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The Dynamics of Production and Destruction: Analytic Insight Into Complex Behavior

Uwe an der Heiden[1] and Michael C. Mackey[2]

[1] Forschungsschwerpunkt "Stabilitätsgrenzen biologischer Systeme", Universität Bremen NW2, D-2800 Bremen, Federal Republic of Germany
[2] Department of Physiology, McGill University, Montreal H3 G1 Y6, Quebec, Canada

Abstract. This paper analytically explores the properties of simple differential-difference equations that represent dynamic processes with feedback dependent on prior states of the system. Systems with pure negative and positive feedback are examined, as well as those with mixed (positive/negative) feedback characteristics. Very complex time dependent behaviors may arise from these processes. Indeed, the same mechanism may, depending on system parameters and initial conditions, produce simple, regular, repetitive patterns and completely irregular random-like fluctuations.

For the differential-delay equations considered here we prove the existence of: (i) stable and unstable limit cycles, where the stable cycles may have an arbitrary number of extrema per period; and (ii) chaos, meaning the presence of infinitely many periodic solutions of different period and of infinitely many irregular and mixing solutions.

Key words: Differential-difference equations – Positive feedback – Negative feedback – Mixed feedback – Limit cycles – Chaos – Physiological control processes

1. Introduction

The hallmark of our universe is its ubiquitous complexity. Though we are accustomed to the near chaos in our social systems, in the sciences we are conditioned to accept data with apparent regularities. Indeed, the rejection of sufficiently irregular (complex) data "for cause" is common. Historically the avoidance of complex behavior by theoreticians was rooted in the lack of mathematical techniques appropriate for the study of nonlinear systems.

However, irregular behavior has been the subject of a flurry of attention by scientists with diverse interests in the last few years (cf. May, 1976; Gurel and Rössler, 1978; and Helleman, 1980 for representative reviews and work). This interest was in part kindled by the work of Lorenz (1964), and by the rediscovery (Li and Yorke, 1975) and popularization of some of the analytic results of Sharkovski (1964) related to the regular and irregular behavior exhibited by the solutions of

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simple difference equation models of the type

$$x_{i+1} = H(x_i), \qquad i = 1, 2, 3, \ldots \tag{1.1}$$

for suitably defined nonlinear functions H.

Interest in discrete processes like (1.1) was heightened by applications in population biology, x_i representing population size in the ith generation for species with separated generations. With $H(x_i) = rx_i(1 - x_i)$ the discrete logistic equation is obtained (for this and more general discrete population models see Guckenheimer et al., 1977).

For one-dimensional nonlinear discrete time models of the type (1.1) there are a few techniques available from ergodic theory that give sufficient conditions for the appearance of completely irregular behavior (Lasota and Yorke, 1977; Pianigiani, 1979a, b; Ruelle, 1977; Collet and Eckmann, 1980). These techniques do not apply to discrete time systems of two or more dimensions, e.g. of the type

$$\begin{aligned} x_{i+1} &= F(x_i, y_i), \\ y_{i+1} &= G(x_i, y_i). \end{aligned} \tag{1.2}$$

The situation is even bleaker for the understanding of the behavior of continuous time systems modeled by nonlinear differential equations. Few general techniques are available to predict the behavior of a given system and the investigator is forced to rely on numerical simulations.

However, a class of delay-differential equations has recently been discovered that is amenable to analytic treatment (Peters, 1980; Walther, 1981a; an der Heiden et al., 1981; an der Heiden and Walther, 1982). In the present paper we consider a prototypical model from this class which is particularly relevant to applications and outstanding in its diversity of behavior.

Section 2 offers a description of processes resulting from the interaction of production and destruction. We identify a variety of biological systems to which these concepts are applicable and which are related to our specific model. There are several types of feedback involved. In the cases of pure negative or pure positive feedback a nearly complete picture of the dynamics can be obtained. This is given in Sections 3 and 4 respectively. These also serve as an introduction to the discussion of the much more complicated behavior with mixed feedback considered in Sections 5–7. We show how the combination of negative and positive feedback creates an infinity of new types of dynamical behavior.

Section 5 demonstrates the existence of stable limit cycles showing a spiral structure. These orbits spiral around an unstable limit cycle. The number of revolutions ranges from one to infinity depending on the value of a single parameter. In the limit the cycles evolve into an orbit which is homoclinic to the unstable limit cycle.

In Section 6 a large domain of parameters is determined where depending on initial conditions, the equation has infinitely many periodic and uncountably many aperiodic solutions. The erratic structure of most of the aperiodic solutions and their mixing trajectories come to what Li and Yorke (1975) called "chaotic" motion.

Section 7 reveals that there is a realm of arbitrary complex and irregular time behavior still to be discovered. New types of bifurcations are discussed in parallel with a series of illustrations showing the evolution of complexity. This section may be read independently of Sections 4–6.

2. Modeling Processes of Production and Destruction

Consider a process characterized by a time dependent quantity x (a single variable, a vector or a function of space), $x = x(t)$. Often it is possible to equate the rate of change (dx/dt) to a balance between the production rate p, and the destruction (or decay or consumption) rate d, of x:

$$dx/dt = p - d.$$

Table 1 lists a few biological examples, with references, where this concept has been or may be applied.

In many cases, through feedback or other interactive mechanisms production and destruction depend on the quantity x itself: $p = p(x)$ or $q = q(x)$. In general the dependence is complicated since the production (or destruction) at time t depends not only on $x(t)$ but also on the past history $x(\tilde{t})$, $\tilde{t} \leqslant t$, of the variable x. This may be formally represented by

$$dx/dt = p(x_t) - q(x_t), \tag{2.1}$$

where x_t denotes the function defined by $x_t(t') = x(t - t')$, $t' \geqslant 0$. Here p and q are functionals and (2.1) represents a functional-differential equation (Hale, 1977).

For most applications the dependence on the past history may be made explicit by an integral

$$y_i(t) = \int_{-\infty}^{t} K_i(t, t', x(t), x(t'))\, dt', \qquad i = p, q,$$

or more specifically

$$y_i(t) = \int_{-\infty}^{t} g_i(t - t') f_i(x(t'))\, dt', \qquad i = p, q,$$

where the function y_p (or y_q) gives the prescription for evaluating the total influence of the history of x on the present production (or destruction). The function g_i (the kernel) specifies the weight to be attached to some function f_i of x at each point of time in the past.

If the kernel functions g_p, g_q can be written as sums of exponentials then the integro-differential equation $dx/dt = p(y_p) - q(y_q)$ is equivalent to a system of ordinary differential equations possessing no explicit dependence on prior history (MacDonald, 1978).

If the present effects stem from a very narrow region in the past, the kernel g is sharply peaked about some previous time, say $t - \tau$, with, in the simplest case, a constant delay $\tau > 0$. In the limit $g(t - t') = \delta(t - t' - \tau)$, the Dirac delta function, and Eq. (2.1) acquires the form of a difference-differential equation

$$dx/dt = p(x(t - \tau_1)) - q(x(t - \tau_2)). \tag{2.2}$$

It will be evident to the reader after having read this paper that a general discussion

Table 1. Some biological and economic topics, where production (p) and destruction (d) play a role for the dynamics of a relevant quantity (x). The numbers refer to literature, where mathematical models of type (2.3), (A) have been applied to the processes

Area	x	p	d	τ (delay)
Population biology [6], [18]	Population size	Reproduction rate	Death rate	Generation time
Hematopoiesis[2], [19], [20], [29]	Concentration of blood cells	Rate of stem cell differentiation	1/blood cell life·time	Cell cycle time
Neurophysiology [8], [10], [11], [3]	Impulse frequency	Excitatory potentials	Inhibitory potentials	Synaptic and conduction time delays
Respiration [5], [19]	CO_2-concentration	Whole body CO_2-production rate	Respiratory CO_2-elimination rate	Circulation time from lungs to brain
Metabolic regulation [10], [18]	Concentration of end product	mRNA-synthesis	End product consumption	Transformation mRNA to end product
Agricultural commodity markets [1]	Commodity price	Demand for commodity	Supply of commodity	Commodity production time

of the behavior of solutions to systems like (2.1) is out of reach. Instead we study here a special case of (2.2), namely the equation

$$dx/dt = f(x(t-\tau)) - \alpha x(t) \tag{2.3}$$

with a constant $\alpha > 0$.

This equation has been used to understand a variety of problems in various areas of biology, as listed in Table 1 (an der Heiden, 1979; Glass and Mackey, 1979; Mackey and Glass, 1977; Wazewska-Czyewska and Lasota, 1976).

Equation (2.3) may be thought of as representing a process where the single state variable x decays with a rate α proportional to x at the present, and is produced with a rate dependent on the value of x some time in the past. There are several different forms that the production rate function f may take. Two extreme types are those where f is a monotone decreasing or increasing function of $\xi = x(t-\tau)$. The first case corresponds to a pure negative feedback, the second to positive feedback. Generally, the maximal production rate is attained neither at $\xi = 0$ (negative feedback) nor for very large values of ξ (positive feedback), but rather at some intermediate value. Thus the general situation is characterized by a production function f that is a "humped" function of $x(t-\tau)$. This type we call mixed feedback and it is the main subject of this paper.

Since $[-\alpha x(t)]$ describes destruction the quantity x must be nonnegative, $x(t) \geqslant 0$. Since f describes production it is logical to assume that $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative function. By rescaling the time axis it can always be ensured that $\tau = 1$. Therefore we study the equation

$$\frac{dx}{dt}(t) = f(x(t-1)) - \alpha x(t), \tag{A}$$

with a constant $\alpha > 0$.

A solution to Eq. (A) is a continuous function $x\colon [-1, \infty) \to \mathbb{R}_+$ obeying (A) for all $t > 0$. The continuous function $\varphi\colon [-1, 0] \to \mathbb{R}_+$, $\varphi(t) = x(t)$ for all $t \in [-1, 0]$, is called the initial condition to x.

It is well known that if f is piecewise continuous, nonnegative and bounded then to any continuous, nonnegative initial condition φ there corresponds a unique, nonnegative, continuous solution $x = x_\varphi$ to Eq. (A), which is defined for all $t > 0$.

3. Negative Feedback

Throughout this section f is assumed to be monotone decreasing:

$$\xi_1 \leqslant \xi_2 \qquad \text{implies} \qquad f(\xi_1) \geqslant f(\xi_2).$$

The main results in the literature on Eq. (A) under this condition are summarized as follows: If f is additionally differentiable then there is a critical value $\beta = \beta(\alpha)$ such that the unique stationary solution $x = \bar{x}$, $d\bar{x}/dt = 0$, $f(\bar{x}) = \alpha\bar{x}$, is

(i) locally asymptotically stable if $f'(\bar{x}) < \beta(\alpha)$,

(ii) unstable if $f'(\bar{x}) > \beta(\alpha)$.

