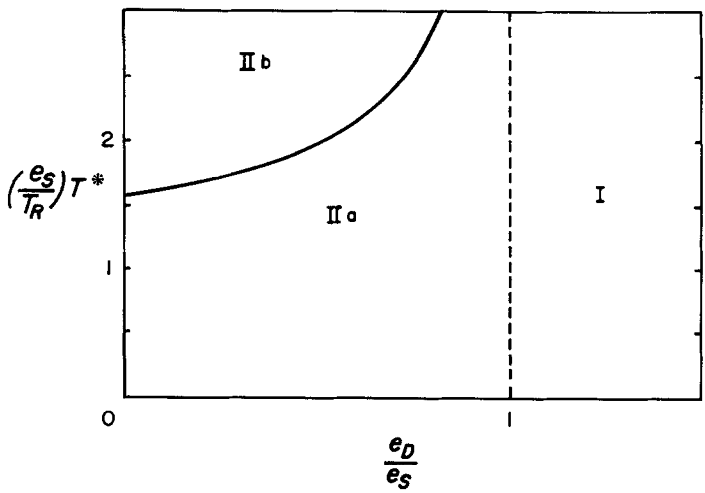

FIG. 1. A graphical representation of the Hayes criteria for the local stability of the equilibrium market price. Any combination of the parameters (e_d, e_S, T_R, T^*) falling into region I or IIa corresponds to locally stable equilibrium prices P^*, while parameter sets in region IIb correspond to an unstable equilibrium price. The solid concave up curve is the graph of Eq. (7c).

it is clear that although the linearized Eq. (4) contains four essential economic parameters (e_D, e_S, T_R, and T^*), the stability of the equilibrium market price P^* depends only on the two ratios e_D/e_S and $e_S T^*/T_R$. Thus, the Hayes criteria may be examined by plotting them in the $e_S T^*/T_R$ vs. (e_D/e_S) plane. Since both e_D and e_S are positive, attention need only be confined to the first quadrant of this plane which is naturally divided into two separate regions by conditions (7) and (7a). The division between these two areas is indicated by the dashed vertical line at $(e_D/e_S) = 1$ in Fig. 1. From condition (6), whenever the ratio (e_D/e_S) falls into region I (cf. Fig. 1), the equilibrium price is locally stable irrespective of the value of T^*. However, in region II, the situation is more complicated. Region II, by virtue of Eq. (8) and the inequality $e_S T^*/T_R < e_S T_{\text{crit}}/T_R$, which follows from (7b), is naturally divided into two subregions, IIa and IIb, and the boundary between them is indicated by the solid curved line which is the graph of Eq. (8). In the limit as $(e_D/e_S) \to 0$, $e_S T_{\text{crit}}/T_R \to \pi/2$ as indicated on the graph of (8). Thus, from inequality 7c, for all values of the parameters such that a point $(e_D/e_S, e_S T^*/T_R)$ lies in region IIa, the equilibrium price will be stable. Once the point passes into region IIb, for whatever reason, the equilibrium price becomes unstable.

The graphical representation of stability in Fig. 1 may be used to examine the effect of an alteration of model parameters on an initially stable equilibrium price as one parameter is varied at a time, holding the other three constant. The conclusions are summarized in Table I.

To this point, only variations in the several economic parameters leading to a loss of stability of the equilibrium price P^* have been considered. What happens when P^* becomes unstable? Exactly when stability is lost, $\text{Re}(\lambda) = 0$ and this condition is defined by the relation $T^* = T_{\text{crit}}$ between the production lag T^* and the critical combination of the elasticities of

