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Wednesday, May 25th, 2022

Ph.D. Comprehensive Exam
Macroeconomics

INSTRUCTIONS :
Answer both long questions and one of the two short questions. Be concise and to the point.

Provide explanations for your answers to receive full credit. You may state additional assumptions.

The total points for each question are shown below.

LONG QUESTIONS
Answer both long questions below.

Question 1. (40 pts.) Two-sector growth model. Consider an infinite-horizon economy with two sectors. The first sector produces the consumption good y using the technology :

$$y_t = Ak_t^{\alpha} h_t^{\gamma} n_{y,t}^{1-\alpha-\gamma}$$

where $k_t \geq 0$ is physical capital at t, $h_t \geq 0$ is human capital at t, and $n_{y,t} \geq 0$ are labor hours used in the consumption-good sector at t. The parameters A, α, γ are positive and $\alpha + \gamma < 1$. The second sector is the human-capital production sector. Human capital is accumulated according to

$$h_{t+1} = (1 - \delta_h)h_t + n_{h,t},$$

where $n_{h,t} \geq 0$ are labour hours used in the human-capital production sector, and $\delta_h \in (0, 1)$ is a parameter. The representative household has an endowment of one unit of labor time at each point in time and so $n_{y,t} + n_{h,t} \leq 1$ for all t. The resource constraint is given by

$$k_{t+1} + c_t \leq y_t + (1 - \delta_k)k_t,$$

where c_t is consumption at t and $\delta_k \in (0, 1)$. Preferences of the representative household are given by

$$\sum_{t=0}^{\infty} \beta^t \sqrt{c_t},$$

where $\beta \in (0, 1)$. The economy starts off at $t = 0$ with given endowments $k_0 > 0$ and $h_0 > 0$.

(a) State the sequential formulation of the planner's problem. State all constraints including non-negativity constraints.

(b) Which of the non-negativity constraints will not bind, and which may bind ? Briefly argue why.

(c) Find the first-order conditions for the optimal accumulation of physical capital and of human capital (assuming an interior solution) and interpret them briefly. Obtain these from the Lagrangian formulation of the planner's problem.

(d) State the planner's problem for this economy in dynamic programming form : Say what the state, the control(s), the feasible-set correspondence and the return functions are.

(e) Formulate the planner's Bellman equation. Briefly explain how to interpret this functional equation.

Question 2 (40 pts). Money in the utility function. Consider an economy where infinitely-lived agents solve the problem

$$\underset{\{c_t^i, M_t, k_t\}_{t=s}^{\infty}}{Max} \quad \sum_{t=s}^{\infty} \beta^{t-s} \left(\log(C_t) + a \log(m_t)\right),$$

where $\beta \in (0,1)$ is the subjective discount rate, C_t is consumption, $m_t = M_t/P_t$ is the real money stock, M_t is the nominal money stock, P_t is the price level, and $a > 0$. Consumption, C_t, is the aggregate

$$C_t = \left(\int_0^1 (c_t^i)^{(\theta-1)/\theta} di \right)^{\theta/(\theta-1)},$$

where c_t^i is consumption of good i, $i \in [0,1]$ and $\theta > 1$. Utility maximization is subject to the constraint

$$\int_0^1 \left(\frac{p_t^i c_t^i}{P_t} \right) di + m_t + k_t = r_{t-1}k_{t-1} + \frac{m_{t-1}}{1+\pi_t} + (1-\delta)k_{t-1} + \tau_t + d_t,$$

where π_t is the inflation rate, r_t is the rental rate of capital, k_t is the capital stock, τ_t is a lump-sum transfer, d_t are profits, p_t^i is the price of good i, and

$$P_t = \left(\int_0^1 (p_t^i)^{1-\theta} di \right)^{1/(1-\theta)},$$

is the aggregate price index. The production function of good i is

$$y_t^i = \left(k_t^i\right)^{\alpha}$$

where $0 < \alpha < 1$. Capital is perfectly mobile between sectors.

a) Write the representative consumer's problem in recursive form. Justify your choice of state variables.

b) Derive the consumption function of good c_t^i as a function of its real price and total consumption. Interpret your results.

c) Derive the money demand function. Interpret your results.

d) Find the value of money in this model.

SHORT QUESTIONS (20 points)
Answer <u>one of the two</u> short questions below.

Question 3. Operators on function spaces. Consider the state space $x \in X = [0,1]$ and the space of continuous and bounded functions $f : X \to R$ defined on that state space. Consider the following operator T :

$$Tf(x) = \max_{x' \in [0,x]} \left\{ -\alpha(1 - x - x'^2 + \beta f(x') \right\}$$

where $\alpha > 0$ and $\beta \in (0,1)$ are parameters.

(a) Show that the operator maps $C(X)$, the space of continuous and bounded functions on the interval $[0,1]$ endowed with the sup-norm, to itself, i.e. $T : C(X) \to C(X)$.

(b) Does the operator T have a unique fixed point ? Show your results.

Question 3. Exact solution of growth model. Consider an economy where an infinitely-lived representative agent has preferences

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

where $\beta \in (0,1)$, c_t is consumption, and E_0 denotes the expectation conditional on information known at time 0. The aggregate production function is $y_t = z_t k_t^{\alpha}$, where $\alpha \in (0,1)$, z_t is the stochastic level of productivity, k_t is the capital stock, and y_t is output. Assume that $\ln(z_t)$ is independently and identically distributed with mean zero and variance σ^2. The capital depreciates completely during production and, thus, the aggregate resource constrain is

$$c_t + k_{t+1} = y_t.$$

The initial capital stock is given.

(a) Find the value function that solves this problem using the conjecture that value function is log-linear. What are the coefficients of the state(s) variables ?

(b) Find the policy rules for the choice of current consumption and future capital.