# Cartel Stability in a Dynamic Oligopoly with Sticky Prices*

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**Abstract**

We study the stability of cartels in a differential game model of oligopoly with sticky prices (Fershtman and Kamien 1987). We show that when firms use closed-loop strategies and the rate of increase of the marginal cost is "small enough", the grand coalition (i.e., when the cartel includes all firms) is stable: it is unprofitable for a firm to exit the cartel. Moreover, a cartel of 3 firms is stable for any positive rate of increase of the marginal cost: it is not profitable for an insider firm to exit the coalition, nor it is profitable for an outsider firm to join the coalition. When firms use open-loop strategies the grand coalition is never stable; moreover, we show that only a cartel of size 2 can be stable and it is so only when the rate of increase of the marginal cost is large enough.

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**Keywords:** Stable Cartel, Mergers, Dynamic Oligopoly, Differential Games.

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# 1 Introduction

Intuitively, in an oligopolistic market the formation of a cartel that acts as a multiplant firm and competes against the other firms should be beneficial for the cartel members. This intuition can be wrong, however. When firms compete *a la* Cournot it is well known from the merger literature[1] that the formation of a cartel by a subset of firms may reduce its members' profits. Indeed, in a static Cournot oligopoly with linear demand and constant marginal cost, a cartel (or merger) of a subset of firms is profitable only if they represent a significant share of the market prior to the formation of the cartel (see Salant, Switzer and Reynolds, 1983, SSR henceforth). In particular, if there are three or more firms in the industry, a cartel formed by two firms always decreases their total profits. This remains to be true when marginal cost is not constant as long as the cartel does not experience "large" efficiency gain, such is the case when the cost function takes the quadratic form of $cq + \frac{1}{2}q^2$ (where $c$ is constant and $q$ is the output). This result can be intuitively explained by the following: The cartel has an incentive to reduce its members' output relative to their production prior to the cartel. Then the outsiders react by increasing their production, which reduces the profits of the cartel members.

The above important result in static oligopoly theory does not generalize to dynamic oligopolies. In a model of dynamic oligopoly with price dynamics (Fershtman and Kamien, 1987), linear demand and quadratic cost function of the form $cq+\frac{1}{2}q^2$, Dockner and Gaunersdorfer (2001) conduct numerical simulations and obtain a very interesting result: all cartels are profitable[2]. This is in sharp contrast with the conclusions in the static framework.

Since all cartels are profitable it is natural to ask which cartel size is more likely to emerge. To address this question we consider a stability criterion. A cartel is stable if no insider firm has an incentive to exit the cartel and no outsider firm would wish to join the cartel. Such a stability notion was first introduced by D'Aspremont, Jacquemin, Gabsewicz, and Weymark (1983) in a price leadership model in which the dominant cartel acts as the leader and firms in the competitive fringe take price as given. Although this stability notion can be adapted to the context of static Cournot competition, it is easy to see that with three or more firms, no cartel is stable in the case of linear demand and constant marginal cost.

[1] The objective of a cartel is identical to the objective of a merged firm, adopted in the merger literature (Salant, Switzer and Reynolds 1983), when a subset of firms merge. Throughout this paper, the term cartel can be substituted by the term merger and vice versa.

[2] Benchekroun (2003) shows analytically that such a result generalizes to the case where the number of firms is arbitrarily large: a cartel of 2 firms is profitable even when the number of firms tends to infinity.

We study the stability of cartels in the dynamic oligopoly model with sticky prices. We consider a generalized version the framework of Fershtman and Kamien (1987) and Dockner and Gaunersdorfer (2001). They focus on the case where the coefficient of the quadratic term is one (i.e., the rate of change of the marginal cost is one). We allow for a general quadratic cost function and such a generalization turns out to have important implications in determining the stability of cartels (although all cartels remain profitable). In particular, when the coefficient of the quadratic term is sufficiently low, the grand coalition (i.e., cartel of all firms) is stable; that is, no firm can benefit from exiting the grand coalition. When the rate of change of the marginal cost is one, only coalitions of three firms are stable, regardless of the total number firms. In fact, size-three coalitions remain stable for any general quadratic cost function.

The above results hold when firms use close-loop strategies whereby a firm's strategy specifies a production rate at a given moment as a function of that moment and the level of the price (the state variable) at that moment. For comparison, we also consider open-loop strategies where a firm's strategy corresponds to a production path announced at time zero and, defined over the whole infinite time horizon. It is well known that the steady state of an open-loop equilibrium "coincides with the Cournot equilibrium of the corresponding static game" (Dockner 1992, see also Dockner et al. 2000). An open-loop Nash equilibrium is in general not subgame perfect but a closed-loop Nash equilibrium is by construction subgame perfect. As a benchmark, the open-loop case allows us to isolate the impact of feedback strategies on the stability of cartels. We show that when firms use open-loop strategies, the grand coalition is never stable. We also show that only a cartel of two firms can be stable and it is so only when the cost function is sufficiently convex.

The rest of the paper is organized as follows. In section 2 we present the model and the formal definition of the stability of a cartel. In section 3 we provide the open-loop and the closed-loop Nash equilibrium after a cartel forms. In section 4 we examine the stability of cartels.

# 2 The model

Consider an industry with $N$ identical firms producing an homogeneous product. The production of firm $i$ at time $t$ is denoted $q_i(t)$. The production cost of each firm is given by

$$cq_i(t) + \frac{\gamma}{2}[q_i(t)]^2$$

where $c \geq 0$ and $\gamma > 0$. Note that $\gamma = 1$ in Fershtman and Kamien (1987) and Dockner and Gaunersdorfer (2001).