If $f'(\bar{x}) > \beta(\alpha)$ then Eq. (A) has a non-constant periodic solution. Proofs of this periodicity result are rather lengthy and intricate (Chow, 1974; Kaplan and Yorke, 1977; Hadeler and Tomiuk, 1977). Also it is not known whether the periodic solution is unique and stable. However, as we show, there is a special, though admittedly extreme, case where a complete characterization is easily obtainable. Another reason to investigate this case is that it facilitates understanding of the situation with mixed feedback.

We define a piecewise constant function f: $\mathbb{R}_+ \to \mathbb{R}_+$ by

$$f(\xi) = \begin{cases} c & \text{if} \quad 0 \leqslant \xi \leqslant b, \\ 0 & \text{if} \quad b < \xi, \end{cases} \tag{F1}$$

where b and c are positive constants. (We shall point out in the end that the results below are not an artefact of the discontinuity.)

With the nonlinearity (F1) any solution to Eq. (A) satisfies, whenever $0 \leqslant t_0 < t$,

$$x(t) = x(t_0)\exp(-\alpha(t - t_0)) \qquad \text{if} \qquad x(s-1) > b \qquad \text{for all} \qquad s \in (t_0, t), \tag{3.1a}$$

and

$$x(t) = \gamma - (\gamma - x(t_0))\exp(-\alpha(t - t_0)) \qquad \text{if} \qquad x(s-1) \leqslant b \qquad \text{for all} \qquad s \in (t_0, t), \tag{3.1b}$$

where $\gamma = c/\alpha$. Thus any solution is a continuous function which is piecewise composed of two types of exponentials, one approaching 0, the other approaching $\gamma = c/\alpha$ at a rate α. It is easy to see that solutions starting with values between 0 and γ are bounded between 0 and γ forever. Moreover, for arbitrary initial conditions, $x(t)$ approaches the interval $[0, \gamma]$ as $t \to \infty$.

The following two theorems characterize the behavior of the solutions to (A) in case of the nonlinearity (F1).

Theorem 3.1. *Let the function f be given by* (F1). *If $\gamma = c/\alpha < b$ then $x = \gamma$ is a globally asymptotically stable stationary solution to Eq.* (A).

Proof. Obviously $x = \gamma$ is the only constant solution. Let x be any solution. Then x obeys (3.1) and, since $\gamma < b$, there is $t_0 \geqslant 0$ such that $x(t) < b$ for all $t \in [t_0 - 1, t_0]$. Hence x obeys (3.1b) for all $t > t_0$. Q.E.D.

Theorem 3.2. *Let the function f be given by* (F1). *If $\gamma > b$ then Eq.* (A) *has an asymptotically orbitally stable periodic solution $x = \tilde{x}$. The period of $\tilde{x}$ is larger than 2. The orbit of $\tilde{x}$ attracts all orbits corresponding to monotone initial conditions.*

Proof. Let φ be such that there is some $w \in (-1, 0)$ with $0 \leqslant \varphi(t) < b$ for all $t \in (-1, w)$, $\varphi(t) > b$ for all $t \in (w, 0)$. For $t \in (0, w+1)$ the solution $x = x_\varphi$ obeys (3.1b), $t_0 = 0$. Since $x(0) > b$ and $\gamma > b$ we obtain $x(t) > b$ for all $t \in (w, w+1)$, $x(w+1) > 1$. It follows from (3.1a) with $t_0 = w + 1$ that

$$x(t) = x(w+1)\exp(-\alpha(t - w - 1))$$

for all $t \in (w+1, t_1+1)$, where $t = t_1$ is the first time satisfying $t_1 > w + 1$ and $x(t_1) = b$. In particular

$$x(t) = b\exp(-\alpha(t - t_1)) \quad \text{for all} \quad t \in [t_1, t_1 + 1], \quad x(t_1 + 1) = b\exp(-\alpha). \tag{3.2}$$

This result shows that the solution is independent of φ for all $t > t_1$ (up to a shift in time).

Since $x(t) < b$ for all $t \in (t_1, t_1 + 1)$, the solution obeys (3.1b) with $t_0 = t_1 + 1$ for all $t \in [t_1 + 1, t_2 + 1]$, where t_2 denotes the first time satisfying $t_2 > t_1 + 1$ and $x(t_2) = b$. Again, since $x(t) > b$ for all $t \in (t_2, t_2 + 1)$, we have $x(t) = x(t_2 + 1)\exp(-\alpha(t - t_2 - 1))$ for all $t \in [t_2 + 1, t_3 + 1]$, where t_3 is the first t satisfying $t_3 > t_2 + 1$, $x(t_3) = b$. In particular

$$x(t) = b\exp(-\alpha(t - t_3)) \qquad \text{for all} \qquad t \in [t_3, t_3 + 1]. \tag{3.3}$$

Comparing (3.3) and (3.2) shows that x on the interval $[t_1, t_3]$ is just one cycle of a periodic solution $\tilde{x}$.

Analogous considerations hold for solutions corresponding to initial conditions where the presumed inequality on φ is reversed or which are totally above or below b. All their orbits converge to the orbit of $\tilde{x}$.

To prove the asymptotic orbital stability of $\tilde{x}$, let T be the period of $\tilde{x}$ and let $\varepsilon > 0$ be such that $\tilde{x}(t_3 + t) - \varepsilon > b$ for all $t \in [t_3 - 1 - \varepsilon, t_3 - \varepsilon]$, $t_3 - 1 - \varepsilon > t_2$. If $\delta = \delta(\varepsilon) > 0$ is sufficiently small and the initial condition ψ near to the orbit of $\tilde{x}$, i.e. there is $t^* > 0$ with $|\psi(t) - \tilde{x}(t^* + t)| < \delta$ for all $t \in [-1, 0]$, then

$$|x_\psi(t) - \tilde{x}(t^* + t)| < \varepsilon \qquad \text{for all} \qquad t \in [0, T]$$

and hence there is some time $\tilde{t} \in [t^*, t^* + T]$ with

$$x_\psi(t) > b \qquad \text{for all} \qquad t \in [\tilde{t}, \tilde{t} + T]. \tag{3.4}$$

This inequality implies that x_ψ satisfies a relation as in (3.3) for some time t^{**} instead of t_3, i.e. the orbit of x_ψ is on the orbit of $\tilde{x}$ for all $t > t^{**}$. Q.E.D.

An illustration of the periodic solution $\tilde{x}$ is given by Fig. 5(a) where the parameters are $b = 2$, $c = 4$, $\alpha = 0.6$.

Remark 3.3. In a similar, but technically more complicated, fashion it is provable that Theorems 3.1 and 3.2 also hold for continuous nonlinearities $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(\xi) = 0$ for all $\xi > b + \delta$, $f(\xi) = c$ for all $\xi \in [0, b]$ and f monotone on $[b, b + \delta]$, if δ is positive and sufficiently small. (This type of a continuous approximation to a step function we owe to H.-O. Walther who used it in [27]).

The considerations in the proof of Theorem 3.2 can be extended to obtain the following *criterion for periodicity*, which will be helpful later to show the existence of complicated limit cycles.

Lemma 3.4. *Let $b > 0$. Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a piecewise continuous function satisfying $f(\xi) = 0$ for all $\xi > b$. Assume x is a solution to Eq.* (A) *having the following property*:

> *There are two times t_1 and t_2, $t_1 < t_2$ such that $x(t) > b$ for all $t \in [t_1, t_1 + 1]$ and for all $t \in [t_2, t_2 + 1]$; additionally $x(t) < b$ for some $t \in [t_1, t_2]$.* (P)

Then x is periodic for $t > t_2$. The corresponding periodic orbit is asymptotically orbitally stable.

Proof. The solutions to (A) obey (3.1a). In particular

$$x(t) = x(t_i + 1)\exp(-\alpha(t - t_i - 1)) \qquad \text{for all} \qquad t \in [t_i + 1, \tilde{t}_i + 1],$$

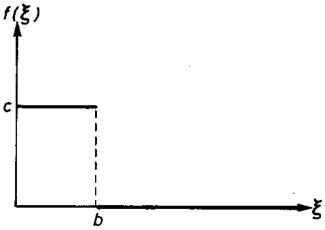

Fig.1

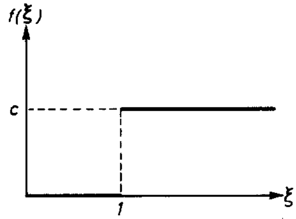

Fig.2

Fig. 1. Extreme case of a negative feedback nonlinearity
Fig. 2. Extreme case of a positive feedback nonlinearity

where $\tilde{t}_i$ is the first time obeying $\tilde{t}_i > t_i$ and $x(\tilde{t}_i) = b$, $i = 1, 2$. Since $x(t) < b$ for some $t \in [t_1, t_2]$, we have $\tilde{t}_2 > \tilde{t}_1$. The time courses of x on the two intervals $[t_i, t_i + 1]$, $i = 1, 2$, are identical. Hence $[\tilde{t}_1, \tilde{t}_2]$ corresponds to one period of x, which is periodic for all $t > \tilde{t}_1$. The stability proposition is proved as that of Theorem 3.2. Q.E.D.

4. Positive Feedback

In this section f: $\mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be a monotone increasing, bounded function. There seems to be no investigation of Eq. (A) with respect to positive feedback in the literature. Here again we study the extreme situation where f is a step function or a smooth function near to a step function, allowing us to obtain nearly complete insight.

Let f be defined by

$$f(\xi) = \begin{cases} 0 & \text{if } \ 0 \leqslant \xi \leqslant 1, \\ c & \text{if } \ \xi > 1, \end{cases} \tag{F2}$$

where $c > 0$ is constant. Then for any solution x to Eq. (A) and for any two times t_0, t with $0 \leqslant t_0 < t$, the following relations hold:

$$x(t) = x(t_0)\exp(-\alpha(t - t_0)) \quad \text{if} \quad x(s) < 1 \quad \text{for all} \quad s \in [t_0 - 1, t - 1], \tag{4.1a}$$

and

$$x(t) = \gamma - (\gamma - x(t_0))\exp(-\alpha(t - t_0)) \quad \text{if} \quad x(s) > 1$$
$$\text{for all} \quad s \in [t_0 - 1, t - 1], \qquad \gamma = c/\alpha. \tag{4.1b}$$

For all $\alpha > 0$ and all $c > 0$ Eq. (A) has the constant solution $\bar{x}_1 = 0$.

Theorem 4.1. *Let the function f be given by* (F2). *If $\gamma = c/\alpha < 1$ then the stationary solution $\bar{x}_1$ is globally asymptotically stable.*

Proof. Since $\gamma < 1$ it follows from (4.1) that to each solution x there is a time $T > 0$ such that $x(t) < 1$ for all $t > T$. There $x(t)$ for all $t > T + 1$ evolves according to (4.1a) with $t_0 = T + 1$. Hence x approaches 0 as $t \to \infty$. Q.E.D.

If $\gamma > 1$ then there is a second constant solution $\bar{x}_2 = \gamma$.