TABLE I

Summary of Stability Results

Parameter varied	Regions visited	Stability of Equilibrium
Increased production delay	I → I	Always stable
	IIA → IIB	Eventually unstable
Decreased demand elasticity	I → IIA	Always stable
or	if $e_S T^*/T_R < \frac{1}{2}\pi$	
Increased price relaxation	I → IIA → IIB	Eventually unstable
	if $e_S T^*/T_R > \frac{1}{2}\pi$	
Increased supply elasticity		
or	I → IIA → IIB	Eventually unstable
Decreased price relaxation		

supply and demand and the supply and demand relaxation times. Using Eq. (7c) gives the explicit relation

$$T^* = \frac{T_R \cos^{-1}(-e_D/e_S)}{[(e_S)^2 - (e_D)^2]^{1/2}} \tag{9}$$

defining when $\mathrm{Re}(\lambda) = 0$. Remember that the graph of Eq. (8), the concave up curve in Fig. 1, defines a locus of points in parameter space for which $\mathrm{Re}(\lambda) = 0$. Below and to the right of this curve $\mathrm{Re}(\lambda) < 0$ (local stability), while above the curve $\mathrm{Re}(\lambda) > 0$ (local instability). Whenever the four parameters e_D, e_S, T_R, and T^* satisfy (9) the eigenvalue Eq. (5) has a purely imaginary solution $\lambda = \pm i\omega_H$. Thus for the very special combination of parameters defined by Eq. (9) the linear variational equation has an oscillatory solution and the period may be calculated exactly.

To see how this works, refer again to Eq. (4). Set $\lambda = \mu + i\omega$ and separate the real and imaginary parts to give the two equations

$$\mu T_R + e_D = -e_S \exp(-\mu T^*) \cos(\omega T^*) \tag{10a}$$

and

$$\omega T_R = e_S \exp(-\mu T^*) \sin(\omega T^*). \tag{10b}$$

Since $\mathrm{Re}(\lambda) = 0$ is the point of interest, set $\mu = 0$, square both equations, and add them to find that when $\mathrm{Re}(\lambda) = 0$, $\omega_H T_R = [(e_S)^2 - (e_D)^2]^{1/2}$. Thus the period of the periodic solution that ensues when the four parameters of the model satisfy (9) is given exactly by $T_H = 2\pi/\omega_H$, or by

$$T_H = \frac{2\pi T_R}{[(e_S)^2 - (e_D)^2]^{1/2}}. \tag{11}$$

Equation (11) makes the role of the elasticities of supply and demand and the price relaxation time in determining the period of the periodic solution totally explicit.

This period is denoted by T_H because it is known as the Hopf period. Exactly when Eq. (9) is satisfied a Hopf bifurcation takes place [20, Chapter 4]. (Depending on auxiliary conditions involving higher order nonlinear terms in the expansion about P^*, this bifurcation may be either subcritical or supercritical, but all of the numerical experiments carried out (see below) indicate that the bifurcation is supercritical.) This Hopf bifurcation is marked by the passing of a pair of complex conjugate eigenvalues from the left hand to the right hand side of the complex plane. Just as they cross the imaginary axis (when Eq. 9 is satisfied) the loss of stability of the equilibrium price is accompanied by the birth of a cyclic oscillation of period T_H in the market price $P(t)$ near P^*. When stability of P^* is lost, the amplitude of this oscillation varies as the square root of the parameter being changed, i.e., e_S, e_D, T_R, T^*, that gives rise to this crossing of the roots [20].

In addition to the explicit relation between T_H and the elasticities and relaxation time given by Eq. (11), it is possible to assign upper and lower bounds to the period of the oscillation in market price that occurs at the Hopf bifurcation. This is easily accomplished by noting from Eq. (10a) that at $\mu = 0$, $\omega_H T^* = T_R \cos^{-1}(-e_D/e_S)$. Thus Eq. (11) may be written in the alternate equivalent form

$$T_H = \frac{2\pi T^*}{\cos^{-1}(-e_D/e_S)}.$$

Since $0 \leqslant (e_D/e_S) \leqslant 1$ (cf. inequality 7a), $\frac{1}{2}\pi \leqslant \cos^{-1}(-e_D/e_S) \leqslant \pi$, and thus

$$2T^* \leqslant T_H \leqslant 4T^*. \tag{12}$$

Therefore, this commodity market model predicts that when the equilibrium price becomes unstable there will be an oscillation in market price with a period that is between two and four times the production plus storage lag T^* evaluated at the steady state. These results, coupled with the well documented instability of many commodity markets, suggest that highly responsive and well informed commodity marketing schemes with elasticities of supply exceeding elasticities of demand may be primary contributors to commodity price fluctuations whose periods are of the order of the inherent production delays.