Let $p(t)$ denote the price of the output. Due to price stickiness, the adjustment process of the price to a change in quantity is given

$$\dot{p}(t) = s\left(a - \sum_i q_i(t) - p(t)\right) \text{ with } p(0) = p_0 \geq 0 \tag{1}$$

where $a > c$ and $0 < s \leq \infty$ is a parameter that captures the speed of the adjustment of the price. (1) implies a linear inverse demand function.

The objective of firm $i$ is to maximize the discounted sum of profits

$$J_i = \int_0^\infty \left(p(t) - c - \frac{\gamma}{2} q_i(t)\right) q_i(t) e^{-rt} dt \tag{2}$$

subject to (1) where $r > 0$ denotes the interest rate.

We consider two strategy spaces. The first set of strategies is the open-loop strategy set where one firm's strategy defines this firm's production path for the whole time horizon. The second set of strategies considered is the set of closed-loop strategies where the production of firm $i$ at time $t$ is allowed to depend on $t$ and the price level at time $t$, $p(t)$[3].

A closed-loop (open-loop) Nash equilibrium for the pre-cartel game, i.e. the game without formation of a cartel, is defined by a vector of $N$ closed-loop (open-loop) strategies $(\phi_1^*, .., \phi_N^*)$ such that each strategy $\phi_i^*$ is a best closed-loop (open-loop) response to $\Phi_{-i}^* \equiv \left(\phi_1^*, .., \phi_{i-1}^*, \phi_{i+1}^*, .., \phi_N^*\right)$.

**The Cartel**

We then consider the possibility of a cartel of $M$ firms, $M \leq N$. Without loss of generality assume that in the case of a cartel of $M$ firms, the insider firms (firms that form the cartel) are the first $M$ firms, i.e. firm $j = 1, .., M$. If $M < N$, the outsider firms are firms $k$ , $k = M + 1, .., N$.

The objective of a cartel of $M$ firms is to maximize the joint discounted sum of profits of the $M$ firms denoted $J^C$

$$J^C = \sum_{j=1}^{M} J_j.$$

[3]For a formal definition of these strategy sets see for example Fershtman and Kamien (1987), Definition 1 and Definition 3, page 1154.

The cartel takes the production strategies, $\phi_k$ with $k = M+1, .., N$, of the outsider firms as given and chooses an $M$-tuple vector of production strategies $(\phi_1, .., \phi_M)$ that solves

$$\max \; J^C$$

subject to (1). The problem of an outsider firm $k$, with $M+1 \leq k \leq N$, is to maximize $J_k$ subject to (1) by choosing a strategy while taking the strategies of the cartel and $N-M-1$ other outsiders as given.

A closed-loop (open-loop) Nash equilibrium of the game when a cartel of $M$ firms forms is thus an $N$-tuple vector of closed-loop (open-loop) production strategies $\left(\phi_1^M, .., \phi_N^M\right)$ such that

$$J^C\left(\phi_1^M, .., \phi_N^M\right) \geq J^C\left(\phi_1, .., \phi_M, \phi_{M+1}^M .., \phi_N^M\right) \text{ for all } (\phi_1, .., \phi_M)$$

and

$$J_k\left(\phi_1^M, .., \phi_k^M, .., \phi_N^M\right) \geq J_k\left(\phi_1^M, .., \phi_k, .., \phi_N^M\right) \text{ for all } \phi_k \text{ and all } k = M+1, .., N.$$

A cartel of $M$ firms is said to be profitable (for the constituent firms) if

$$J^C\left(\phi_1^M, .., \phi_N^M\right) \geq \sum_{j=1}^{M} J_j\left(\phi_1^*, .., \phi_N^*\right)$$

where $(\phi_1^*, .., \phi_N^*)$ is a closed-loop (open-loop) Nash equilibrium of the game without a cartel (when $M=1$) and $\left(\phi_1^M, .., \phi_N^M\right)$ is a closed-loop (open-loop) Nash equilibrium when $M$ firms form a cartel. Since we assume that the firms are symmetric we focus on the symmetric equilibrium, i.e. an equilibrium with a cartel of $M$ firms such that

$$\phi_j = \phi_{ins} \text{ for all } j = 1, .., M$$

and

$$\phi_k = \phi_{out} \text{ for all } k = M+1, .., N$$

where $\phi_{ins}$ and $\phi_{out}$ are strategies of insiders and outsiders respectively. Given that the marginal cost of a firm is increasing with the quantity produced and that firms are symmetric, the cartel equally splits the overall production and (hence) profits among its members.

Note that by setting $M=1$ we obtain the case of a symmetric oligopoly with $N$ firms[4].

[4]The symmetric Cournot oligopoly can also be obtained as a special case of this game with 0 insiders and $N$ outsiders.

**Stability of a Cartel**

When all cartel sizes are profitable, a natural follow up question is: which of the profitable cartels are more likely to emerge? We use a stability criterion to answer this question.

The concept of stability used here is the one proposed by D'Aspremont et al. (1983). It will be convenient to introduce the following notations for insiders' and outsiders' profits at the equilibrium of the game with a cartel of $M$ firms in an $N$-firm oligopoly. Let

$$\Pi_{ins}(M,N) \equiv \frac{J^C\left(\phi_1^M,..,\phi_N^M\right)}{M}$$

and

$$\Pi_{out}(M,N) \equiv J^j\left(\phi_1^M,..,\phi_N^M\right) \text{ for any } j = M+1,..,N.$$

A cartel of $M$ firms is *internally stable* if

$$\Pi_{ins}(M,N) \geq \Pi_{out}(M-1,N); \tag{3}$$

i.e., no insider of cartel has an incentive to unilaterally exit.