Theorem 4.2. *Let the function f be given by* (F2). *If* $\gamma > 1$ *then the constant solutions* $\bar{x}_1 = 0$ *and* $\bar{x}_2 = \gamma$ *to Eq.* (A) *are* (*locally*) *asymptotically stable.*

Proof. It follows from (4.1) that solutions x_φ converge exponentially towards $\bar{x}_1$ if their initial condition φ satisfies $0 \leqslant \varphi(t) < 1$ for all $t \in [-1, 0]$. Similarly x_φ converges exponentially towards $\bar{x}_2$ if $\varphi(t) > 1$ for all $t \in [-1, 0]$. Q.E.D.

The proof of the last theorem shows that, if $\gamma > 1$, solutions to initial conditions larger than 1 converge to $\bar{x}_2$, and those to initial conditions smaller than 1 converge to $\bar{x}_1$. However, what happens if the initial condition oscillates around the level 1? Answering this question leads to the discovery of unstable periodic solutions playing an important role for the understanding of more complicated behavior in the following sections.

Theorem 4.3. *Let the function f be given by* (F2). *If* $\gamma > 1$ *then Eq.* (A) *has an unstable periodic solution* $\tilde{\bar{x}}$. *This solution separates domains of attraction of the stationary solutions* $\bar{x}_1$ *and* $\bar{x}_2$; *its* (*smallest*) *period is between* 1 *and* 2. *There is exactly one minimum within one cycle of* $\tilde{\bar{x}}$.

Proof. A set D of (continuous) initial conditions is defined as follows: $\varphi \in D$ if there is some $w = w(\varphi) \in [0, 1]$ such that $\varphi(t) > 1$ for all $t \in [-1, -1 + w)$, $\varphi(t) < 1$ for all $t \in (-1 + w, 0)$, $\varphi(0) = 1$.

A map $V\colon D \to [0, 1]$ is induced by $V(\varphi) = w$. Because of (4.1) a solution to $\varphi \in D$, $V(\varphi) = w$, satisfies

$$x(t) = \gamma - (\gamma - 1)\exp(-\alpha t) \qquad \text{for all} \qquad t \in [0, w], \tag{4.2}$$

$$x(t) = x(w)\exp(-\alpha(t - w)) \qquad \text{for all} \qquad t \in [w, 1]. \tag{4.3}$$

Observe that the solution x on $[0, 1]$ is uniquely determined by w. Hence x for all $t > 0$ only depends on w. This is denoted by writing $x = x_w$. The values

$$x_w(w) = \gamma - (\gamma - 1)\exp(-\alpha w) \tag{4.4}$$

and

$$x_w(1) = x_w(w)\exp(-\alpha(1 - w)) = [\gamma(e^{\alpha w} - 1) + 1]\exp(-\alpha) \tag{4.5}$$

are increasing with respect to w.

There is a unique $w = w_2 \in (0, 1)$ obeying $x_{w_2}(1) = 1$. For all $w \geqslant w_2$ we have $x_w(t) > 1$ for all $t \in (0, 1)$, hence Eq. (4.1b) with $t_0 = 1$ applies for all $t > 1$ and $x_w(t)$ converges to $\bar{x}_2 = \gamma$ as $t \to \infty$.

Assume $w \leqslant w_2$. Then there is $t_1 \in [w, 1]$ such that $x_w(t_1) = 1$. If w increases from 0 to w_2 then $t_1 = t_1(w)$ increases from 0 to 1.

For $w \in [0, w_2]$ and $t \in [1, 1 + t_1]$ we have

$$x_w(t) = \gamma - (\gamma - x_w(1))\exp(-\alpha(t - 1)).$$

It follows that $x_w(1 + t_1)$ is an increasing function of w;

$$x_0(1 + t_1) = \exp(-\alpha) < 1, \qquad x_{w_2}(1 + t_1) = \gamma - (\gamma - 1)\exp(-\alpha) > 1.$$

There is a unique $w_1 \in (0, w_2)$ obeying $x_{w_1}(1 + t_1) = 1$. For $w \in [0, w_1]$ we have $x(t) < 1$ for all $t \in (t_1, t_1 + 1)$. Therefore for these w the corresponding solutions converge to 0 as $t \to \infty$.

Assume $w \in [w_1, w_2]$. Then there is $t_2 = t_2(w) \in [1, 1 + t_1]$ obeying $x_w(t_2) = 1$. Define another initial condition $\psi = \psi(w)$ by

$$\psi_w(t) = x_w(t_2 + t) \qquad \text{for all} \qquad t \in [-1, 0].$$

Obviously $\psi_w \in D$ and $\psi_w(1 - (t_2(w) - t_1(w))) = 1$. Define a continuous map

$$F\colon [w_1, w_2] \to [0, 1] \qquad \text{by} \qquad F(w) = V(\psi_w) = 1 - (t_2(w) - t_1(w)). \quad (4.6)$$

It follows from

$$x_w(1) = \exp(-\alpha(1 - t_1)) \qquad \text{and} \qquad 1 = \gamma - (\gamma - x_w(1))\exp(-\alpha(t_2 - 1))$$

that

$$\exp(-\alpha(t_2 - t_1)) = x_w(1)(\gamma - 1)/(\gamma - x_w(1)),$$

hence

$$\exp(\alpha F(w)) = \exp(\alpha)\, x_w(1)(\gamma - 1)/(\gamma - x_w(1)), \quad (4.7)$$

$$F(w) = w + (\log(\gamma - 1) + \log x_w(w) - \log(\gamma - x_w(1))/\alpha. \quad (4.8)$$

The function F allows to determine the time course of the solutions to $\varphi \in D$ with $V(\varphi) = w \in [w_1, w_2]$: If $F(w) \leqslant w_1$ then $x_\varphi(t)$ converges to 0 as $t \to \infty$. If $F(w) \geqslant w_2$ then $x_\varphi(t)$ converges to γ as $t \to \infty$. If $F(w) \in (w_1, w_2)$ then x_φ evolves for $t > t_2(w)$ just as $x_\psi(t)$ evolves for $t > 0$, where $\psi \in D$ obeys $V(\psi) = F(w)$. Therefore the iterates of F determine the evolution of x_φ. The function F has the properties:

$$F(w_1) = 0, \quad F(w_2) = 1, \quad F \text{ is differentiable, and}$$

$$dF/dw > 1 \quad \text{for all} \quad w \in [w_1, w_2]. \quad (4.9)$$

It follows that F has a unique fixed point $w_p \in (w_1, w_2)$. With respect to iterates $F^n(w)$, $n = 1, 2, \ldots$, this fixed point is unstable.

Thus, we have the following conclusions: If $\varphi \in D$ obeys $V(\varphi) = w_p$ then for $t > t_2(w_p)$ the solution x_φ is periodic with minimal period between 1 and 2. This periodic solution is unstable since $\varphi \in D$, $V(\varphi) < w_p$ implies that $x_\varphi(t) \to 0$, and $\varphi \in D$, $V(\varphi) > w_p$ implies $x_\varphi(t) \to \gamma$ as $t \to \infty$. Q.E.D.

Remark 4.4. Theorems 4.1–4.3 also hold if the nonlinearity f is continuous and obeys $f(\xi) = 0$ for all $\xi \in [0, 1 - \delta]$, $f(\xi) = c$ for all $\xi \geqslant 1 + \delta$, f monotone on $[1 - \delta, 1 + \delta]$, under the conditions that $c > \alpha$ and $\delta = \delta(c, \alpha)$ is a positive, sufficiently small number. For details of the proof the reader is referred to an der Heiden and Walther (1982).

For the next section we need

Lemma 4.5. *The minimal value of the periodic solution* $\tilde{\tilde{x}}$ *in Theorem 4.3 is the positive root of the quadratic*

$$z^2 - (\gamma - (\gamma - 1)e^{-\alpha} - \gamma^2)z - e^{-\alpha}\gamma(\gamma - 1) = 0. \quad (4.10)$$

Proof. According to the proof of Theorem 4.3 the minimal value of $\tilde{\tilde{x}}$ is $z = x_{w_p}(1)$. Equation (4.5), $w_p = F(w_p)$, and Eq. (4.7) imply

$$z/\gamma + e^{-\alpha}/(1 - \gamma) = \exp(-\alpha(1 - w_p)) = z(\gamma - 1)/(\gamma - z),$$

implying Eq. (4.10). Q.E.D.

5. Mixed Feedback: Stable Limit Cycles of Spiral Type

The behavior of solutions to Eq. (A) is much more complicated in the situation of mixed feedback. Here the nonlinearity f is not monotone, but has at least one "hump". Mackey and Glass (1977), when they discovered the bewildering range of solution types by numerical experiments, used the smooth function

$$f(\xi) = K\frac{\xi}{1+\xi^n}, \qquad n > 1.$$

However, until now all efforts to analytically verify these computer results were without success. Therefore, as a paradigm of mixed feedback we consider the piecewise constant function f: $\mathbb{R} \to \mathbb{R}$ given by

$$f(\xi) = \begin{cases} 0 & \text{if} \quad 0 \leqslant \xi < 1 \quad \text{or} \quad \xi > b, \\ c & \text{if} \quad 1 \leqslant \xi \leqslant b, \end{cases} \tag{F3}$$

where b, c are positive constants satisfying $\gamma = c/\alpha > b > 1$. As in the previous sections generalizations to smooth, nearly piecewise constant nonlinearities are possible. Any solution x to (A) with (F3) obeys

$$x(t) = x(t_0)\exp(-\alpha(t-t_0)) \quad \text{if} \quad x(s-1) > b \quad \text{or} \quad x(s-1) < 1 \quad \text{for all} \quad s \in (t_0, t), \tag{5.1a}$$

$$x(t) = \gamma - (\gamma - x(t_0))\exp(-\alpha(t-t_0)) \quad \text{if} \quad 1 < x(s-1) < b \quad \text{for all} \quad s \in (t_0, t), \tag{5.1b}$$

whenever $0 \leqslant t_0 < t$. The nonlinearity (F3) can be viewed upon as a synthesis of (F1) and (F3). This observation gives some guide to the analysis. First it is easy to see that the periodic solutions $\tilde{x}$ and $\tilde{\tilde{x}}$ present in case of (F1) and (F2) respectively also exist for (F3) if the thresholds 1 and b are sufficiently separated. More precisely *a stable limit cycle $\tilde{x}$ oscillating around the level b exists if*

$$be^{-\alpha} > 1. \tag{5.2}$$

The proof proceeds as that of Theorem 3.2, where it was shown that $\min_t \tilde{x}(t) = be^{-\alpha}$. Similarly *an unstable limit cycle $\tilde{\tilde{x}}$ exists as a solution to* (A) *with* (F3) *if*

$$\gamma > 1 \qquad and \qquad \gamma - (\gamma - 1)e^{-\alpha} \leqslant b. \tag{5.3}$$

These conditions are sufficient since it was shown in the proof of Theorem 4.3 that

$$\max_t \tilde{\tilde{x}}(t) = \tilde{\tilde{x}}(w_p) < \gamma - (\gamma - 1)e^{-\alpha}.$$

These cycles are simple in that they have only one minimum within one (smallest) period. If condition (5.2) is violated (and hence $\tilde{x}$ is no longer a solution) but (5.3) still holds, then limit cycles with arbitrarily many minima within one period are possible as described by the following theorem. The proof explains that they wind n times around $\tilde{\tilde{x}}$, followed by a large excursion away from $\tilde{\tilde{x}}$ towards an arc of $\tilde{x}$, afterwards reentering the spiral structure ($n \to \infty$ as $be^{-\alpha} \to z$).

