Ph.D. Comprehensive Exam
Macroeconomics

Prof. Barczyk & Prof. Castro

May 10, 2023

INSTRUCTIONS:
Answer both long questions (40 points each) and one of the two short questions (20 points) for a total of 100 points. Be concise and to the point. Provide explanations for your answers to receive full credit.

LONG QUESTIONS
Answer *both* long questions below.

Question 1. Consumption-savings model with a no-borrowing constraint. Consider the following consumption-savings problem with a no-borrowing constraint in a deterministic environment:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + \frac{a_{t+1}}{R} \leq a_t + y_t, \quad \forall t$$
$$a_{t+1} \geq 0, c_t \geq 0, \quad \forall t$$

where $\beta \in (0,1)$ and $R\beta = 1$. The flow utility function is standard with $u'(c) > 0, u''(c) < 0, \lim_{c\to 0} u'(c) = +\infty$ and $\lim_{c\to\infty} u'(c) = 0$. Labor income y_t is exogenously determined. The initial asset stock $a_0 \geq 0$ is given.

1. (8 pts.) Use the sequence problem to find the optimal consumption sequence, $\{c_t^*\}_{t=0}^{\infty}$, asset sequence, $\{a_{t+1}^*\}_{t=0}^{\infty}$, and value function, assuming $y_t = \bar{y} > 0, \forall t$. Comment on your results.

2. (2 pts.) Find the value function using the recursive problem.

3. (8 pts.) We now extend the model to accommodate a life-cycle earnings profile. Specifically, assume that for periods $t = 0, 1, \ldots, T-1$ the agent faces a time-varying income profile, y_t, which from time T on remains $\bar{y}$ forever. At time T (what we have referred to as $t = 0$ above) the value function is given by the one you have just found.

   a.) (2 pts.) Write down the Bellman equation for this extended model.

   b.) (4 pts.) Find optimal consumption and savings in period $T-1$ (the penultimate period, i.e. before entering the forever-constant-income regime). Assume that $y_{T-1} > \bar{y}$. Comment on your results.

   c.) (2 pts.) Find the value function in period $T-1$.

4. (7 pts.) Now go backwards one additional period to $T-2$. To make the notation lighter set $T = 2$ which means that we are now in period $t = 0$. Write down the Bellman equation for this period. Find consumption c_0 and savings a_1 assuming $a_0 = 0$. Do so for both the case in which the agent would like to borrow and to save. Comment on your results.

5. (15 pts.) Return to the initial sequence problem. Savings are bounded by exogenous upper and lower bounds: $a_t \in [\underline{a}, \bar{a}]$ for all t and a_0 is given.

   a.) (5 pts.) Bring the problem into dynamic-programming form (e.g. return function, feasible-set correspondence) and formulate the Bellman equation.

   b.) (10 pts.) Argue carefully – using Theorems when appropriate – that there is a unique solution V to the Bellman equation in the space of bounded and continuous functions, $\mathcal{C}(X)$, equipped with the supremum norm.

Question 2. The Hopenhayn model of industry dynamics. Consider the Hopenhayn model of industry dynamics. Firms are atomistic and produce a homogeneous product at price $p = 1$, according to $y = zn^{\gamma}$, $\gamma \in (0, 1)$, where productivity z follows a Markov chain, and n is labor employment. There is a per-period fixed cost of operation c_f. Firms discount future profits at rate $\beta \in (0, 1)$. A firm may decide to exit the industry at the end of each period, after production but before observing next period's productivity. Every period there is a large number of ex-ante identical potential entrants, facing an entry cost

c_e. Upon entry, initial productivity is drawn from a discrete distribution $G(z)$. Assume aggregate labor supply is some strictly increasing function $L^s(w)$.

1. Formulate the problem of an incumbent firm in recursive fashion.
2. Show that firm size (n) is strictly increasing in z. Show that profits (gross of c_f) are strictly positive, and also strictly increasing in z. Explain the importance of decreasing returns to scale for these results (in contrast to constant returns to scale).
3. Characterize the optimal exit decision rule.
4. Define a stationary recursive competitive equilibrium.
5. Describe the steady-state effects of an increase in c_e on wages, average firm size, mass of entrants, and mass of operating firms. Explain your reasoning.
6. Describe a numerical algorithm to solve for the transitional dynamics associated with a surprise permanent positive shock to c_e.

SHORT QUESTIONS

Answer *one of the two* short questions below.

Question 3. Consumption-savings model with a consumption floor. Consider the following two-period model. At time $t = 0$, the agent is endowed with wealth $a_0 > 0$ but does not receive an endowment at time $t = 1$. The agent has access to a savings technology with gross interest rate R and is subject to a no-borrowing constraint $a_1 \geq 0$. Preferences are standard and represented by $u(c_0) + \beta u(c_1)$. The per-period utility function u is twice continuously differentiable with $u'(c) > 0$, $u''(c) < 0$ and Inada condition $\lim_{c \to 0} u'(c) = \infty$.

The consumption floor enters in the following way: It ensures that the agent can consume at least $\underline{c} > 0$ in the final period. It does so by conditioning on resources the agent enters the last period with. If $a_1 < \underline{c}$, the agent receives a transfer equal to $\underline{c} - a_1$. If $a_1 > \underline{c}$, there is no transfer. In summary, the transfer function is given by

$$T(a_1) = \max\{\underline{c} - a_1, 0\}$$

1. (5 pts) State the agent's optimal consumption policy in the final period and sketch it. State the agent's value function at the beginning of the final period and sketch it. Which properties of the value function are non-standard (i.e. different from the standard model in which $\underline{c} = 0$)?

2. (10 pts) Write down a payoff function $J(a_1|a_0)$ that, conditional on the initial state a_0, gives the lifetime payoff from a feasible savings choice a_1 (i.e. this is not a value function but rather evaluates the payoff from any *feasible* savings choice). Sketch the payoff function for some arbitrary initial values for $a_0 > 0$. Comment on the function in regards to continuity, monotonicity and concavity.

3. (5 pts) Consider an auxiliary problem in which the agent is restricted to savings choices $a_1 \leq \underline{c}$. Find the value function $\underline{V}_0$ and the optimal savings policy $\underline{g}$.

Question 3. Neoclassical growth model with two types of agents. Consider the neoclassical growth model with two types of agents, rich and poor, in measure μ and $1 - \mu$, respectively. Agents are identical in all respects except that each rich agent has a higher initial endowment of capital. Leisure is not valued, and each agent has an endowment of one unit of time every period. Preferences for consumption are standard. Show that there are multiple capital distributions consistent with steady-state, even though aggregate capital is fully determined.