Micro Comprehensive Exam

Winter 2018
Final Exam

Part A

(Please answer BOTH questions from this part.)

1. [***Utility representation.***] Let $X = \mathbb{R}^2_+$. Assume that a preference relation $\succsim$ satisfies the following three properties:

- **Additivity:** If $(x_1, x_2) \succsim (y_1, y_2)$ then $(x_1 + s, x_2 + t) \succsim (y_1 + s, y_2 + t)$ for all s, t.
- **Strong Monotonicity:** If $x_1 \geq y_1$ and $x_2 \geq y_2$ then $(x_1, x_2) \succsim (y_1, y_2)$; if in addition either $x_1 > y_1$ or $x_2 > y_2$ then $(x_1, x_2) \succ (y_1, y_2)$.
- **Continuity:** For all (x_1, x_2), sets $\{(y_1, y_2) : (y_1, y_2) \succsim (x_1, x_2)\}$ and $\{(y_1, y_2) : (y_1, y_2) \precsim (x_1, x_2)\}$ are closed.

1. A **positive linear utility representation** is a utility function such that $u(x_1, x_2) = \alpha x_1 + \beta x_2$ for some $\alpha, \beta > 0$. Show that if $\succsim$ has a positive linear representation then it must satisfy the three properties above.
2. For any pair of the three properties give an example of a preference relation that does not satisfy the third property.
3. Show that if $\succsim$ satisfies the three properties, then it must have a positive linear representation. [Harder. Make sure you finish the other questions first.]

2. [***General equilibrium.***] Consider a pure exchange economy with I agents and L commodities. The consumption space is $X = \mathbb{R}^L_+$. Each agent i has some initial endowment $\omega_i \in \mathbb{R}^L_+$ and a utility function $u_i : \mathbb{R}^L_+ \mapsto \mathbb{R}$, which is assumed to be continuous and strongly monotone. Assume $\sum_i \omega_i >> 0$. Recall that a feasible allocation is a consumption vector $\mathbf{x} = (x_1, \ldots, x_I) \in (\mathbb{R}^L_+)^I$ such that $x_i \in \mathbb{R}^L_+$ and $\sum_i x_i = \sum_i \omega_i$. Recall that a feasible allocation $\mathbf{x}^*$ is Pareto optimal if there does not exist another feasible allocation $\mathbf{x}$ such that $u_i(x_i) \geq u_i(x_i^*)$ for all $i = 1, \ldots, I$ and $u_j(x_j) > u_j(x_j^*)$ for some j.

Define the **utility possibility set** $\mathcal{U}$ as

$$\mathcal{U} = \{\mathbf{u} \in \mathbb{R}^I | u_i \leq u_i(x_i) \text{ for } i = 1, \ldots, I, \text{ for some feasible allocation } \mathbf{x}\}.$$

It describes the attainable vectors of utility levels for this economy.

Define the **Pareto frontier** as

$$\mathcal{UP} = \{\mathbf{u} \in \mathcal{U} | \text{ there is no } \mathbf{u}' \in \mathcal{U} \text{ such that } u'_i \geq u_i \text{ for all } i \text{ and } u'_j > u_j \text{ for some } j\},$$

which is equivalent to the set of utility vectors of all Pareto optimal allocations. You can use the fact that $\mathcal{UP} \subseteq$ the boundary of $\mathcal{U}$.

Now consider a social welfare maximization problem. For some non-negative weights $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_I) \in \mathbb{R}^I_+$, $\alpha \neq 0$, a social planner aims to maximize a linear social welfare function:

$$\max_{\mathbf{x} \in \mathbb{R}^{IL}_+} \quad \sum_i \alpha_i u_i(x_i)$$
$$s.t. \sum_i x_i \leq \sum_i \omega_i$$

Prove **directly** the following statements.

1. If $\mathbf{x}^*$ is a solution to the social welfare maximization problem with some strictly positive $\alpha \in \mathbb{R}^I_{++}$, then it is a Pareto optimal allocation.

2. If each agent's utility function is concave, then the set $\mathcal{U}$ is convex.

3. Suppose $\mathcal{U}$ is convex. If $\mathbf{x}^*$ is a Pareto optimal allocaiton, then there exists some vector $\alpha \in \mathbb{R}^I_+$, $\alpha \neq 0$, such that $\mathbf{x}^*$ is a solution to the social welfare maximization problem with α.

Part B

(Please answer BOTH questions from this part.)

3. ***VCG***

Consider the allocation of three identical and indivisible objects to three agents, $N = \{1, 2, 3\}$, each of whom can consume multiple units. Agent i values the consumption of k units of the object at v^i_k. Her valuation profile $\theta^i = (v^i_1, v^i_2, v^i_3)$ is her private information. All that is publicly known is that for each player i, $0 \leq v^i_1 \leq v^i_2 \leq v^i_3$. Suppose the true valuations of the three players are as follows,

	θ^1	θ^2	θ^3
v_1	5	4	7
v_2	10	9	10
v_3	15	14	12

The table above shows, for instance, that player 1 has valuations $v^1_1 = 5$, $v^1_2 = 10$ and $v^1_3 = 15$.
Suppose the three players report their respective valuation profile truthfully,
a) What is the allocation of the three objects that maximizes the sum of values(utilities) across all players, for such a profile? (Who gets what?)
b) What are the transfers according to the VCG mechanism for this valuation profile?
(Note: You are only required to state the allocation and transfers for *this particular* valuation profile and not for any general valuation profile. The correct answer involves explicit numbers and not algebraic expressions.)
Now suppose instead that Player 1 chooses to lie about his valuation profile by reporting $\theta^1 = (11, 12, 20)$.

All the others report truthfully.
c) What is the allocation of the three objects that maximizes the sum of *reported* values(utilities) across all players, for such a *reported* profile? (Who gets what?)
d) What are the transfers according to the VCG mechanism for this *reported* valuation profile?
e) How much does Player 1 gain or lose by lying?

4. *Information Transmission with Lying Costs*
Bob needs to hire someone to do all the math for important theorems he needs to prove. Ann is applying for the position. Ann could either be proficient at math or not with equal probability. If Ann is proficient in math then she has access to a certificate (for free) which states so. If Ann is not proficient, then the cost of getting such a certificate to her is 1. Ann's choice is simply whether to send the certificate or not. After Bob observes Ann's decision about the certificate, he decides whether to hire Ann or not. Hiring a mathematically proficient person gives Bob a payoff of 2 while hiring someone without the requisite skills gives him a payoff of -3. Not hiring anyone gives him a payoff of 0. Ann gets a payoff of 3 from being hired and 0 from staying unemployed irrespective of whether she is mathematically skilled or not.
a) Model this situation as a sequential game and write down the extensive form.
b) Is there a sequential equilibrium in which Ann provides a certificate *if and only if* she is proficient and Bob hires (with positive probability) only if a certificate is provided? If so, state the equilibrium
c) Is there a pooling equilibrium in this game? If so, state the equilibrium.
d) Is there a sequential equilibrium in which Bob hires (with positive probability) only if he receives a certificate? If so, state the equilibrium.
Note: Parts b and d are different (but perhaps related) questions.