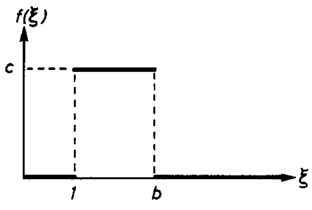

Fig.3

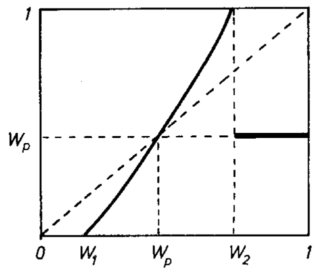

Fig.4

Fig. 3. Extreme case of a mixed feedback nonlinearity
Fig. 4. The map $\tilde{F}$: $[w_1, 1] \to [0, 1]$ in the situation, where $w_p = w(b)$ is a homoclinic point

Theorem 5.1. *Let the function f be given by* (F3). *Let* $\gamma = c/\alpha > 1$. *Let the positive root z of Eq.* (4.10) *satisfy*

$$\gamma - (\gamma - 1)e^{-\alpha} \leqslant ze^{\alpha} < \gamma \qquad \text{and} \qquad \gamma(\gamma - 1 + e^{-\alpha})^{-1} \leqslant ze^{\alpha}. \tag{5.4}$$

Then there is a sequence $(b_n)_{n=1,2,\dots}$ *of numbers* $b_n = b_n(\alpha, c)$ *satisfying the following conditions*:

(i) $\gamma - (\gamma - 1)e^{-\alpha} < b_n$, $b_{n+1} < b_n$, $b_\infty = \lim_{n\to\infty} b_n = ze^{\alpha}$, $b_n \leqslant \min(e^{\alpha}, \gamma)$,

(ii) *for every* $b \in [b_\infty, b_1)$ *there is an asymptotically orbitally stable periodic solution* $x = x_b$ *to Eq.* (A),

(iii) *if* $b \in (b_{n+1}, b_n]$ *then* x_b *has exactly* $n_0 + n$ *minima within one smallest period, where* n_0 *is a fixed number not depending on* b,

(iv) *the* (*smallest*) *period of* x_b *converges to* ∞ *as* $b \to b_\infty$.

(v) *there is exactly one maximum of* x_b *the value of which is larger than* b,

(vi) *for all* $b \in [b_\infty, b_1]$ *the periodic function* $\tilde{\tilde{x}}$, *given in Theorem 4.3, is an unstable solution to Eq.* (A) *with* (F3). *For* $b = b_\infty$ *the orbit of* $\tilde{\tilde{x}}$ *attracts all orbits corresponding to initial conditions* $\varphi > b$.

Proof. Let $b \in [ze^{\alpha}, b_{\max})$, $b_{\max} = \min(\gamma, e^{\alpha})$. Consider the initial condition $\varphi(t) = \gamma - (\gamma - b)\exp(-\alpha(t+1))$, $t \in [-1, 0]$ (which is on the orbit $\tilde{x}$ of Theorem 3.2). Then $x_\varphi(0) = \gamma - (\gamma - b)e^{-\alpha} > b$ and $x_\varphi(t) = x(0)\exp(-\alpha t)$ for all $t \in [0, t_1 + 1]$, where t_1 is the first $t > 0$ obeying $x(t_1) = b$. In particular $x_\varphi(t_1 + 1) = be^{-\alpha}$. Since $b < e^{\alpha}$ we have $x_\varphi(t_1 + 1) < 1$, and there is $t_2 \in (t_1, t_1 + 1)$ with $x_\varphi(t_2) = 1$, $x_\varphi(t) \in (1, b)$ for all $t \in (t_1, t_2)$. Therefore

$$x_\varphi(t) = \gamma - (\gamma - x(t_1 + 1))\exp(-\alpha(t - t_1 - 1)) \quad \text{for all} \quad t \in [t_1 + 1, t_2 + 1],$$

implying $x_\varphi(t_2 + 1) = \gamma - (\gamma - be^{-\alpha})/b$, since $1 = b\exp(-\alpha(t_2 - t_1))$.

$b \geqslant \gamma(\gamma - 1 + e^{-\alpha})^{-1}$ implies $x_\varphi(t_2 + 1) \geqslant 1$, and there exists $t_3 \in (t_2, t_2 + 1)$ such that $x_\varphi(t_3) = 1$. For $t > t_3$ the solution x_φ evolves just as the solution x_ψ corresponding to the initial condition $\psi = \psi(b)$ satisfying $\psi(t) = x_\varphi(t_3 + t)$, $t \in [-1, 0]$. Obviously $\psi \in D$, where D is defined in the proof of Theorem 4.3, the reader is assumed to be familiar with here. $w(b) = V(\psi) = 1 - (t_3 - t_2)$ is considered as a function of b.

Since

$$be^{-\alpha} = \exp(-\alpha(t_1 + 1 - t_2))$$

and

$$1 = \gamma - (\gamma - be^{-\alpha})\exp(-\alpha(t_3 - t_1 - 1))$$

it follows that

$$\exp(-\alpha(t_3 - t_2)) = be^{-\alpha}(\gamma - 1)/(\gamma - be^{-\alpha}).$$

Hence $w(b) = \alpha^{-1}\log[b(\gamma - 1)/(\gamma - be^{-\alpha})]$ is an increasing function of b obeying $w(ze^{\alpha}) = w_p$ (because of Lemma 4.5) and

$$w_{\max} = w(b_{\max}) = \begin{cases} 1 & \text{if} \quad e^{\alpha} \leqslant \gamma, \\ \alpha^{-1}\log(\gamma - 1)/(1 - e^{-\alpha}) < 1 & \text{if} \quad \gamma < e^{\alpha}. \end{cases}$$

The solution $x_b = x_\psi(t)$ evolves as described in the proof of Theorem 4.3 for all $t \in [0, t^* + 1]$, where $t^* > 0$ is the first time observing $x_b(t^*) = b$. As $b \geqslant \gamma - (\gamma - 1)e^{-\alpha}$ the following holds: For $b \in (ze^{\alpha}, b_{\max})$ let $n(b) \in \mathbb{N} \cup \{0\}$ be the smallest number such that $F^{n(b)}(w(b)) \in (w_2, 1]$. Then between 0 and t^* the solution x_b intersects the level 1 exactly at $2n(b)$ times $0 < \tau_1 < \cdots < \tau_{2n(b)} < t^*$, $\tau_{2i} - \tau_{2i-1} = F^i(w(b)) \leqslant 1$. For $t \in (\tau_{2n(b)}, \tau_{2n(b)} + 1)$ we have $b > x_b(t) > 1$. Hence

$$x_\varphi(t) = \gamma - (\gamma - x_b(\tau_{2n(b)} + 1))\exp(-\alpha(t - \tau_{2n(b)} - 1))$$

for all $t \in [\tau_{2n(b)} + 1, t^* + 1]$. In particular on $[t^*, t^* + 1]$ the solution x_b equals φ (up to a time shift), therefore x_b is periodic with period $t^* + t_3 + 1$. Lemma 3.4 implies that x_b is asymptotically orbitally stable. x_φ has exactly $n(b) + 1$ or $n(b) + 2$ minima per period. Let $n_0 \in \mathbb{N} \cup \{0\}$ be the minimal number such that

$$F^{-n_0}(1) \in (w(ze^{\alpha}), w_{\max}], \qquad w_n = F^{-n-n_0}(1), \qquad n = 0, 1, 2, \ldots .$$

Then there are values $b_n \in (ze^{\alpha}, b_{\max}]$ obeying $w_n = w(b_n)$ and having the properties described in the theorem. Q.E.D.

Remark 5.2. By essentially the same, but technically difficult, arguments it can be proved that Theorem 5.1 also holds for continuous functions f obeying

$$f(\xi) = \begin{cases} 0 & \text{if} \quad 0 \leqslant \xi < 1 - \delta \text{ or } \xi \geqslant b + \delta, \\ c & \text{if} \quad 1 - \delta \leqslant \xi \leqslant b - \delta, \end{cases}$$

f arbitrary, but monotone on the intervals $[1 - \delta, 1 + \delta]$, $[b - \delta, b + \delta]$, if $\delta > 0$ is sufficiently small.

Remark 5.3. The conditions of Theorem 5.1 are not contradictory. E.g. if $\gamma = 2$ and $\alpha > \log 3$ then they are simultaneously verified. Let the conditions of Theorem 5.1 hold and let $b \in [ze^{\alpha}, \min(\gamma, e^{\alpha})]$. The proof of this theorem shows that for every initial condition $\varphi \in D$ obeying $V(\varphi) \geqslant w_p$ and $\varphi(t) < b$ for all $t \in [-1, 0]$, the solution x_φ oscillates around the threshold 1. The times $0 < \tau_1 < \tau_2 < \cdots$ with $x_\varphi(\tau_i) = 1$ satisfy the one-dimensional difference equation

$$\tau_{2i} - \tau_{2i-1} = \tilde{F}(\tau_{2(i-1)} - \tau_{2(i-1)-1}),$$

where $\tilde{F}$: $[w_1, 1] \to [0, 1]$ is defined by

$$\tilde{F}(w) = \begin{cases} F(w) & \text{if } w \in [w_1, w_2] \quad \text{(see (4.8))}, \\ w(b) = \alpha^{-1} \log[b(\gamma - 1)/(\gamma - be^{-\alpha})] & \text{if } w \in (w_2, 1]. \end{cases}$$

Note that for $b = b_\infty$ we have $\tilde{F}(w) = w_p$ for all $w \in (w_2, 1]$. In this case w_p is an unstable, but with respect to all $w \in [w_p, 1]$ attractive fixed point of $\tilde{F}$.

This situation is of particular interest, since we have here an intersection of the unstable and stable manifold of the fixed point w_p, meaning that w_p is a *homoclinic point*. As may be imagined from Fig. 4.1, the smaller a perturbation w of w_p (i.e. the smaller $w - w_p, w > w_p$) the more iterations of $\tilde{F}$ are necessary to return to w_p (i.e. the larger is $n = n(w)$ with $\tilde{F}^n(w) = w_p$, indeed $\lim_{w \to w_p} n(w) = \infty$). Consequently there are arbitrarily small perturbations of $\tilde{\tilde{x}}$ leading to arbitrarily long lasting excursions away from $\tilde{\tilde{x}}$ followed by a rapid return to $\tilde{\tilde{x}}$.

Remark 5.4. It is often argued that unstable structures are either never or only exceptionally observable in nature. However, as the above example shows, it may be that the unstable manifold associated with the unstable structure is connected with (i.e. intersects) the stable manifold. In this situation the process again and again approaches the unstable structure arbitrarily close. Clearly, each time this occurs the unstable structure can be observed.