A cartel of $M$ firms is *externally stable* if

$$\Pi_{out}(M,N) \geq \Pi_{ins}(M+1,N); \tag{4}$$

i.e., no outsider firm has an incentive to join the cartel of the $M$ firms.

A cartel of $M$ firms is *stable* if it is both internally and externally stable.

*Remark 1*: If a cartel of $M$ firms is *strictly* internally stable (i.e. $\Pi_{ins}(M,N) > \Pi_{out}(M-1,N)$) then a cartel of $M-1$ is, necessarily, externally unstable. Conversely, if a cartel of $M$ firms is internally unstable then a cartel of size $M-1$ is externally stable.

*Remark 2*: In the case of the grand coalition, i.e. when $M = N$, only internal stability is relevant.

*Remark 3*: A cartel of two firms is profitable if and only if a cartel of size 2 is internally stable.

The objective of this paper is to determine, among the profitable cartels, the ones that are stable. To this end we must characterize the equilibrium profits of firms (insiders and outsiders) when a cartel forms.

# 3 The equilibrium with a cartel

In this section, we determine the equilibrium of the game when $M$ firms form a cartel. For simplicity we set $c = 0$. This is innocuous but it simplifies the expressions that characterize the close-loop equilibrium we study.

### 3.1 The case of open-loop strategies

**Proposition 1** *There exists an asymptotically stable steady-state open-loop Nash equilibrium of the post-cartel game. The equilibrium production rates and the price at the steady state are given by*

$$q_{ins}^{OL} = \delta\left(\gamma + R\right) \tag{5}$$

$$q_{out}^{OL} = \delta\left(\gamma + MR\right) \tag{6}$$

*and*

$$p^{OL} = a - \left(N - M\right) q_{out}^{OL} - M q_{ins}^{OL} \tag{7}$$

*where* $R \equiv \frac{s}{r+s}$ *and* $\delta \equiv \frac{a}{\gamma^2+\gamma N+\gamma R+MR+MR\gamma+MRN-M^2R+MR^2} > 0$.

**Proof.** The proof is omitted. It is a straightforward extension of Proposition 1 in Benchekroun (2003) to arbitrary $\gamma > 0$. ■

The profits of a merging firm at the steady state are given by:

$$\pi_{ins}^{OL}\left(M, N\right) \equiv \left(p^{OL} - \frac{\gamma}{2} q_{ins}^{OL}\right) q_{ins}^{OL}$$

that is

$$\pi_{ins}^{OL}\left(M, N\right) = \frac{1}{2}\delta^2\left(2MR\gamma + \gamma R + 2MR^2 + \gamma^2\right)\left(\gamma + R\right) \tag{8}$$

It can be verified that in the limit case where the adjustment speed is infinite, i.e. when price adjusts instantaneously, the steady-state equilibrium price, quantities and profits correspond exactly to the outcome of a one shot static Cournot game. This is in line with Fershtman and Kamien (1987) and is similar to the result in Dockner (1992) where, for an adjustment cost differential game, it is shown that the steady state open-loop equilibrium "coincides with the Cournot equilibrium of the corresponding static game".

In this context a cartel is profitable only if the pre-cartel market share of the member firms is large enough. In particular, when $\gamma$ tends to 0, the results of SSR emerge.

In the remainder of the paper, as in Fershtman and Kamien (1987) and Dockner and Gaunersdorfer (2001), we shall focus on the limit case where $s$ tends to $\infty$. This facilitates the tractability of the equilibria we study. It can be shown that[5]

$$\Pi_{ins}\left(M, N\right) = \frac{1}{2r} a^2 \frac{\left(\gamma + 1\right)\left(2M\gamma + \gamma + 2M + \gamma^2\right)}{\left(\gamma^2 + \gamma N + \gamma + 2M + M\gamma + MN - M^2\right)^2} \tag{9}$$

[5] If $c > 0$, $a^2$ should be replaced with $\left(a - c\right)^2$ in (9) and (10).

and

$$\Pi_{out}(M,N) = \frac{1}{2r} a^2 \frac{(\gamma+M)(2M\gamma+\gamma+2M+\gamma^2)}{(\gamma^2+\gamma N+\gamma+2M+M\gamma+MN-M^2)^2} \tag{10}$$

The results of this section will serve as a benchmark that will allow us to isolate and identify the role played by closed-loop strategies.

### 3.2 The case of closed-loop strategies

We shall focus on the equilibrium production strategies when there is an interior solution.

**Proposition 2** *Let*

$$\phi^*_{ins}(p) = \frac{(1-K_c)p+E_c}{\gamma} \text{ and } \phi^*_{out}(p) = \frac{(1-K_k)p+E_k}{\gamma}$$

*where* $(K_k, K_c)$ *is a pair that solves the following system* $(S_K)$

$$(S_K)\begin{cases} \frac{M}{\gamma}(1-K_c) - \frac{1}{2}\frac{M}{\gamma}(1-K_c)^2 + K_c\left(-\frac{M}{\gamma}(1-K_c) - \frac{1}{\gamma}(N-M)(1-K_k) - 1\right) = 0 \\ \frac{1}{\gamma}(1-K_k) - \frac{1}{2\gamma}(1-K_k)^2 + K_k\left(-\frac{M}{\gamma}(1-K_c) - \frac{1}{\gamma}(N-M)(1-K_k) - 1\right) = 0 \end{cases}$$

*such that*

$$MK_c + (N-M)K_k - 1 - N < 0. \tag{11}$$

*and* $(E_k, E_c)$ *is the unique solution to the following linear system* $(S_E)$

$$(S_E)\begin{cases} K_k M\frac{E_c}{\gamma} + ((2(N-M)-1)K_k + MK_c - N - \gamma)\frac{E_k}{\gamma} = aK_k \\ (MK_c - N + (N-M)K_k - \gamma)\frac{E_c}{\gamma} + (N-M)K_c\frac{E_k}{\gamma} = aK_c. \end{cases}$$

*The strategies* $\phi^*_{ins}(.)$, $\phi^*_{out}(.)$ *constitute close-loop equilibrium strategies for the insider and outsider firms respectively.*

**Proof.** The proof is omitted. It is a straightforward extension of Proposition 3 in Dockner and Gaunersdorfer (2001) to the case of an arbitrary $\gamma > 0$. ■

*Remark 4*: Condition (11) ensures that the steady-state equilibrium price is asymptotically stable.