This analysis gives information about the stability of the commodity market in response to small deviations away from equilibrium. In order to examine the full behaviour of the model, we must specify the demand and supply schedules as well as the form of the state dependent delay. Then the local analysis plus numerical simulations may be used to examine the complete behaviour of the model.

To investigate the global behaviour of this commodity market model, we chose a demand schedule D that was a smooth monotone decreasing function of P and a supply schedule S that was a monotone increasing function of P_S, and set the minimum delay $T_{\min} > 0$ and the storage delay $\Delta(P)$ to be either zero or a monotone decreasing non-negative function of current market price. With the functions selected it was possible to completely characterize the local stability of the single equilibrium price P^*. An implementation of the scheme of Feldstein and Neves [12], specifically designed for obtaining the numerical solutions of delay differential equations with state (P) dependent delays, was used to explore the full range of commodity price dynamics.

In every case, the linear analysis accurately predicts the set of parameter values for which the equilibrium price is unstable. It furthermore accurately predicts the period of the ensuing oscillation found in the numerical solutions when stability is lost. The numerical results support the analytic conclusion that production delays are destabilizing factors, and that the introduction of storage delays through improved technology can lead to an unstable situation. The amplitude of the oscillations in the numerically computed market price is increased by the storage policy, as is the period of the oscillations. The numerical solutions offer no indication that there is any other behaviour to be observed in this model save for a single (apparently supercritical) Hopf bifurcation from a stable equilibrium price to an apparently stable limit cycle oscillation in market price. There is no numerical evidence for the existence of higher order bifurcations to other periodic or chaotic solutions, as may be found in some delay differential equations and systems of ordinary differential equations. This indicates that this model offers a reasonable explanation for the cyclical behaviour seen in many commodity markets, and suggests that the pronounced higher frequency fluctuations seen in these markets may be the consequence of extraneous market noise.

VI. Discussion

The discrete time cobweb models that have been so widely exploited in economic modelling are, in some sense, limiting cases of the model developed here. To see this, note that when the delay T is constant and the price relaxation time T_R is very short so price adjustment is quite rapid, then $(dP/dt)/P \simeq 0$. Thus, in this limiting case Eq. (1) reduces to an implicit (generally nonlinear) difference equation in $P(t)$ given by $D(P(t)) = S(P(t-T))$. If the demand schedule D is monotone, and thus invertible, then this implicit relation may be rewritten in the explicit form

$$P(t) = F(P(t-T)), \tag{13}$$

where $F = D^{-1} \circ S$ and "$\circ$" denotes composition. When time t is measured in units of the constant lag T, then the explicit Eq. (13) is a limiting cobweb version of the present model that should hold under conditions of very rapid price adjustment. However, the correspondence between the behaviour of difference equations such as Eq. (13), derived in this manner, and delay differential equations such as Eq. (1) may be severely limited [46].

It is well known [5] that the equilibrium price P^*, given by the solution of $D(P^*) = S(P^*)$, is locally asymptotically stable when $|F'(P^*)| < 1$ and unstable when $|F'(P^*)| \geqslant 1$. Now $F'(P^*) = S'(P^*)/D'(P^*) = e_S/e_D$ so these conditions are simply the discrete time limiting cases of the Hayes criteria for the stability of the continuous time model as would be expected.

The periodic and chaotic properties of time series generated by maps such as Eq. (12) have been the subject of intense mathematical investigation during the last decade [5, 40, 42, 53]. Recently a variety of discrete time economic models that may be case into the form of the map (13), or higher dimensional versions, have been studied because of their analogy with similar maps having a period doubling bifurcation structure leading to the generation of chaotic time series [2, 6–8, 11, 49, 56].