6. Mixed Feedback: Chaotic Behavior

It is a peculiarity of the choice of parameters α, b, c in Theorem 5.1 that $\tilde{F}$ is constant on the interval $(w_2, 1]$. For other parameters but with more computations it should be possible to construct a map $\tilde{\tilde{F}}$ for the differences $\tau_{2i} - \tau_{2i-1}$ which is not constant on $(w_2, 1]$ and equal to F on $[w_1, w_2]$.

Indeed, an der Heiden and Walther (1982) succeeded in constructing such a map. However, in order to avoid too long computations of trajectories they presupposed a nonlinearity f with three steps instead of two with (F3). It was then possible to find in the neighborhood of a homoclinic orbit an infinity of periodic and aperiodic solutions, characteristic of chaotic behavior. The chaoticity found was roughly of the following kind:

Given any sequence $(n_1, n_2, \ldots)$ of natural numbers, then there is an initial condition φ such that the corresponding solution x_φ oscillates n_1 times with (relatively) small amplitude around some level, afterwards x_φ describes a large amplitude excursion, followed by n_2 small amplitude oscillations, followed by a large excursion etc. with $n_3, n_4, \ldots$. Embedded in the regime of such aperiodic solutions there are infinitely many periodic solutions of spiral type described in the previous section. The proof was given for continuous nonlinearities near to a 3-step function.

In this section we prove the existence of *another type* of chaotic motion in which the relation between large and small amplitudes is reversed. Moreover it is sufficient to consider a two-step nonlinearity. Compared with the previous paper the discussion here has a more global character. We do not rely on the concept of a homoclinic orbit, but apply the one-dimensional folding condition (6.7) of Li and Yorke (1975). However, despite *substantial* differences in the analysis and techniques all approaches [Peters, 1980; Walther, 1981; an der Heiden and Walther, 1982 (the first two deal with $dx/dt = g(x(t-1))$] have in common the reduction of the problem to a discrete system of dimension one in order to apply known criteria. The condition of Li and Yorke was first applied by Peters (1980).

Again, for simplicity we restrict the investigation to piecewise constant functions f in Eq. (A). By techniques used in Walther (1981) and an der Heiden, Walther (1982) the results may be generalized to smooth nonlinearities.

Contrary to the previous section we now choose b, the threshold of negative feedback, near the maximum of $\tilde{\tilde{x}}$, the unstable periodic solution associated with positive feedback (Theorem 4.3). This maximal value is given by

$$\tilde{\tilde{x}}_{\max} = (\gamma - z)/(\gamma - 1), \tag{6.1}$$

where z is the positive root of the quadratic (4.10). (Expression (6.1) follows from (4.8) by $\tilde{\tilde{x}}_{\max} = x_{w_p}(w_p)$, $z = \tilde{\tilde{x}}_{\min} = x_{w_p}(1)$ and $w_p = F(w_p)$).

The nonlinearity of Eq. (A) considered in this section is slightly more general than (F3), namely

$$f(\xi) = \begin{cases} 0 & \text{if } \xi < 1, \\ c & \text{if } 1 \leqslant \xi \leqslant b, \\ d & \text{if } b < \xi, \end{cases} \tag{F4}$$

where the constants obey $b > 1$, $d \leqslant c$, and $\gamma = c/\alpha > b$. For technical reasons we allow $d < 0$. The transformation $y = x - d/\alpha$ again leads to an equation with nonnegative nonlinearity.

Theorem 6.1. *Let the function f be given by* (F4). *Let α and $\gamma = c/\alpha$ satisfy*

$$\gamma/(\gamma - 1)^2 + z < 1, \tag{6.2}$$

where z is the positive root of the quadratic (4.10). *Then there are numbers $\mu = \mu(\alpha, \gamma) > 0$ and $d^* = d^*(\alpha, \gamma, b) \leqslant 1$ with the following property: If*

$$(\gamma - z)/(\gamma - 1) < b < (\gamma - z)/(\gamma - 1) + \mu \qquad \text{and} \qquad d \leqslant d^* \tag{6.3}$$

then Eq. (A) *has infinitely many periodic solutions and uncountably many aperiodic, mixing solutions.*

The conclusion may be stated more explicitly as follows:

There is a sequence $T_1 = \{x_k;\ k = 1, 2, \ldots\}$ of periodic solutions and an uncountable set T_2 of non-periodic solutions to Eq. (A) *satisfying:*

(i) *to each $x \in T_1 \cup T_2$ there corresponds a sequence $0 < t_1 < t_2 < \cdots$, $\lim_{i\to\infty} t_i = \infty$, such that $x(t_i) = 1$, $x(t) \neq 1$ for all $t \neq t_i$, $i = 1, 2, \ldots,$*

(ii) *there is a continuous map $G\colon I \to J$ of intervals $I \subset J \subset [0, 1]$ such that to every $x \in T_1 \cup T_2$ there corresponds a $v_x \in I$ obeying*

$$t_{2i} - t_{2i-1} = G^i(v_x), \qquad i = 1, 2, \ldots,$$

(iii) *for $x = x_k \in T_1$*

$$G^k(v_x) = v_x, \qquad G^i(v_x) \neq v_x, \qquad 1 \leqslant i < k,$$

$t_{2k+1} - t_1$ is the (smallest) period of x_k,

(iv) *the sets* $S_1 = \{v_x\colon x \in T_1\}$, $S_2 = \{v_x\colon x \in T_2\}$ *obey*

$$\left.\begin{array}{l}\limsup_{i\to\infty} |G^i(v) - G^i(v')| > 0 \\ \liminf_{i\to\infty} |G^i(v) - G^i(v')| = 0\end{array}\right\} \quad \text{if} \quad v, v' \in S_2, \quad v \neq v',$$
$$\limsup_{i\to\infty} |G^i(v) - G^i(v')| > 0 \qquad \text{if} \quad v \in S_1, \quad v' \in S_2. \tag{LY}$$

Remark 6.2. (LY) specifies the conditions of Li and Yorke (1975) for chaotic motion with respect to one-dimensional maps. It also specifies a type of mixing behavior.

Proof of Theorem 6.1. We use here the material of the proof to Theorem 4.3. Let $\tilde{\tilde{x}}$ be the periodic solution to Eq. (A) in case of (F2). This solution also solves (A) in case of (F4) if $b \geqslant \tilde{\tilde{x}}_{\max}$. Assume $b > \tilde{\tilde{x}}_{\max}$. Let

$$\varphi \in D_b = \{\varphi \in D\colon \varphi(t) < b,\ t \in [-1, 0]\}, \qquad w_2 > w = V(\varphi) > w_p,$$

and x_w the corresponding solution to (A) with (F4). Then

$$x_w(w) = \gamma - (\gamma - 1)\exp(-\alpha w) > x_{w_p}(w_p) = \tilde{\tilde{x}}_{\max}, \qquad x_w(w) < x_{w_2}(w_2).$$

Assume

$$\tilde{\tilde{x}}_{\max} < b < x_{w_2}(w_2). \tag{6.4}$$

Then there is a unique $\bar{w} = \bar{w}(b) \in (w_p, w_2)$ obeying $x_{\bar{w}}(\bar{w}) = b$. The values $\bar{w}(b)$ and b are related by $\exp(-\alpha\bar{w}) = (\gamma - b)/(\gamma - 1)$.

Let $w \in [w_p, \bar{w})$. Then $x_w(t) < b$ for all $t \in [0, 1]$ and (4.5) implies that $x_w(1)$ is an increasing function of w; $x_{\bar{w}}(1) = be^{-\alpha}(\gamma - 1)/(\gamma - b) < 1$. For $t \in [1, 1 + t_1(w)]$ the solution x_w increases, and $x_w(1 + t_1(w))$ is an increasing function of w; $x_{\bar{w}}(1 + t_1(\bar{w})) = \gamma^2(b - 1)/(b(\gamma - 1)) + e^{-\alpha} > b$. For $\bar{\bar{w}} = \bar{\bar{w}}(b) = F^{-1}(\bar{w})$ we have $x_{\bar{\bar{w}}}(1 + t_1(\bar{\bar{w}})) = b$.

Let $0 < t_1(w) < t_2(w) < \cdots$ denote the successive times obeying $x_w(t_i(w)) = 1$, and $1 < \tau_1(w) < \tau_2(w) < \cdots$ the successive times with $x_w(\tau_i(w)) = b$.

For $w \in [w_p, \bar{w}]$ we have $t_2(w) - t_1(w) = 1 - F(w)$. For $w \in [w_p, \bar{\bar{w}})$ the inequality $x_w(t) < b$ holds for all $t \in (1, t_4(w))$, and $t_4(w) - t_3(w) = 1 - F^2(w)$, in particular $t_4(\bar{\bar{w}}) - t_3(\bar{\bar{w}}) = 1 - F^2(\bar{\bar{w}}) = 1 - F(\bar{w})$. For $w \in [\bar{\bar{w}}, \bar{w}]$ it follows from (5.1) that

$$x_w(t_2(w) + 1) = x_w(t_1(w) + 1)\exp(-\alpha(1 - F(w))),$$

which is increasing with respect to w. For these w we also observe $t_2(w) < \tau_1(w) < t_1(w) + 1$. Hence $x_w(t)$ increases as a function of $t \in (t_2(w) + 1, \tau_1(w) + 1)$ and

$$x_w(\tau_1(w) + 1) = \gamma - (\gamma - x(t_2(w) + 1))(\gamma - b)/(\gamma - 1)$$

(since $\exp(-\alpha(\tau_1(w) - t_2(w))) = (\gamma - b)/(\gamma - 1)$) increases as a function of w.