To determine the steady-state profits of an insider firm, we need to determine the steady-state production of each firm (insider and outsider firms) and the steady-state price as a function of $M$ and $N$. This requires us to determine solutions to the system $(S_K)$. However, the system $(S_K)$ is nonlinear and an analytical solution of $(K_c, K_k)$ as explicit functions of $N$ and $M$, when it exists, is in general impossible to obtain[6]. Dockner and Gaunersdorfer

[6]For example, substitution of $K_c$ from the second equation of the system $(S_K)$ into the first equation yields a polynomial of degree 4 in $K_k$ for which the roots can be determined explicitly but are too complex to offer any insight, see Dockner and Gaunersdorfer (2001).

(2001) solve the system numerically for specific values of $N$ and $M$ while Benchekroun (2003) shows the existence of a solution to this system.

The steady-state equilibrium price is the solution to

$$a - M\phi_{ins}(p) - (N-M)\,\phi_{out}(p) - p = 0$$

that is

$$p^{CL} = \frac{a\gamma - ME_c + ME_k - NE_k}{N + \gamma - MK_c + MK_k - NK_k}$$

where $K_c$, $K_k$, $E_c$ and $E_k$ are respectively given by $(S_K)$ and $(S_E)$.

The profits of an insider firm at the steady state are thus given by

$$\begin{aligned}
\pi_{ins}^{CL}(M,N) &= \left(p^{CL} - \frac{\gamma}{2}\phi_{ins}\left(p^{CL}\right)\right)\phi_{ins}\left(p^{CL}\right) \\
\pi_{out}^{CL}(M,N) &= \left(p^{CL} - \frac{\gamma}{2}\phi_{out}\left(p^{CL}\right)\right)\phi_{out}\left(p^{CL}\right)
\end{aligned}$$

Dockner and Gaunersdorfer (2001), numerically establish that all cartels are profitable in a 10 firm industry and that a cartel of 2 firms remains profitable when the total number of firms varies between 2 and 10. Benchekroun (2003) shows analytically that this remains true even when the total number of firms in an industry is arbitrarily large.

It can be shown that the discounted sum of profits of an insider firm and an outsider firm can be respectively written as

$$\Pi_{ins} = \frac{1}{2}\frac{(ME_c - 2a\gamma - 2ME_k + 2NE_k)\,E_c}{M\gamma r} \tag{12}$$

and

$$\Pi_{out} = \frac{1}{2}\frac{(2ME_c - E_k - 2a\gamma - 2ME_k + 2NE_k)\,E_k}{\gamma r}. \tag{13}$$

We now turn to the main question of the paper: Are these cartels stable?

# 4 Stability

## 4.1 The case of open-loop strategies

When firms use open-loop strategies, profitability of a cartel depends on $\gamma$. With small $\gamma$, for a cartel to be profitable, it has to represent a significant pre-cartel market share. For example, when $\gamma = 1$, it can be shown that a minimum market share of 69% is necessary for a cartel to be profitable. However, only size-2 cartel can be stable and it is so only when $\gamma$ is sufficiently large.

**Proposition 3** *If $N \geq M \geq 3$, then no cartel is stable.*

**Proof.** See Appendix 1. ■

The proof of Proposition 3 detailed in Appendix 1 consists of showing that no cartel of size 3 or larger is internally stable. An insider firm always gains by leaving the cartel. Even if a merger of $M$ firms ($M \geq 3$) may be profitable it is even profitable not to be a member of the merged entity.

**Proposition 4** *A cartel of size 2 is not stable if $\gamma \leq \tilde{\gamma}_N \equiv 3\left(N - 2.6\right)$ and is stable if $\gamma \geq \hat{\gamma}_N \equiv 3\left(N - 2.4\right)$.*

**Proof.** See Appendix 2. ■

Propositions 3 and 4 also imply the following results.

**Corollary 1** *If $N \geq 3$, the grand coalition is never stable.*

**Corollary 2** *If $\gamma \leq 1$ and $N \geq 3$, then no cartel is stable.*

From this analysis we conclude that the only possible size of a stable cartel is 2. Indeed, from Proposition 3 a cartel of size 3 is not stable because it is internally unstable and therefore a cartel of size 2 is always externally stable (see Remark 1). The proof of Proposition 4, detailed in Appendix 2, consists of showing that a cartel of size 2 can be internally stable (or profitable, see Remark 3) when $\gamma$ is below some threshold. For a given total number of firms $N$, a cartel of two firms can only be stable if the cost function is sufficiently convex ($\gamma \geq \hat{\gamma}_N$). For example, when $N = 5$ and $\gamma \geq 7.8$, a cartel of two firms is stable. A large rate of increase of firms' marginal cost ensures that the increase in production of the outsider firms following a merger will be moderate, while the gain from a decrease in production of the merged firms is large. Therefore when the cost function is sufficiently convex a merger of two firms becomes profitable and a size 2 cartel is stable. When the cost function is not sufficiently convex no cartel is stable.