In an apparently little known paper, the British physiologist J. B. S. Haldane developed a linear model for a single commodity market [19] which is a special case of that presented here. Some 30 years later, apparently unaware of Haldane's work, Larson [38] proposed a linear "harmonic motion" model for the cyclical behaviour of the pork market. The equations of Larson's model are completely equivalent to the linearized Eq. (4) under the assumption that the elasticity of demand is identically zero or that the price relaxation time is infinitely rapid.

It is clear from the present work that production delays in commodity markets are potentially destabilizing factors, as has been pointed out previously [15, 19, 31–35, 38]. However, the analysis presented here is the first in which there has been such an explicit consideration of the roles played by a variety of economic parameters in determining the stability of a single commodity market, and the relation of the period of the oscillation to various economic parameters when the market becomes unstable. Further, this seems to be the first consideration of the effect of adding price dependent storage policies in commodity market models. From the results presented here, such policies are highly destabilizing. They may either destabilize a previously stable market, or exacerbate an unstable market situations by increasing the amplitude and period of oscillations in commodity prices.

REFERENCES

1. J. BELAIR AND M. C. MACKEY, A model for the regulations of mammalian platelet production, *Ann. N. Y. Acad. Sci.* **504** (1987), 280–282.
2. J. BENHABIB AND R. H. DAY, A characterization of erratic dynamics in the overlapping generations model, *J. Econ. Dynam. Control* **4** (1982), 37–55.
3. W. A. BROCK, Distinguishing random and deterministic systems: Abridged version, *J. Econ. Theory* **40** (1986), 168–195.
4. W. W. CHANG AND D. J. SMYTH, The existence and persistence of cycles in a non-linear model: Kaldor's 1940 model re-examined, *Rev. Econ. Stud.* **38** (1971), 37–44.
5. P. COLLET AND J.-P. ECKMANN, "Iterated Maps on the Interval as Dynamical Systems," Birkhäuser, Basel, 1980.
6. F. CUGNO AND L. MONTRUCCHIO, Some new techniques for modelling non-linear