Let us write $b = b(\varepsilon) = \tilde{\tilde{x}}_{\max} + \varepsilon$, $\varepsilon > 0$. Then

$$\lim_{\varepsilon\to 0} x_{\bar{w}(b)}(\tau_1(\bar{w}(b)) + 1) = x_{w_p}(t_2(w_p) + 1 + w_p) = x_{w_p}(1 + w_p) = \gamma - (\gamma - z)e^{-\alpha w_p},$$

where z is the minimal value of $\tilde{\tilde{x}}$ (see Lemma 4.5). To obtain an estimate of $\exp(-\alpha w_p)$ observe that w_p as a fixed point of F satisfies

$$\gamma - (\gamma - 1)\exp(-\alpha w_p) = x_{w_p}(w_p) = (\gamma - x_{w_p}(1))/(\gamma - 1) = (\gamma - z)/(\gamma - 1),$$

hence

$$\exp(-\alpha w_p) = (\gamma^2 - 2\gamma + z)/(\gamma - 1)^2 > (\gamma^2 - 2\gamma)/(\gamma - 1)^2.$$

Condition (6.2) implies

$$x_{w_p}(1 + w_p) < \gamma(1 - \exp(-\alpha w_p)) + z < \gamma/(\gamma - 1)^2 + z < 1.$$

Thus there is a $\mu = \mu(\alpha, c) > 0$ such that $b \in (\tilde{\tilde{x}}_{\max}, \tilde{\tilde{x}}_{\max} + \mu)$ implies (6.4) and

$$x_w(\tau_1(w) + 1) < 1 \qquad \text{for all} \qquad w \in [\bar{\bar{w}}, \bar{w}].$$

With this μ let the first condition of (6.3) hold (note (6.1)) in the following:

For $w \in (\bar{\bar{w}}, \bar{w}]$ the solution x_w satisfies

$$x_w(t) = d/\alpha - (d/\alpha - x_w(\tau_1(w) + 1))\exp(-\alpha(t - \tau_1(w) - 1)),$$

$$t \in (\tau_1(w) + 1, \tau_2(w) + 1),$$

$$x_w(\tau_2(w) + 1) = d/\alpha - (d/\alpha - x_w(\tau_1(w) + 1))\exp(-\alpha(\tau_2(w) - \tau_1(w))). \quad (6.5)$$

For these w there is a unique $t_3(w)$, $\tau_2(w) < t_3(w) < t_2(w) + 1$, obeying $x_w(t_3(w)) = 1$. Hence x_w increases on $(\tau_2(w) + 1, t_3(w) + 1)$ and, since $\exp(-\alpha(t_3(w) - \tau_2(w))) = 1/b$, we obtain

$$x_w(t_3(w) + 1) = \gamma(1 - 1/b) + x_w(\tau_2(w) + 1)/b \qquad \text{for all} \qquad w \in (\bar{\bar{w}}, \bar{w}].$$

For each fixed $w \in (\bar{\bar{w}}, \bar{w}]$ it follows from the first part of (6.3) that $x_w(t_3(w) + 1)$ is a decreasing function of d satisfying

$$x_w(t_3(w) + 1) > 1, \qquad \text{if} \qquad d = c$$

and

$$\lim_{d \to -\infty} x_w(t_3(w) + 1) = -\infty.$$

There is exactly one $d = d(w)$ such that $x_w(t_3 + 1) = 1$. For all $d \in [d(w), c]$ we have $t_3(w) < \tau_1(w) + 1 < t_4(w) \leqslant t_3(w) + 1$ and the quantity $t_4(w) - t_3(w)$ is an increasing function of d (for fixed w);

$$t_4(w) - t_3(w) = 1 - F^2(w) \qquad \text{if} \qquad d = c,$$

$$t_4(w) - t_3(w) = 1 \qquad \text{if} \qquad d = d(w). \quad (6.6)$$

Define for each $d \leqslant c$:

$$F_d(w) = 0 \qquad \text{if} \qquad w \in [0, w_1],$$

$$F_d(w) = F(w) \qquad \text{if} \qquad w \in [w_1, \bar{w}].$$

For $w \in [\bar{w}, F(\bar{w})]$ define

$$F_d(w) = \begin{cases} 1 - [t_4(F^{-1}(w)) - t_3(F^{-1}(w))] & \text{if} \quad d(F^{-1}(w)) \leqslant d \leqslant c, \\ 0 & \text{if} \quad d \leqslant d(F^{-1}(w)). \end{cases}$$

Since the times $t_i(w)$, $\tau_i(w)$ depend continuously on w, for fixed parameters α, b, c, d the function F_d: $[0, F(\bar{w})] \to [0, 1]$ is continuous.

Let us assume $d/\alpha \leqslant 1$. Then $x_w(\tau_2(w) + 1) < 1$ and hence $t_4(w) > \tau_2(w) + 1$ for all $w \in (\bar{\bar{w}}, \bar{w}]$. This implies that for these w the function

$$\psi(t) = \psi_{F(w)}(t) = x_w(t_4(w) + t), \qquad t \in [-1, 0],$$

is an element of D_b. Therefore it follows from the construction of F_d that for every $w \in [w_p, F(\bar{w})]$ satisfying

$$F_d^k(w) \in [w_p, F(\bar{w})], \qquad k = 1, 2, \ldots$$

the corresponding solution x_w obeys

$$x_w(t_{2k} - t_{2k-1}) = 1 - F_d^k(w)$$

(this has just been shown for $k = 1$, and it follows in the same way for $k + 1$ if established for k).

We can now apply the following theorem of Li and Yorke (1975): Let I, J be two intervals, $I \subset J$. Let $G: I \to J$ be a continuous function. Assume there is $r \in I$ such that $G^i(r) \in I$, $i = 1, 2, 3$, and

$$G^3(r) \leqslant r < G(r) < G^2(r). \tag{6.7}$$

Then there is a sequence $S_1 = \{v_1, v_2, \ldots\} \subset I$ and an uncountable set $S_2 \subset I$ such that S_1 and S_2 are invariant with respect to G and

$$G^k(v_k) = v_k, \qquad G^i(v_k) \neq v_k, \qquad 1 \leqslant i < k, \qquad \text{for all} \qquad v_k \in S_1,$$

and (LY) holds (see Theorem).

Note that Li and Yorke originally assumed $I = J$; however inspection of their proof shows that their result can be generalized as indicated.

We now identify

$$I = [0, F(\bar{w})], \qquad G(w) = F_d(w), \qquad J = G(I), \qquad r = \bar{\bar{w}}(b) = F_d^{-1}(\bar{w}(b)).$$

Condition (6.7) is satisfied if $G^3(r) = F_d^2(\bar{w}) \leqslant \bar{\bar{w}}$. It follows from the observation with (6.6) that there is exactly one $d = \tilde{d} = \tilde{d}(\alpha, c, b)$ obeying $F_{\tilde{d}}^2(\bar{w}) = \bar{\bar{w}}$ and $F_d^2(\bar{w}) < \bar{\bar{w}}$ for all $d < \tilde{d}$. Therefore the theorem is proved for all b and d satisfying (6.3) with μ as given above and $d^* = \min(\alpha, \tilde{d})$. Q.E.D.

Remark 6.3. A similar result can be proved for continuous nonlinearities f approximating (F4) just as the function defined in Remark 5.2 approximates (F3).

Theorem 6.1 does not tell whether the chaotic domain is attracting or repelling. A detailed analysis of the function F_d is in preparation and will clarify related questions.

7. Mixed Feedback: More Types of Behavior and of Bifurcations

The patterns of solutions exhibited in Sects. 5 and 6 and also in an der Heiden and Walther (1982) do by far not exhaust the rich dynamics of Eq. (A) with mixed feedback nonlinearities. It has become a well established supposition that among the oscillatory solutions to delay-differential equations such as (A) or the simpler one, $dx/dt = g(x(t-1))$, those are predominant and stable which oscillate slowly, i.e. successive extrema are spaced apart at least the length of the delay. This conviction was substantiated by Walther (1981) who proved the density of slowly oscillating solutions. However, he presupposed nonlinearities of negative feedback type. Already the solutions described in Sects. 5 and 6 are not slowly oscillating in the strict sense, though in a mild form (every second pair of minima has a distance of more than one delay time). In this section we show that very rapidly oscillating solutions do exist. We give an idea how they may arise through a series of bifurcations. However, our understanding is not well developed, since these solutions are by far more intricately structured and irregular than those for which proofs of chaos exist. A reduction to one-dimensional discrete dynamics seems to be impossible in general. We also give a series of illustrations both to help in the analysis and to stimulate comparison with similar structures in nature.

Let us consider Eq. (A) with a nonlinearity given by (F3). To facilitate the calculations we assume $\gamma = c/\alpha = 4$ and $b = 2$. Possible generalizations are indicated at the end of this section.

Let us start with some initial condition φ satisfying $\varphi(t) > b = 2$ for all $t \in (-1, 0)$. Then according to (5.1a) the corresponding solution x decays exponentially in the time interval $[0, t_1 + 1]$, where t_1 is the first time obeying $x(t_1) = 1$. In particular

$$x(t) = 2\exp(-\alpha(t - t_1)) \qquad \text{for all} \qquad t \in [t_1, t_1 + 1]$$

and

$$x(t_1 + 1) = 2\exp(-\alpha) \qquad \text{for all} \qquad \alpha > 0. \tag{7.1}$$

In Figs. 5 – 8, showing x versus t for various values of α, we indicate t_1 (note that the vertical lines are spaced apart just the delay time, here normalized to 1). For $t \in [t_1, t_1 + 1]$ the solution x only depends on α, and hence the same holds for all $t > t_1$. Therefore, if relevant, we shall write x_α instead of x. We meet a first alternative: either $x_\alpha(t_1 + 1) \geqslant 1$ or $x_\alpha(t_1 + 1) < 1$. Let α_1 be defined by $2\exp(-\alpha_1) = 1$.

Assume $\alpha \leqslant \alpha_1$. Then, since $1 < x_\alpha(t) < b$ for all $t \in (t_1, t_1 + 1)$, Condition (5.1b) with $t_0 = t_1 + 1$ implies that x increases after $t_1 + 1$, more precisely

$$x(t) = \gamma - (\gamma - x(t_1 + 1))\exp(-\alpha(t - t_1 - 1)) \qquad \text{for all} \qquad t \in [t_1 + 1, t_2 + 1],$$

where t_2 is the first time obeying $t_2 > t_1 + 1$ and $x(t_2) = b$. In the time interval $(t_2, t_2 + 1)$ the solution x has the same property as the initial condition φ, namely to exceed 1. Therefore the previous arguments prove that there will be a first time $t^* > t_2 + 1$ with $x(t^*) = b$, and x obeying the relation (7.1) with t^* instead of t_1. Thus we have shown that $[t_1, t^*]$ corresponds to one (smallest) period of a periodic solution to (A), which is a stable limit cycle as following from Lemma 3.4. This cycle is simple in that there is just one minimum within one (smallest) period. Figure 5a shows this oscillation for $\alpha = 0.6$. In summary we have proved: *If* $\alpha < \alpha_1$ *then there is a stable limit cycle with exactly one minimum within one period.*

A new type of solution occurs when α exceeds α_1, since if x crosses the threshold 1 from above at time t, then at time $t + 1$ there must be a maximum of x (as following from (5.1)). If x crosses 1 from below at t then there has to be a minimum at $t + 1$: Assume $\alpha > \alpha_1$. Then $x(t_1 + 1) < 1$ and there is a time $\tau_1 \in (t_1, t_1 + 1)$ such that $x(\tau_1) = 1$. For $t \in (t_1 + 1, \tau_1 + 1)$ the solution obeys (5.1b) with $t_0 = t_1 + 1$.