## 4.2 The case of closed-loop strategies

In contrast to the open-loop case, simulations indicate that all cartels are profitable regardless of $\gamma$, when we consider close-loop equilibrium; moreover, stable cartels do exist in this case for all non negative values of $\gamma$. One surprising result we obtain is that the grand coalition (i.e., the cartel of all firms) is stable when $\gamma$ (the rate of change in marginal cost) is low. In addition, cartels of size 3 is stable *regardless* of $\gamma$.

### 4.2.1 Stability of the grand coalition

We show that the stability of the grand coalition depends on the value of the rate of change in the marginal cost, $\gamma$.

**Proposition 5** *For any $N \geq 2$ there exists $\bar{\gamma}_N > 0$ such that for all $\gamma \in (0, \bar{\gamma}_N)$ the grand coalition is stable.*

**Proof.** To prove the above proposition, we first note that for any $0 < \gamma < \infty$ a multiplant monopoly earns strictly positive profits per plant and moreover, the profits per plant increases as $\gamma$ decreases[7]. Next, we show that when $\gamma$ tends to zero, the profit function, as determined by (13) in Section 3.2, for the firm that exits the grand coalition (i.e., the outsider of a cartel of size $N - 1$) tends to zero. Therefore when $\gamma$ is close enough to zero, the outsider's profits become arbitrarily small. Consequently, no firm would wish to exit the grand coalition. Thus the proof consists of showing that when $M = N - 1$, the discounted sum of profits of the outsider tends to zero when $\gamma$ becomes arbitrarily small. This is done in Appendix 3. ■

We note that this result is in contrast with Corollary 1 derived for the open-loop case. The difference is solely due to the nature of the strategies used. Kamien and Fershtman (1987) show that the outcome of a closed-loop game is closer to a competitive outcome than the outcome of an open-loop game[8]. This remains true in the case of a finite horizon differential game with sticky prices when the time horizon is long enough (see Kamien and Fershtman (1990)).

As long as a cartel has at least one fringe firm, the smaller the value of the positive cost parameter $\gamma$ the closer is the closed-loop equilibrium outcome of the game to price-taking behavior. When $\gamma$ tends to zero price tends to the marginal cost and profits tend to zero. That is, when $\gamma$ is small enough the profits of all competing firms and cartel in the closed-loop

[7] We can also obtain the closed form solution for the per plant profits.

[8] Dockner (1988) generalizes this result to the case of an oligopoly.

equilibrium become very small (see Table 1)[9]. If a firm exits the grand-coalition, it would face an exacerbated competition due to the feedback effect and would get lower profits than if it had remained in the coalition.

We note however that as $\gamma$ becomes larger, the grand coalition is no longer stable. This is illustrated[10] in the table below for the case that $N = 10$ and $a = 100$.

Table 1

| | | | | | |
|---|---|---|---|---|---|
| $\gamma$ | 0.001 | **0.19** | **0.2** | 1 | 10 |
| $\pi_{ins}\,(M = N = 10)$ | 250.0 | **247.65** | **247.52** | 238.10 | 166.67 |
| $\pi_{out}\,(M = 9, N = 10)$ | 1.927 | **240.68** | **248.47** | 415.97 | 197.45 |

From these simulations we get that the critical value of $\gamma$ beyond which the grand coalition in a 10 firms industry stops being stable is approximately equal to 0.20.

### 4.2.2 Another stable cartel size

It is interesting to note that the grand coalition may not be the unique stable cartel. Numerical simulations indicate that *for all* $\gamma > 0$ a cartel of size 3 is stable for all $N \geq 4$.

In the first set of simulations we set $\gamma = 0.1, N = 10$ and $a = 100$. We obtain that 3 and 10 (the grand coalition are the two stable cartels:

Table 2

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $M$ | 1 | 2 | **3** | 4 | 5 | 6 | 7 | 8 | 9 | **10** |
| $\pi_{ins}$ | 6.02 | 6.12 | **6.38** | 6.85 | 7.63 | 8.92 | 11.20 | 15.92 | 30.18 | **248.88** |
| $\pi_{out}$ | 6.02 | 6.32 | **6.97** | 8.13 | 10.17 | 13.97 | 22.01 | 44.07 | 152.60 | |

[9] We actually obtain a rather interesting result. The profits of firms are inversely U shaped functions of the cost parameter $\gamma$. When $\gamma$ is small enough, firms are able to achieve larger profits when $\gamma$ increases. A similar result was obtained in a static framework by Seade (1985).

[10] This result was confirmed by additional simulations that were carried out for other parameter values. Numerical simulations were carried out with MuPad Pro 3.0.

In the second set of simulations we set $\gamma = 1, N = 10$ and $a = 100$. We obtain that 3 is the only stable cartel:

Table 3

| $M$ | 1 | 2 | **3** | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\pi_{ins}$ | 49.02 | 49.61 | **51.29** | 54.26 | 58.98 | 66.27 | 77.87 | 97.73 | 136.90 | 238.10 |
| $\pi_{out}$ | 49.02 | 50.97 | **55.17** | 62.45 | 74.48 | 94.85 | 132.00 | 209.60 | 416.00 | |

In the third set of simulations we set $\gamma = 5, N = 10$ and $a = 100$. We obtain that 3 is the only stable cartel size:

Table 4

| $M$ | 1 | 2 | **3** | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\pi_{ins}$ | 121.4 | 122.0 | **123.9** | 127.2 | 132.0 | 138.7 | 147.7 | 160.0 | 176.7 | 200.0 |
| $\pi_{out}$ | 121.4 | 123.7 | **128.6** | 136.5 | 148.2 | 164.8 | 188.6 | 223.1 | 275.0 | |

Other simulations for different values of $N$ and $\gamma$ confirm that 3 is a stable cartel size for all $\gamma > 0$ This is also in contrast with the case where firms use open-loop strategies, where size 2 cartel is stable only for sufficiently convex production cost ($\gamma \geq \hat{\gamma}_N$). There are comparable results in the literature on the stability of cartels in a static framework where it is shown that only cartels of *small* sizes are stable.