economic fluctuations: A brief survey, *in* "Nonlinear Models of fluctuating Growth" (R. M. Goodwin, M. Kruger, and A. Vercelli, Eds.), pp. 146–165, Lecture Notes in Economics and Mathematical Systems, Vol. 228, Springer-Verlag, Berlin/New York, 1984.
7. R. H. Day, Irregular growth cycles, *Amer. Econ. Rev.* **72** (1982), 406–414.
8. R. H. Day and W. Shafer, Keynesian chaos, *J. Macroecon.* **7** (1986), 277–295.
9. R. Driver, A functional differential system in neutral type arising in a two-body problem of classical electrodynamics, *in* "Nonlinear Differential Equations and Nonlinear Mechanics," (J. P. LaSalle and S. Lefschetz, Eds.), pp. 474–484, Academic Press, New York, 1963.
10. M. Ezekiel, The cobweb theorem, *Quart. J. Econ.* **52** (1938), 255–280.
11 R. E. A. Farmer, Deficits and cycles, *J. Econ. Theory* **40** (1986), 77–88.
12. A. Feldstein and K. W. Neves, Higher order methods for state-dependent delay differential equations with nonsmooth solutions, *SIAM J. Numer. Anal.* **21** (1984), 844–863.
13. G. Gabish and H.-W. Lorenz, "Business Cycle Theory," Lecture Notes in Economics and Mathematical Systems, Vol. 283, springer-Verlag, Berlin/New York, 1987.
14. L. Glass and M. C. Mackey, "From Clocks to Chaos: The Rhythms of Life", Princeton Univ. Press, Princeton, NJ, 1988.
15. R. M. Goodwin, The nonlinear accelerator and the persistence of business cycles, *Econometrica* **19** (1951), 1–17.
16. R. M. Goodwin, M. Kruger, and A. Vercelli, Eds., "Nonlinear Models of Fluctuating Growth," Lecture Notes in Economics and Mathematical Systems, Vol. 228, Springer-Verlag, Berlin-New York, 1984.
17. J.-M. Grandmont and P. Malgrange, Nonlinear economic dynamics: Introduction, *J. Econ. Theory* **40** (1986), 3–12.
18. J. Guckenheimer and P. Holmes, "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields," Springer Verlag, New York/Berlin, 1983.
19. J. B. S. Haldane, A contribution to the theory of price fluctuations, *Rev. Econ. Stud.* **1** (1933), 186–195.
20. B. D. Hassard, N. D. Kazarinoff, and Y-H. Wan, "Theory and Applications of Hopf Bifurcation," Cambridge Univ. Press, Cambridge, 1981.
21. N. D. Hayes, Roots of the transcendental equation associated with a certain difference–differential equation, *J. Lond. Math. Soc.* **25** (1950), 226–232.
22. U. an der Heiden, Delays in physiological systems, *J. Math. Biol.* **8** (1979), 345–364.
23. U. an der Heiden and H.-O. Walther, Existence of chaos in control systems with delayed feedback, *J. Differential Equations* **47** (1983), 273–295.
24. U. an der Heiden, Stochastic properties of simple differential–delay equations, *in* "Delay Equations, Approximation and Application" (G. Meinardus and G. Nurnberger, Eds.), pp. 147–164. Birkhäuser, Basel, 1985.
25. U. an der Heiden and M. C. Mackey, The dynamics of production and destruction: Analytic insight into complex behaviour, *J. Math. Biol.* **16** (1982), 75–101.
26. U. an der Heiden and M. C. Mackey, Mixed feedback: A pardigm for regular and irregular oscillations, *in* "Temporal Disorder in Human Oscillatory Systems" (L. Rensing, U. an der Heiden, and M. C. Mackey, Eds.), Springer-Verlag, New York/Berlin, 1987.
27. J. R. Hicks, "A Contribution to the Theory of the Trade Cycle," Oxford Univ. Press, Oxford, 1950.
28. T. D. Howroyd and A. M. Russell, Cournot oligopoly models with time delays, *J. Math. Econ.* **13** (1984), 97–103.
29. S. Ichimura, Toward a general nonlinear macrodynamic theory of economic fluctuations, *in* "Post Keynesian Economics" (K. K. Kurihara, Ed.), Rutgers Univ. Press, New Brunswick, NJ, 1954.
30. N. Kaldor, A model of the trade cycle, *Econ. J.* **50** (1940), 78–92.