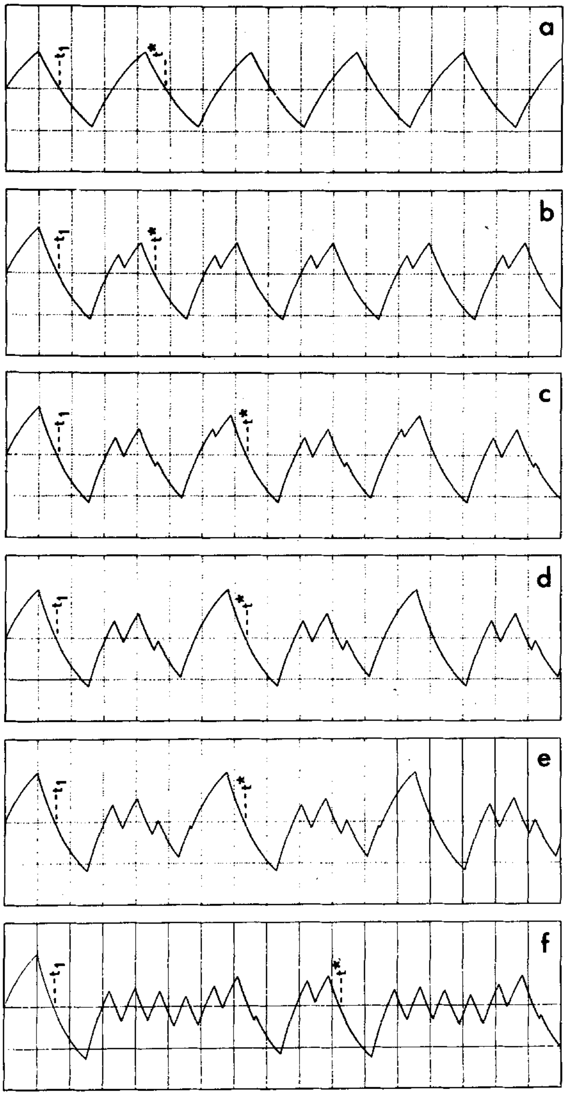

Fig. 5. The analytic solutions to Eq. (A) in conjunction with the mixed feedback nonlinearity (F3), for various values of α, but fixed $b = 2$, $c/\alpha = 4$. The vertical lines here and in Figs. 6–8 are spaced one time unit (= delay time) apart. Horizontal lines in all figures correspond to $x = 0, 1, 2, 4$. **(a)** $\alpha = 0.6$, **(b)** $\alpha = 0.8$, **(c)** $\alpha = 0.86$, **(d)** $\alpha = 0.88$, **(e)** $\alpha = 0.9$, **(f)** $\alpha = 0.98$. Note that, according to Eqs. (5.1), the figures have to satisfy: If at time t the solution x crosses the threshold 1 or 2, then x has an extremum at time $t + 1$. Beginning and end of one period are indicated by t_1 and t^* respectively

Therefore during this time interval x increases from $b\exp(-\alpha)$ to

$$x(\tau_1 + 1) = \gamma - (\gamma - be^{-\alpha})\exp(-\alpha(\tau_1 - t_1)) = 2 + \exp(-\alpha),$$

since $1 = b\exp(-\alpha(\tau_1 - t_1))$. It follows that there are unique times τ_2, $t_2 \in (t_1 + 1, \tau_1 + 1)$ satisfying $x(\tau_2) = 1$, $x(t_2) = b$. Since $x(t) < 1$ for $t \in (\tau_1, \tau_2)$, Condition (5.1a) with $t_0 = \tau_1 + 1$ implies that x decreases during $[\tau_1 + 1, \tau_2 + 1]$ and $x(\tau_2 + 1) = x(\tau_1 + 1)\exp(-\alpha(\tau_2 - \tau_1))$.

Because of

$$x(t_1 + 1) = \exp(-\alpha(t_1 + 1 - \tau_1))$$

and

$$1 = \gamma - (\gamma - x(t_1 + 1))\exp(-\alpha(\tau_2 - t_1 - 1)) \tag{7.2}$$

we obtain

$$\exp(-\alpha(\tau_2 - \tau_1)) = x(t_1 + 1)(\gamma - 1)/(\gamma - x(t_1 + 1)), \tag{7.3}$$

hence

$$x_\alpha(\tau_2 + 1) = (2 + e^{-\alpha})2e^{-\alpha}3/(4 - 2e^{-\alpha}) = 3e^{-\alpha}(2 + e^{-\alpha})/(2 - e^{-\alpha}).$$

If α increases from α_1 to ∞ then $x_\alpha(\tau_2 + 1)$ decreases from $2 + \exp(-\alpha_1)$ to 0. There is $\alpha_2 > \alpha_1$ such that $x_{\alpha_2}(\tau_2 + 1) = 2$.

For $t \in (\tau_2 + 1, t_2 + 1)$ the solution again increases and

$$x_\alpha(t_2 + 1) = \gamma - (\gamma - x_\alpha(\tau_2 + 1))\exp(-\alpha(t_2 - \tau_2)). \tag{7.4}$$

As $\exp(-\alpha(t_2 - \tau_2)) = (\gamma - b)/(\gamma - 1) = \frac{2}{3}$, we obtain

$$x_\alpha(t_2 + 1) = \tfrac{4}{3} + 2e^{-\alpha}(2 + e^{-\alpha})/(2 - e^{-\alpha}) \qquad \text{for all} \qquad \alpha > \alpha_1.$$

For $\alpha \in (\alpha_1, \alpha_2]$ we have $x_\alpha(t) \geqslant 2$, $t \in (t_2, t_2 + 1)$. Thus for these α, after $t = t_2 + 1$ the solution decreases exponentially to $2\exp(-\alpha)$. This last value is obtained for $t = t^* + 1$, where t^* is the first time with $t^* > t_2 + 1$ and $x(t^*) = 1$.

Again $[t_1, t^*]$ is one period of a stable periodic solution. However, there are exactly two minima within one cycle, namely those at $t = t_1 + 1$ and at $t = \tau_2 + 1$. In summary: *For each* $\alpha \in (\alpha_1, \alpha_2]$ *there is a stable limit cycle having* 2 *minima per (smallest) period.*

Figure 5b shows an example of this type of limit cycle ($\alpha = 0.8$). A second change of behavior takes place at $\alpha = \alpha_2$. At this value the minimum $x_\alpha(\tau_2 + 1)$ crosses the level b from above giving rise to the phenomenon of *period doubling*:

Assume $\alpha > \alpha_2$. Then there is $t_3 \in (\tau_1 + 1, \tau_2 + 1)$ such that $x_\alpha(t_3) = 1$. For $t \in (t_2 + 1, t_3 + 1)$ Condition (5.1a) applies with $t_0 = t_2 + 1$, x decreases on this interval and

$$x_\alpha(t_3 + 1) = x_\alpha(t_2 + 1)\exp(-\alpha(t_3 - t_2)).$$

By calculations similar to (7.2) and (7.3) we arrive at

$$\exp(-\alpha(t_3 - t_2)) = (4 - x(\tau_1 + 1))/x(\tau_1 + 1),$$

hence

$$x_\alpha(t_3 + 1) = \tfrac{4}{3}(2 - e^{-\alpha})/(2 + e^{-\alpha}) + 2e^{-\alpha}.$$

$x_\alpha(t_3 + 1)$ is a decreasing function of α with $x_{\alpha_2}(t_3 + 1) < 2$ and $x_\infty(t_3 + 1) = \frac{4}{3}$. As long as $x_\alpha(\tau_2 + 1) > 1$ we have $x_\alpha(t_2 + 1) > 2$, since $x_\alpha(t_2 + 1) = 2$ if and only if $x_\alpha(\tau_2 + 1) = 1$. Therefore, as long as $x_\alpha(\tau_2 + 1) > 1$ and $\alpha > \alpha_2$ there is

$t_4 \in (\tau_2 + 1, t_2 + 1)$ satisfying $x_\alpha(t_4) = 1$. Since

$$x_\alpha(t_4 + 1) = \gamma - (\gamma - x_\alpha(t_3 + 1)) \exp(-\alpha(t_4 - t_3)),$$

there is another minimum of x at $t_3 + 1$ and a maximum at $t_4 + 1$.

Now assume $\alpha = \alpha_2 + \varepsilon$, $\varepsilon > 0$. Then

$$\lim_{\varepsilon \to 0} x_\alpha(\tau_2 + 1) = 2, \qquad \lim_{\varepsilon \to 0} (t_4 - t_3) = 0, \qquad \lim_{\varepsilon \to 0} x_\alpha(t_4 + 1) = x_{\alpha_2}(t_3 + 1) < 2.$$

For sufficiently small ε it follows that $x_\alpha(t_4 + 1) < 2$ and that $x_\alpha(t) > 1$ for $t \in (t_2, t_2 + 1) - (t_3, t_4)$. The latter relation implies that $x_\alpha(t)$ for $t > t_2$ evolves just in the way as $x_\alpha(t)$ for $t > 0$ if ε is very small. In particular there are first times $1 + t_2 < t_5 < \tau_3 < \tau_4$ obeying $x(t_5) = 2, x(\tau_3) = x(\tau_4) = 1$. The minimum at $t_5 + 1$ satisfies $x_\alpha(t_5 + 1) > x_\alpha(t_1 + 1) = 2\exp(-\alpha)$ (since $x_\alpha(t_4 + 1) > x_\alpha(t_3 + 1)$). Hence $\tau_4 - \tau_3 < \tau_2 - \tau_1$. Moreover $\tau_3 - t_5 > t_2 - \tau_2$. These relations imply for small positive ε that $x_\alpha(\tau_3 + 1) > x_\alpha(\tau_1 + 1)$, $x_\alpha(\tau_4 + 1) > x_\alpha(\tau_2 + 1)$. Indeed, a detailed calculation shows $x_\alpha(\tau_4 + 1) > 2$. There is a time $t_6 \in (\tau_4, \tau_3 + 1)$ with $x(t_6) = 1$. Since x increases during $[\tau_4 + 1, t_6 + 1]$, we obtain $x_\alpha(t) > 2$ for $t \in (t_6, t_6 + 1)$. Hence there is a first $t_7 > t_6 + 1$ with $x(t_7) = 1$ and x on (t_1, t_7) is just one cycle of a periodic solution. The courses of x_α on (t_1, t_5) and on (t_5, t_7) are very similar for small ε. Therefore $t_7 - t_1 \to 2(t_5 - t_1)$ as $\varepsilon \to 0$: when α crosses α_2 from below then there is an *abrupt doubling of the period.* More exactly, if $\pi(\alpha)$ is the period of x_α then we have the relation

$$\lim_{\varepsilon \to 0+} \pi(\alpha_2 + \varepsilon) = 2\pi(\alpha_2). \tag{7.5}$$

In summary: *There is a number $\alpha_3 > \alpha_2$ such that for all $\alpha \in (\alpha_2, \alpha_3)$ Eq.* (A) *has a stable limit cycle with 5 minima per (smallest) period. The period $\pi(\alpha)$ satisfies relation* (7.5). An example of such a periodic solution is shown in Fig. 5c ($\alpha = 0.86$).

It is possible to continue these considerations beyond α_3. However, they become very intricate and lengthy. Instead we present a sequence of computer drawn solutions using the formulae (5.1). The parameter α is varied in small steps from $\alpha = 0.6$ (Fig. 5a) to $\alpha = 6$ (Fig. 8a). The last example (Fig. 8b) is for $\alpha = 20$. Lemma 3.4 is very helpful to detect a periodic solution in the computer plots: We started with an initial condition $\varphi(t) > b = 2$, $t \in [-1, 0]$. Then, clearly, according to (5.1a), there is a first time t_1 satisfying $x(t_1) = 2$, $x(t) > 2$ for all $t \in (t_1 - 1, t_1)$. If there is a second time t^* obeying this condition, namely $x(t^*) = 2$ and $x(t) > 2$ for all $t \in (t^* - 1, t^*)$, then by Lemma 3.4 the solution x on the interval $[t_1, t^*]$ corresponds to one cycle of a stable periodic solution. The criterion is robust against small numerical errors.