Donsimoni et al. (1986) show that in D'Aspremont et al.'s price leadership model with competitive fringe, results similar as ours emerge with linear demand function and quadratic cost function without the linear term. Either there is a unique stable cartel $M < N$ when the firms are not too cost efficient relative to demand or there are two stable cartels $N$ and $M < N$ otherwise. In particular, when cost function of each firm is $C(q) = \frac{1}{2}q^2$ and there are at least four firms, the only stable cartel size is three. Note, however, the demand function used by Donsimoni et al. (1986) is $D(p) = N(a - bp)$, which is a function of $N$. Nonetheless, the similarity between their results and ours is intriguing.

Shaffer (1995) considers the case where the cartel acts as a Stackelberg leader and the outsider firms constitute a Cournot fringe. She investigates how the size of the stable cartel is related to the number of firms in a setting with linear demand and constant marginal cost. Konishi and Lin (1999) proves the existence of a stable cartel with general demand and cost functions. In an example they show that with demand function $D(p) = a - Q$ and cost function $C(q) = \frac{1}{2}q^2$, size of stable cartel increase with $N$. This is in contrast to that in D'Aspremont et al.'s price leadership model with the same demand function and cost function, the size of stable cartel is 3 for $N > 5$.

Diamantoudi (2005) shows that if firms are endowed with foresight, larger cartels are stable. The stability proposed in Diamantoudi (2005) captures the foresight of any firm that contemplates leaving (joining) a cartel. In particular, each firm anticipates that after it leaves the grand coalition, other firms may also leave afterwards and consequently, its profits may decrease ultimately to a level below those of an insider of the grand coalition.

# 5 Concluding Remarks

We have shown that when firms use open-loop strategies, the grand coalition is never stable. We have also shown that only a cartel of size 2 can be stable and it is so only when the cost function is sufficiently convex, i.e., when $\gamma$, the rate of increase of the marginal cost, is sufficiently high. Furthermore, no cartel with 3 or more firms is stable for all values of $\gamma$. Note that a large cartel may be profitable, but each firm earns even larger profits by exiting the cartel. Like the static Cournot equilibrium, open-loop equilibrium in the dynamic model we analyze cannot be used to explain the stability of the grand coalition.

We have shown that when firms use closed-loop strategies, the grand coalition is stable for small enough $\gamma$. The closed-loop effect renders not only all cartels profitable but also the grand coalition stable. For larger values of $\gamma$ the grand coalition is no longer stable; however the coalition size of 3 emerges as the only coalition size that is stable regardless of $\gamma$.

It could also be interesting to investigate the profitability and stability of cartels when firms play a non-linear equilibrium. It is known that the game studied in this paper, i.e. a Cournot competition with sticky prices, admits a continuum of equilibria with non-linear strategies (see Tsutsui and Mino (1990)).

## Appendix 1: Proof of proposition 3

We show that if $N \geq M \geq 3$, then no cartel is stable.

Internal stability of size-$M$ cartel requires that $\pi_{out}^{OL}(M-1, N) - \pi_{ins}^{OL}(M, N) \leq 0$ or $\frac{A}{B} - \frac{C}{D} < 0$ where

$$\begin{aligned}
A &= (\gamma + (M-1))\left(2(M-1)\gamma + \gamma + 2(M-1) + \gamma^2\right) \\
B &= \left(\gamma^2 + \gamma N + \gamma + 2(M-1) + (M-1)\gamma + (M-1)N - (M-1)^2\right)^2 \\
C &= (\gamma + 1)\left(2M\gamma + \gamma + 2M + \gamma^2\right) \\
D &= \left(\gamma^2 + \gamma N + \gamma + 2M + M\gamma + MN - M^2\right)^2.
\end{aligned}$$

Since $B > 0$ and $D > 0$, $\frac{A}{B} - \frac{C}{D} < 0$ if and only if $AD - BC < 0$.$AD - BC$ can be written as

$$AD - BC = (\gamma + 1)\left(\alpha_5\gamma^5 + \alpha_4\gamma^4 + \alpha_3\gamma^3 + \alpha_2\gamma^2 + \alpha_1\gamma + \alpha_0\right)$$

where

$$\begin{aligned}
\alpha_5 &= M - 2 \\
\alpha_4 &= 2MN - 4N - 10M + 4M^2 + 3 \\
\alpha_3 &= 6N - 5M - 18MN - 8M^2 + 3M^3 - 2N^2 + MN^2 + 8M^2N + 7 \\
\alpha_2 &= 38M + 4N + 6MN - 48M^2 + 20M^3 + 3N^2 - 4M^4 - 8MN^2 \\
&\quad -20M^2N + 8M^3N + 4M^2N^2 - 7 \\
\alpha_1 &= 14M - 6N + 26MN + 14M^2 - 32M^3 - N^2 + 16M^4 - 3M^5 \\
&\quad +8MN^2 - 30M^2N + 12M^3N - 2M^4N - 12M^2N^2 + 5M^3N^2 - 9 \\
\alpha_0 &= 56M^2 - 12MN - 18M - 68M^3 + 42M^4 - 14M^5 + 2M^6 - 2MN^2 + 36M^2N \\
&\quad -40M^3N + 20M^4N - 4M^5N + 6M^2N^2 - 6M^3N^2 + 2M^4N^2
\end{aligned}$$

To show that no cartel is stable when $N \geq M \geq 3$, it suffices to show that size-$M$ cartel is internally unstable, i.e., $AD - BC > 0$. To this end, we shall show that $\alpha_\ell > 0$ for all $\ell = 0, 1, \ldots, 5$ if $N \geq M \geq 3$.