31. M. Kalecki, A macroeconomic theory of the business cycle, *Econometrica* **3** (1935), 327–344.
32. M. Kalecki, A theory of the business cycle, *Rev. Econ. Stud.* **4** (1937), 77–97.
33. M. Kalecki, A theory of the business cycle (1939), *in* "Essays in the Theory of Economic Fluctuations" (M. Kalecki, Ed.), Allen & Unwin, London, 1972.
34. M. Kalecki, "Studies in Economic Dynamics," Allen & Unwin, London, 1943.
35. M. Kalecki, "Theory of Economic Dynamics," Unwin University Books, London, 1952.
36. L. R. Klein and R. S. Preston, Stochastic nonlinear models, *Econometrica* **37** (1969), 95–106.
37. M. G. Kuczynski, Recent developments in business cycle theory, *J. Econ. Dynam. Control* **10** (1986), 255–260.
38. A. B. Larson, The hog cycle as harmonic motion, *J. Farm Econ.* **46** (1964), 375–386.
39. A. Lasota, Ergodic problems in biology, *Astérisque* **50** (1977), 239–250.
40. A. Lasota and M. C. Mackey, "Probabilistic Properties of Deterministic Systems," Cambridge Univ. Press, London/New York, 1985.
41. W. W. Leontief, Verzögerte Angebotsanpassung und Partielles Gleichgewicht, *Z. Nationalökon.* **5** (1934), 670–676.
42. A. J. Lichtenberg and M. A. Lieberman, "Regular and Stochastic Motion," Springer-Verlag, New York/Berlin, 1983.
43. R. E. Lucas, An equilibrium theory of the business cycle, *J. Pol. Econ.* **63** (1975), 1113–144.
44. M. C. Mackey and L. Glass, Oscillation and chaos in physiological control systems, *Science* **197** (1977), 287–289.
45. M. C. Mackey and U. an der Heiden, The dynamics of recurrent inhibition, *J. Math. Biol.* **19** (1984), 211–225.
46. J. Mallet-Paret and R. D. Nussbaum, A bifurcation gap for a singularly perturbed delay equation, *in* "Chaotic Dynamics and Fractals" (M. F. Barnsley and S. G. Demko, Eds.), pp. 263–286, Academic Press, New York, 1986.
47. L. Rensing, U. an der Heiden, and M. C. Mackey, Eds., "Temporal Disorder in Human Oscillatory Systems," Springer-Verlag, New York-Berlin, 1987.
48. U. Ricci, Die "Synthetische Ökonomie" von Henry Ludwell Moore, *Z. Nationalökon.* **1** (1930), 649–668.
49. D. G. Saari, Dynamical systems and mathematical economics, *in* "Models of Economic Dynamics" (H. F. Sonnenschein, Ed.), pp. 1–24, Lecture Notes in Economics and Mathematical Systems, Vol. 264, Springer-Verlag, Berlin/New York, 1986.
50. T. Sargent and N. Wallace, Rational expectations, the optimal monetary investment and the optimal money supply role, *J. Pol. Econ.* **83** (1975), 241–255.
51. G. J. Schinasi, A nonlinear dynamic model of short run fluctuations, *Rev. Econ. Stud.* **48** (1981), 649–656.
52. H. Schultz, "Der Sin der Statistischen Nachfragen," Veroffentlichungen der Frankfurter Gesellschaft für Konjunkturforschung, Vol. 10, Kurt Schroeder Verlag, Bonn, 1930.
53. H. G. Schuster, "Deterministic Chaos," Physik Verlag, Weinheim, Federal Republic of Germany, 1984.
54. W. Semmler, On a microdynamics of a nonlinear macrocycle model, *in* "Competition, Instability, and Nonlinear Cycles, (W. Semmler, Ed.), pp. 170–199, Lecture Notes in Economics and Mathematical Systems, Vol. 275, 170–199, Springer-Verlag, Berlin/New York, 1966.
55. E. Slutzky, The summation of random causes as the source of cyclic processes, *Econometrica* **5** (1937), 105–146.
56. M. J. Stutzer, Chaotic dynamics and bifurcation in a macro model, *J. Econ. Dynam. Control* **2** (1980), 353–376.

57. J. TINBERGEN, Bestimmung und Deutung von Angebotskurven, *Z. Nationalökon.* **1** (1930), 669–679.
58. V. TORRE, The existence of limit cycles and control in complete Keynesian system by theory of bifurcations, *Econometrica* **45** (1977), 1457–1466.
59. H. R. VARIAN, Catasrophe theory and the business cycle, *Econ. Inquiry* **17** (1979), 14–28.
60. H.-O. WALTHER, Homoclinic solutions and chaos in $\dot{x}(t) = f(x(t-1))$, *J. Nonlinear Anal.* **5** (1981), 775–788.
61. H. O. WALTHER, Bifurcations from a heteroclinic solution in differential delay equations, *Trans. Amer. Math. Soc.* **290** (1980) (1985), 213–233.
62. F. V. WAUGH, Cobweb models, *J. Farm. Econ.* **46** (1964), 732–750.
63. V. ZARNOWITZ, Recent work on business cycles in historical perpective: A review of theories and evidence, *J. Econ. Lit.* **23** (1985), 523–580.