The numerical results may be interpreted as follows: We calculated $\alpha_2 \approx 0.85$, $\alpha_3 \approx 0.87$. As α crosses α_3 from below, at the fourth minimum the value of x_α crosses the threshold 1 from below. Therefore at α_3 one minimum and one maximum disappear by merging. This is the reverse process we observed at α_1. Thus *at values of α slightly above α_3 a periodic solution with 4 minima per period occurs.* The period does not change discontinuously at α_3. For an example see Fig. 5d ($\alpha = 0.88$).

As α increases beyond $\alpha_4 \approx 0.9$ the fourth minimum of x_α crosses the level b from below, creating a new minimum one time unit later. Therefore Fig. 5e shows a stable periodic solution with five minima within one period at $\alpha = 0.905$. The periods for α between α_2 and α_4 are nearly equal, namely $\pi(\alpha) \approx 6$. The next dramatic change in the period occurs at $\alpha = 0.98$, where the sixth minimum of x_α crosses b from above. *The period jumps from about* 6 *to about* 9 (see Fig. 5f, $\alpha = 0.98$). Therefore the factor 2 obtained at α_2 is not universal.

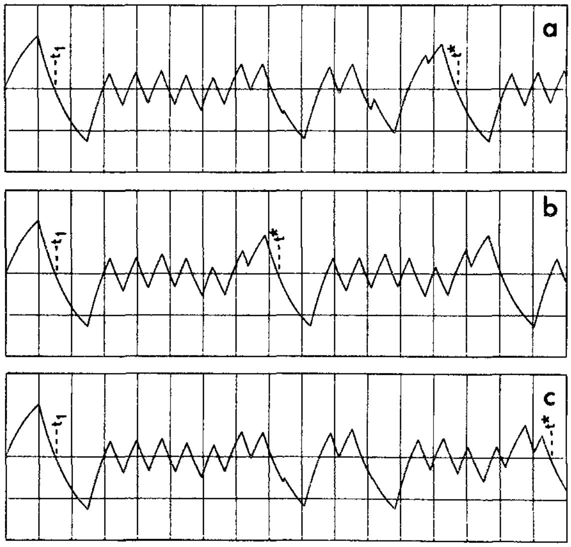

Fig. 6. As in Fig. 5 with **(a)** $\alpha = 1.0015$, **(b)** $\alpha = 1.0125$, **(c)** $\alpha = 1.001$

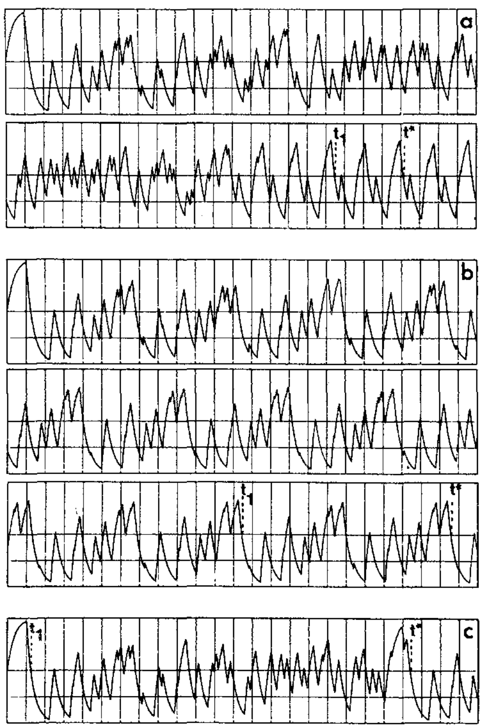

Fig. 7. As in Fig. 5 with **(a)** $\alpha = 2.75$, $t = 0$ to 50, **(b)** $\alpha = 2.7$, $t = 0$ to 75, **(c)** $\alpha = 2.775$, $t = 0$ to 25

There may also be *discrete reductions in period* as α increases. E.g. for $\alpha = 1.0015$ there is a periodic solution with $\pi(\alpha) \approx 12.2$ and 13 minima per period (see Fig. 6a) (the beginning and end of a period are always denoted by t_1 and t^* respectively). However, for $\alpha = 1.0125$ the period is only about 7 (see Fig. 6b). This reduction in period is due to the crossing of the threshold b from below by the 7th minimum of x_α as α increases from 1.0015 to 1.0125. Another example of period reduction is given by the pair $\alpha = 1.001$ and $\alpha = 1.0015$ (see Fig. 6c, a respectively). For $\alpha = 1.001$ the period is about 15 (16 minima per period).

It should be noted that Lemma 3.4 is a sufficient, but not necessary, criterion for periodicity. Indeed for $\alpha = 2.75$ the solution, after a long transitory oscillation, approached a periodic cycle (as far as Fig. 7a shows), not obeying this criterion. Its period is about 3.6. Another example with $\alpha = 2.7$ is shown in Fig. 7b, where a period $\pi(\alpha) \approx 11.2$ seems to be present. However for $\alpha = 2.775$ again a stable limit cycle obeying Lemma 3.4 occurs with $\pi(\alpha) \approx 20.2$.

The last phenomena to be discussed here are made apparent by Figs. 8a ($\alpha = 6$) and 8b ($\alpha = 20$) and occur for large α. Even after a long time the solutions do not show a repetitive pattern. Instead the motion acquires the character of randomness

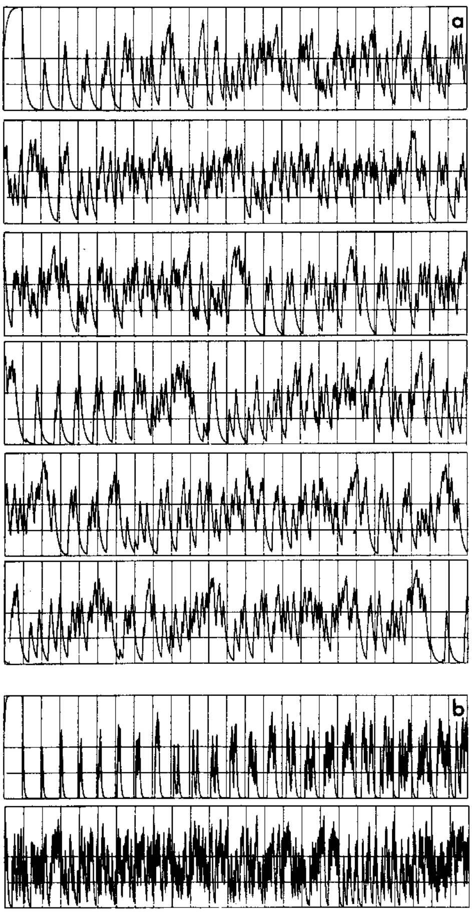

Fig. 8. As in Fig. 5 with **(a)** $\alpha = 6$, $t = 0$ to 150, **(b)** $\alpha = 20$, $t = 0$ to 50

and a high degree of irregularity. Two other features should be observed in these figures. First there may be many oscillations within one time unit (= the normalized delay), even if the initial condition is monotone. Therefore within one delay time there may be a very rich structure, which is sustained despite (or because?) of the delay.

Second, there is a fairly regular building up of irregularity from a simple initial condition. The details of successive creation of new "spikes" during the first time units can be well understood on the basis that every crossing of the thresholds 1 or b implies an extremum one time unit later (see figures).

However the complexity of structures, as illustrated by Figs. 8a, b, not only seems to prohibit analytical understanding, but rather to make it meaningless. A statistical approach to these deterministic systems would perhaps be more suitable.

Finally we like to give up the restriction $b = 2$, $\gamma = 4$. To that purpose the following theorem gives a condition that the solutions do not fade away towards the stable steady state 0, but oscillate permanently around the upper threshold b.

Theorem 7.1. *Let the function f be given by* (F3). *Let*

$$\gamma \geqslant b^2/(b-1). \tag{7.6}$$

If the initial condition φ satisfies $b < \varphi(t) \leqslant \gamma$ for all $t \in [-1, 0]$, then there is a sequence $(t_i)_{i=1,2,\ldots}$, $0 < t_i < t_{i+1}$, $\lim_{i\to\infty} t_i = \infty$, such that

$$x_\varphi(t_i) = b, \qquad x_\varphi(t) \neq b \quad \text{if} \quad t \neq t_i, \qquad i = 1, 2, \ldots .$$

Proof. Condition (5.1a) implies that there is a first time $t_1 > 0$ with $x(t_1) = b$. Let $\tilde{t}$ be some time satisfying $x(\tilde{t}) = b$, let τ be the first time satisfying $\tau > \tilde{t}$, $x(\tau) = 1$. Then

$$x(\tau + 1) > 1. \tag{7.7}$$

To prove (7.7) realize that $1 < x(t) < b$ for all $t \in (\tilde{\tau}, \tau)$, where $\tilde{t} \leqslant \tilde{\tau} < \tau$ and $\tau - \tilde{\tau}$ is at least as large as the time needed for x to decay according to (5.1a) from b to 1. This means

$$\exp(-\alpha(\tilde{\tau} - \tau)) \leqslant 1/b.$$

It follows from (5.1b) that

$$x(\tau + 1) = \gamma - (\gamma - x(\tilde{\tau} + 1)) \exp(-\alpha(\tau - \tilde{\tau})) > \gamma - \gamma/b \geqslant b,$$

since $x(\tilde{\tau} + 1) > 0$.

From (7.7) we can conclude that if there is some finite time $T > 0$ with $x(t) \neq b$ for all $t > T$ then $x(t) > 1$ for all $t > T$. Hence either $x(t) > b$ for all $t > T$, or $1 < x(t) < b$ for all $t > T$. In both cases Eqs. (5.1a, b) imply that there is some $t > T$ with $x(t) = b$, contradicting the definition of T. Q.E.D.

Condition (7.6) of Theorem 7.1 is independent of the decay rate α in Eq. (A). It is our impression that in a large domain of values b and γ obeying (7.6) a kaleidoscope of different solution types occur when α varies from 0 to ∞, ranging from very simple patterns for small values of α to solutions with unbounded complexity as α tends to ∞.

We finish with a speculation on nonlinearities f which are not necessarily near to a step-function.

Conjecture. Let f: $\mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, bounded function satisfying
(i) f has exactly two inflection points, namely $f''(1) = f''(b) = 0$, $b > 1$,
(ii) $\lim_{\xi \to 0} f(\xi) = 0 = \lim_{\xi \to \infty} f(\xi)$,
(iii) $\int_0^\infty f(\xi)\, d\xi \geqslant b^2$.
Then it may be that as α increases from 0 to ∞ the equation

$$dx(t)/dt = \alpha f(x(t-1)) - \alpha x(t)$$

shows increasing, unbounded complexity in the behavior of its solutions.

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