Obviously, $\alpha_5 > 0$. We now proceed to show that $\alpha_4 > 0$. $\alpha_4$ is an increasing function of $N$ given $N \geq 3$. If $N = M$, we have $\alpha_4 = 6M^2 - 14M + 3 > 0$. Thus, $\alpha_4 > 0$. Now, consider $\alpha_3$. Observe first that if $M = 3$, $\alpha_3 = 24N + N^2 + 1 > 0$. It can be show that when $N \geq M \geq 3$,

$$\frac{d\alpha_3}{dM} = 16MN - 18N - 16M + 9M^2 + N^2 - 5 > 0.$$

To show that $\alpha_2 > 0$, when $N = M$, $\alpha_2 = 42.0M - 39.0M^2 - 8.0M^3 + 8.0M^4 - 7.0 > 0$ as $M \geq 3$. Also, we observe that

$$\frac{d\alpha_2}{dN} = 6M + 6N - 16MN - 20M^2 + 8M^3 + 8M^2N + 4$$

which is positive given that $N \geq M \geq 3$. Thus, $\alpha_2 > 0$.

Next, $\alpha_1$ can be written as

$$\alpha_1 = (M-1)\,F$$

where

$$\begin{aligned} F &= 6N - 5M - 20MN - 19M^2 + 13M^3 + N^2 - \\ & \quad 3M^4 - 7MN^2 + 10M^2N - 2M^3N + 5M^2N^2 + 9 \end{aligned}$$

To show that $F > 0$, we first examine

$$\frac{dF}{dN} = 2N - 20M - 14MN + 10M^2 - 2M^3 + 10M^2N + 6.$$

When $N = M$, $F = M - 38M^2 + 16M^3 + 9 > 0$ if $M \geq 3$. Thus, we need only to show that $\frac{dF}{dN} \geq 0$. When $N = M \geq 3$, $\frac{dF}{dN} = 8M^3 - 4M^2 - 18M + 6 > 0$. Moreover, $\frac{d^2F}{dN^2} = 10M^2 - 14M + 2 > 0$ if $M \geq 3$. Thus, $F > 0$, which implies $\alpha_1 > 0$.

Lastly, $a_0$ can be written as

$$\alpha_0 = 2M\,(M-1)^2 \left[N^2\,(M-1) + N\left(6M - 2M^2 - 6\right) + 10M - 5M^2 + M^3 - 9\right].$$

This quadratic function of $N$ in the last bracket is strictly increasing in $N$. Indeed,

$$\begin{aligned} 2\,(M-1)\,N + \left(6M - 2M^2 - 6\right) &> 2\,(M-1)\,M + \left(6M - 2M^2 - 6\right) \\ &= 4M - 6 > 0 \text{ for } M \geq 2. \end{aligned}$$

Since the quadratic function is strictly increasing in $N$ for $N \geq M \geq 2$ we have

$$\begin{aligned} & (M-1)\,N^2 + \left(6M - 2M^2 - 6\right)N + M^3 - 5M^2 - 9 + 10M \\ > & \ (M-1)\,M^2 + \left(6M - 2M^2 - 6\right)M + M^3 - 5M^2 - 9 + 10M \\ = & \ 4M - 9 > 0 \text{ for all } M \geq 3 \end{aligned}$$

Therefore, $\alpha_0 > 0$ ■

**Appendix 2: Proof of Proposition 4**

We show that if $N \geq 2.6 + \frac{1}{3}\gamma$, cartel of size 2 is not stable.

If $M = 2$, we have

$$AD - BC = \left(7\gamma + 3\gamma^2 + 4\right) N^2 + \left(2\gamma^3 - 10\gamma - 8\right) N - 27\gamma^2 - 11\gamma^3 - \gamma^4 - 21\gamma - 4.$$

The larger root of the above quadratic equation is given by

$$\bar{N} = \frac{1}{3\gamma + 4}\left(\gamma - \gamma^2 + 2\left(\gamma + 1\right)\sqrt{\left(\gamma + 2\right)\left(\gamma + 4\right)} + 4\right).$$

It is easy to see that

$$\frac{1}{3\gamma + 4}\left(\gamma - \gamma^2 + 2\left(\gamma + 1\right)\left(\gamma + 2\right) + 4\right) < \bar{N} < \frac{1}{3\gamma + 4}\left(\gamma - \gamma^2 + 2\left(\gamma + 1\right)\sqrt{\left(6\gamma + \gamma^2 + 9\right)} + 4\right)$$

$$2.4 + \frac{1}{3}\gamma < \frac{1}{3\gamma + 4}\left(7\gamma + \gamma^2 + 8\right) < \bar{N} < \frac{1}{3\gamma + 4}\left(9\gamma + \gamma^2 + 10\right) < 2.6 + \frac{1}{3}\gamma$$

Thus, if $N \geq 2.6 + \frac{1}{3}\gamma$, then $AD - BC > 0$, implying that size-2 cartel is not internally stable and if $N \leq 2.4 + \frac{1}{3}\gamma$ size-2 cartel is not internally stable ■

**Appendix 3: Proof of Proposition 5**

We show in this appendix that $\lim_{\gamma \to 0} G_{out} = 0$ where $G_{out}$ is given by (13). From Proposition 2, the equations that determine the parameters are obtained by setting $M = N - 1$, which yields:

for $K_c$ and $K_k$,

$$\frac{1}{2}\frac{N - 2NK_c - 2\gamma K_c + 2K_cK_k - K_c^2 + NK_c^2 - 1}{\gamma} = 0 \tag{14}$$

$$\frac{1}{2}\frac{2NK_cK_k - 2\gamma K_k - 2K_cK_k - 2NK_k + K_k^2 + 1}{\gamma} = 0 \tag{15}$$

for $E_c$ and $E_k$,

$$K_k\left(N - 1\right)\frac{E_c}{\gamma} + \left(K_k + \left(N - 1\right)K_c - N - \gamma\right)\frac{E_k}{\gamma} = aK_k \tag{16}$$

$$\left(\left(N - 1\right)K_c - N + K_k - \gamma\right)\frac{E_c}{\gamma} + K_c\frac{E_k}{\gamma} = aK_c \tag{17}$$

and for $G_{ins}$ and $G_{out}$,

$$G_{ins} = \frac{1}{2}\frac{(2E_k - E_c - 2a\gamma + NE_c)\,E_c}{(N-1)\gamma r} \tag{18}$$

$$G_{out} = \frac{1}{2}\frac{(E_k - 2E_c - 2a\gamma + 2NE_c)\,E_k}{\gamma r} \tag{19}$$

To show that $\lim_{\gamma\to 0} G_{ins} = 0$ and $\lim_{\gamma\to 0} G_{out} = 0$, it suffices to show that $\lim_{\gamma\to 0}\frac{E_c}{\gamma} < \infty$ and $\lim_{\gamma\to 0}\frac{E_k}{\gamma} < \infty$ as these would imply that $\lim_{\gamma\to 0} E_c = 0$ and $\lim_{\gamma\to 0} E_k = 0$. To this end, we first solve for $\frac{E_c}{\gamma}$ and $\frac{E_k}{\gamma}$ for equations (16) and (17) and then compute $\lim_{\gamma\to 0}\frac{E_c}{\gamma}$ and $\lim_{\gamma\to 0}\frac{E_c}{\gamma}$:

$$\begin{aligned}
\lim_{\gamma\to 0}\frac{E_c}{\gamma} &= a\frac{K_c\left(\left(N-1\right)K_c - N\right)}{D} \\
\lim_{\gamma\to 0}\frac{E_k}{\gamma} &= \lim a\frac{\left(K_k - N\right)K_k}{D}
\end{aligned}$$

where $D \equiv 2NK_c - 2N^2K_c + (N-1)\,K_cK_k + (N-K_k)^2 + (N-1)^2\,K_c^2$. It can be shown that the system for $K_k$ and $K_c$ admits only finite solutions[11]. Then it suffices to show that $D \neq 0$.

Assume in negation that $D = 0$. Equations (14) and (15) imply that

$$N - 2NK_c - 2\gamma K_c + 2K_cK_k - K_c^2 + NK_c^2 - 1 = 0$$

and

$$2NK_cK_k - 2\gamma K_k - 2K_cK_k - 2NK_k + K_k^2 + 1 = 0.$$

When $\gamma$ tends to zero we have

$$N - 2NK_c + 2K_cK_k - K_c^2 + NK_c^2 - 1 = 0 \tag{20}$$

and

$$2\,(N-1)\,K_cK_k - 2NK_k + K_k^2 + 1 = 0. \tag{21}$$

From (20), we can obtain

$$2K_cK_k = 2NK_c - N + K_c^2 - NK_c^2 + 1 \tag{22}$$

[11]As noted in footnote 4 any solution $(K_k, K_c)$ to the system $(S)$ is such that $K_k$ (or $K_c$) corresponds to a root of a polynomial of degree 4 in $K_k$ (or $K_c$). Infinity is not a root to a polynomial of non negative degree.

and from (21),

$$\left(N-K_k\right)^2 = N^2 - 1 - 2\left(N-1\right)K_c K_k \tag{23}$$

Substituting (20) and (21) into $D = 0$ gives

$$\frac{1}{2}\left(N-1\right)\left(3N - 6NK_c - 3K_c^2 + 3NK_c^2 + 1\right) = 0.$$

If $N > 1$, we have

$$3N - 6NK_c - 3K_c^2 + 3NK_c^2 + 1 = 0,$$

which implies that

$$K_c = \frac{1}{3N-3}\left(3N - \sqrt{3}\sqrt{2N+1}\right). \tag{24}$$

Substituting the above into equation (20) gives

$$K_k = \frac{1}{3N+1}\left(2N + \frac{2}{3}\sqrt{6N+3}\right). \tag{25}$$

Substituting (24) into equation (21) gives

$$K_k^2 - 2NK_k + K_k\left(2N - \frac{2}{3}\sqrt{6N+3}\right) + 1 = 0.$$

The above equation has two roots

$$\begin{aligned} K_k &= \frac{1}{3}\sqrt{3}\sqrt{2N+1} - \frac{1}{3}\sqrt{6}\sqrt{N-1} \text{ and} \\ K_k &= \frac{1}{3}\sqrt{3}\sqrt{2N+1} + \frac{1}{3}\sqrt{6}\sqrt{N-1}. \end{aligned}$$

It is easy to verify that neither of these roots coincides with (25). This leads to the conclusion that $D \neq 0$. Therefore $\lim_{\gamma\to 0}\frac{E_c}{\gamma} < \infty$ and $\lim_{\gamma\to 0}\frac{E_k}{\gamma} < \infty$ ■

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